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A CLASS OF ARITHMETIC FUNCTIONS ON $\mathbf{PSL_2}(\mathbb{Z})$

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Abstract. In [3] and [2], Atanassov introduced the two arithmetic functions

$$I(n) = \prod_{p^\alpha || n} p^{1/\alpha} \quad \text{and} \quad R(n) = \prod_{p^\alpha || n} p^{\alpha - 1}$$

called the irrational factor and the restrictive factor, respectively. Alkan, Ledoan, Panaitopol, and the authors explore properties of these arithmetic functions in [1], [7], [8] and [9]. In the present paper, we generalize these functions to a larger class of elements of $\mathrm{PSL}_2(\mathbb{Z})$, and explore some of the properties of these maps.

1. Introduction

In [3] and [2], Atanassov introduced the two arithmetic functions

$$I(n) = \prod_{p^\alpha || n} p^{1/\alpha} \quad \text{and} \quad R(n) = \prod_{p^\alpha || n} p^{\alpha - 1}$$

called the irrational factor and the strong restrictive factor, respectively. These functions are multiplicative, and satisfy the inequality

$$I(n)R(n)^2 \geq n,$$

with equality if and only if n is squarefree. In [8], Panaitopol showed that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\phi(n)} < e^2,$$

and proved that the function

$$G(n) = \prod_{\nu=1}^{n} I(\nu)^{1/n}$$

satisfies the inequalities

$$e^{-7} n < G(n) < n.$$

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In [1], Alkan, Ledoan and one of the authors describe a precise asymptotic for the function $G(n)$, and establish further results showing that the function $I(n)$ is very regular on average.

In [7], asymptotic formulas are established for certain weighted real moments of the restrictive factor $R(n)$. In [9], the authors establish asymptotic formulas for weighted combinations $I(n)^\alpha R(n)^\beta$.

In the present paper we consider for a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{PSL}_2(\mathbb{Z})$ the fractional linear transformation Az given by

$$Az = \frac{az+b}{cz+d}.$$

For each positive integer n, define

$$f_A(n) = \prod_{p^\alpha || n} p^{\frac{a\alpha+b}{c\alpha+d}}.$$

As an example, the function $I(n)$ is equal to $f_{A_0}(n)$ for

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We shall consider weighted averages of the functions $f_A(n)$. Let

$$M_A(x) = \frac{1}{x} \sum_{n \le x} \left(1 - \frac{n}{x}\right) f_A(n).$$

Consider the subset $\mathcal{A} \subset \mathrm{PSL}_2(\mathbb{Z})$ given by

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \det A = -1, a, b, d \ge 0, c \ge 1 \right\}.$$

Define for each positive rational number r

$$E_r = \{A \in \mathcal{A} : M_A(x) \asymp x^r \text{ as } x \to \infty\}.$$

Note that if $r_1 \ne r_2$, then $E_{r_1} \cap E_{r_2} = \emptyset$. We will prove that each E_r with $r > 0$ consists of exactly one element.

For each matrix A in $\mathcal{A}$ we define the *associated series* $(A_n)_{n\in\mathbb{N}}$ by

$$A_n = An = \frac{an+b}{cn+d}.$$

As we shall see, the associated series plays an important role in our computations. Clearly, if $A \in \mathcal{A}$, then A_n is monotone decreasing and has the finite limit $A_\infty := a/c$.

We have the following result.

Theorem 1.1. *Given $A \in \mathcal{A}$, if $A_1 > 0$, then there are positive real-valued constants K_A and c such that*

$$M_A(x) = K_A x^{A_1} + O_A\left(x^{A_1-1/2}\exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}\right).$$

We remark that under the Riemann hypothesis, for a restricted class of matrices one has an asymptotic formula for the error term in Theorem 1.1 of the form

$$M_A(x) - K_A x^{A_1} \sim \widetilde{K}_A x^{\frac{1}{2}(A_2-1)} \tag{1}$$

for a real-valued constant $\widetilde{K}_A$. This naturally leads one to consider the maps $\psi_j : \mathcal{A} \to \mathbb{Q}_+$ for $j = 1, 2$ given by

$$\psi_j(A) = A_j. \tag{2}$$

Since, as mentioned above, each E_r consists of exactly one element, it follows that there is a well-defined map $s : \mathbb{Q}_+ \to \mathbb{Q}_+$ given by

$$s(r) = \psi_2 \circ \psi_1^{-1}(r). \tag{3}$$

The map $s(r)$ tells us how accurately the main term $K_A x^{A_1}$ approximates $M_A(x)$ in (1), in the sense that it gives the exact order of magnitude of the error $M_A(x) - K_A x^{A_1}$.

Although it can be shown that this map is nowhere continuous, one can obtain asymptotic formulas for the average value of $s(r)$, with r in various ranges. For example, define the height function for each rational $r = p/q$ with $q \geq 1$ and $(p, q) = 1$ by

$$h(r) := \max\{|p|, |q|\}.$$

We have the following result.

Theorem 1.2. *For any $\delta > 0$,*

$$\sum_{\substack{r\in\mathbb{Q}_+\cap[0,1]\\ h(r)\leq X}} s(r) = \frac{3}{2\pi^2}X^2 + O_\delta(X^{11/6+\delta}).$$

2. Asymptotics of the average

Consider the Dirichlet series

$$F_A(s) = \sum_{n=1}^{\infty} \frac{f_A(n)}{n^s}.$$

We will take advantage of the meromorphic continuation of $F_A(s)$ in the case where $\det A = -1$.

Proof of Theorem 1.1. We prove the result with

$$K_A = \frac{1}{(1+A_1)(2+A_1)\zeta(2)} T_A(1+A_1).$$

If $\det A = -1$, then $p^{A\alpha} \le p^{A_1}$ for all $\alpha \ge 1$, so $f_A(n) \le n^{A_1}$, hence $F_A(s)$ converges in the half plane $\Re s = \sigma > 1 + A_1$. Moreover, $F_A(s)$ has an Euler product in that region. Write

$$F_A(s) = \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} \prod_p (1 + g_p(s)),$$

where

$$g_p(s) = \left(1 + \frac{p^{A_1}}{p^s}\right)^{-1} \sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}}.$$

Note that if $\det(A) = -1$, then $A_1 - A_2 = \frac{1}{(c+d)(2c+d)} \le \frac{1}{2}$. Take $\epsilon > 0$. For $\sigma \ge A_1 + \epsilon$ we have

$$\left(1 + \frac{p^{A_1}}{p^s}\right)^{-1} \ll_\epsilon 1.$$

Also, for $\sigma \ge \frac{1}{2}(1 + A_2 + \epsilon)$ we have

$$\sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}} \ll \frac{p^{A_2}}{p^{2s}} \ll_\epsilon \frac{1}{p^{1+\epsilon}}.$$

Thus for $\sigma \ge \max\{A_1 + \epsilon, \frac{1}{2}(1 + A_2 + \epsilon)\}$ the sum $\sum_p |g_p(s)|$ converges, hence

$$T_A(s) = \prod_p (1 + g_p(s))$$

is analytic for $\sigma > \sigma_0 = \max\left\{A_1, \frac{1}{2}(1 + A_2)\right\}$, so $F_A(s)$ is meromorphic there, with a poles at $s = 1 + A_1$.

To continue, we utilize a variant of Perron's formula and write

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) f_A(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} T_A(s) \frac{x^s}{s(s+1)}\, ds,$$

where $1 + A_1 < c \le 5/4 + A_1$.

We apply the zero-free region for $\zeta(s)$ due to Korobov [6] and Vinogradov [12] (see Chapters 2 and 5 of the reference by Walfisz [13] for an alternative treatment)

$$\sigma \ge 1 - c_0 (\log t)^{-2/3} (\log\log t)^{-1/3}$$

for $t \ge t_0$, in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3} (\log\log t)^{1/3}.$$

Fix $0 < U < T \le x$, let $\nu = 1/2 + A_1$ and

$$\eta = \nu - c_0 (\log U)^{-2/3} (\log\log U)^{-1/3}.$$

Deform the path of integration into the union of the line segments

$$\begin{cases} \gamma_1, \gamma_9 : s = c + it & \text{if } |t| \geq T \\ \gamma_2, \gamma_8 : s = \sigma \pm iT & \text{if } \nu \leq \sigma \leq c \\ \gamma_3, \gamma_7 : s = \nu + it & \text{if } U \leq |t| \leq T \\ \gamma_4, \gamma_6 : s = \sigma \pm iU & \text{if } \eta \leq \sigma \leq \nu \\ \gamma_5 : s = \eta + it & \text{if } |t| \leq U. \end{cases}$$

The integrand is analytic on and within this modified contour, hence by Cauchy's theorem

$$xM_A(x) = \frac{1}{(1+A_1)(2+A_1)\zeta(2)} T_A(1+A_1)x^{1+A_1} + \sum_{k=1}^{9} J_k,$$

with the main terms coming from the residue at the simple pole at $s = 1 + A_1$.

In order to estimate the integral along our modified contour we will make use of the bounds

$$|\zeta(\sigma + it)| = \begin{cases} O(t^{(1-\sigma)/2}, & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1 \\ O(\log t), & \text{if } 1 \leq \sigma \leq 2 \\ O(1), & \text{if } \sigma \geq 2 \end{cases}$$

(see [11], §3.11 and §5.1).

On the line segments on which $s = c + it$, $|t| \geq T$, we have that $\zeta(s - A_1) \ll \log t$ and $1/\zeta(2s - 2A_1) \ll \log t$, so

$$\begin{aligned} |J_1|, |J_9| &\ll \int_T^\infty (\log t)^2 \frac{x^c}{|(c+it)(c+1+it)|}\, dt \\ &\ll \frac{x^c (\log T)^2}{T}. \end{aligned}$$

On the line segments on which $s = \sigma + iT$, $\nu \leq \sigma \leq c$, we have that $1/\zeta(2s - 2A_1) \ll \log T$, $\zeta(s - A_1) \ll T^{(1-\sigma+A_1)/2}$ for $\nu \leq \sigma \leq 1 + A_1$, and $\zeta(s - A_1) \ll \log T$ for $1 + A_1 \leq \sigma \leq c$. So

$$\begin{aligned} |J_2|, |J_8| &\ll \int_\nu^{1+A_1} T^{\frac{1}{2}(1-\sigma+A_1)} \log T \frac{x^\sigma}{T^2} d\sigma + \int_{1+A_1}^{c} (\log T)^2 \frac{x^\sigma}{T^2} d\sigma \\ &\ll T^{\frac{1}{2}(1+A_1)} \log T \max\left\{ \left(\frac{x}{\sqrt{T}}\right)^\nu, \left(\frac{x}{\sqrt{T}}\right)^{1+A_1} \right\} + \frac{(\log T)^2}{T^2} x^c. \end{aligned}$$

On the line segments on which $s = \nu + it$, $U \leq |t| \leq T$, we have that $\zeta(s - A_1) \ll t^{(1-\nu+A_1)/2}$ and $1/\zeta(2s - 2A_1) \ll \log t$, so

$$\begin{aligned} |J_3|, |J_7| &\ll \int_U^T (\log t) t^{\frac{1}{2}(1-\nu+A_1)} \frac{x^\nu}{|(\nu+it)(\nu+1+it)|}\, dt \\ &\ll \frac{\log T}{U^{3/4}} x^\nu. \end{aligned}$$

On the line segments on which $s=\sigma+iU$, $\eta\le\sigma\le\nu$, we have that $\zeta(s-A_1)\ll U^{(1-\sigma+A_1)/2}$ and $1/\zeta(2s-2A_1)\ll\log U$, so

$$\begin{aligned}|J_4|,|J_6| &\ll \int_\eta^\nu (\log U)U^{\frac{1}{2}(1-\sigma+A_1)}\frac{x^\sigma}{U^2}\,d\sigma\\ &\ll U^{\frac{1}{2}(1+A_1)-2}\log U\max\left\{\left(\frac{x}{\sqrt{U}}\right)^\nu,\left(\frac{x}{\sqrt{U}}\right)^\eta\right\}.\end{aligned}$$

On the line segment on which $s=\eta+it$, $|t|\le U$, we have that $\zeta(s-A_1)\ll(|t|+1)^{(1-\eta+A_1)/2}$ and $1/\zeta(2s-2A_1)\ll\log(|t|+1)$, so

$$\begin{aligned}|J_5| &\ll \int_{-U}^{U}(|t|+1)^{1-\eta+A_1}\log(|t|+1)\frac{x^\eta}{|\eta+it||\eta+1+it|}\,dt\\ &\ll x^\eta\int_{-U}^{U}(|t|+1)^{\frac{1}{2}(1-\eta+A_1)-2}\log(|t|+1)\,dt.\end{aligned}$$

Since $\frac{1}{2}(1-\eta+A_1)-2\le-\frac{3}{2}$ for U sufficiently large, the above integral converges, hence $|J_5|\ll x^\eta$.

We collect all estimates, and take $T=x^2$ and

$$U=\exp\{c_2(\log x)^{3/5}(\log\log x)^{-1/5}\}$$

to obtain the desired result. □

One could instead factor

$$\left(1+\frac{p^{A_1}}{p^s}+\frac{p^{A_2}}{p^{2s}}+\frac{p^{A_3}}{p^{3s}}+\cdots\right)=\left(1+\frac{p^{A_1}}{p^s}\right)\left(1+\frac{p^{A_2}}{p^{2s}}\right)(1+g_p(s))$$

with

$$g_p(s)=\left(1+\frac{p^{A_1}}{p^s}\right)^{-1}\left(1+\frac{p^{A_2}}{p^{2s}}\right)^{-1}\left(-\frac{p^{A_1+A_2}}{p^{3s}}+\sum_{k=3}^{\infty}\frac{p^{A_k}}{p^{ks}}\right)$$

so that

$$F_A(s)=\frac{\zeta(s-A_1)}{\zeta(2s-2A_1)}\frac{\zeta(2s-A_2)}{\zeta(4s-2A_2)}\prod_p(1+g_p(s))\,.$$

Under the Riemann hypothesis, we get a second order term of the form $\widetilde{K}_Ax^{A_2}$ in the asymptotic formula for $F_A(s)$ provided that $\frac{1}{4}+A_1<\frac{1}{2}(1+A_2)$. That is, provided that

$$a+b<\frac{c+d}{2}-\frac{1}{2c+d}.$$

This occurs for matrices A in $\mathcal{A}$ with restrictions on c and d. One can see that A_1 will lie in the interval $(0,1/2)$.

3. Mapping through $\mathbf{PSL}_2(\mathbb{Z})$

We now return to the two maps ψ_1 and ψ_2 defined in (2).

Lemma 3.1. *The map ψ_1 is bijective.*

Proof. For $\frac{p}{q} \in \mathbb{Q}_+$, consider the set of matrixes

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$ such that $\psi_1(A) = \frac{p}{q}$. We note that any such quadruple (a, b, c, d) is constrained by $c \geq 0$, $d \geq 0$,

$$ad - bc = -1 \tag{4}$$

and

$$c + d = \frac{q}{p}(a + b) \tag{5}$$

(Note that (4) implies that p cannot be zero). By (5) we have

$$c = \frac{q}{p}(a + b) - d.$$

Inserting this into (4) gives us

$$ad - b(a + b)\frac{q}{p} + bd = -1$$

so

$$(a + b)(pd - qb) = -p.$$

Write $a + b = \pm n$ for some positive integer $n \mid p$. By (5) we have $c + d = \frac{q}{p}(\pm n) \in \mathbb{Z}$ so $p \mid n$, hence $p = n$.

There are two cases: If $a + b = -p$, then $c + d = -q$. This contradicts the assumptions that $q \geq 1$ and c and d are non-negative. On the other hand, if $a + b = p$, then $c + d = q$, so (4) gives us

$$a(q - c) - bc = -1$$

so

$$pc = 1 + aq. \tag{6}$$

So c is uniquely determined by $cp \equiv 1 \pmod{q}$ and $1 \leq c < q$. Then d is uniquely determined by $d = q - c$, and a and b by $a = \frac{1-pc}{q}$ and $b = p - a$. □

In the case where $p/q \in (0, 1]$, we identify p/q as an element of $\mathcal{F}_Q$, the Farey fractions of order Q, with $Q \geq q$. If we consider the "minimal" set of Farey fractions $\mathcal{F}_q$ containing p/q, then elementary properties of Farey fractions (see for example Chapter 3 of [5]) give that the adjacent Farey fractions $p'/q' < p/q < p''/q''$ satisfy $q' = \bar{p}$, $p' = \bar{q}$, $p'' = p - \bar{q}$ and $q'' = q - \bar{p}$. Here $\bar{p}$ is the unique integer $1 \leq \bar{p} < q$ satisfying $p\bar{p} \equiv 1 \pmod{q}$ and $\bar{q}$ is the unique integer $1 \leq \bar{q} < p$ satisfying $q\bar{q} \equiv 1 \pmod{p}$. We can write

$$\psi_1(p/q) = \begin{pmatrix} \bar{q} & p - \bar{q} \\ \bar{p} & q - \bar{p} \end{pmatrix}.$$

That is, the matrix $\psi_1(p/q)$ is comprised of the "parent" Farey fractions in $\mathcal{F}_{q-1}$.

Additionally, we can write the function $s(p/q)$ from (3) uniquely as

$$s(p/q) = \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)}. \tag{7}$$

To prove Theorem 1.2, we will use the following result (see Lemma 2.3 of [4]).

Lemma 3.2. *Assume that $q \geq 1$ and h are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q, and $f : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ is a C^1 function. Then for any integer $T \geq 1$ and any $\delta > 0$*

$$\sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ ab \equiv h \pmod{q} \\ gcd(a,b)=1}} f(a, b) = \frac{\phi(q)}{q^2} \iint_{\mathcal{I} \times \mathcal{J}} f(x, y) dx dy + \mathcal{E},$$

with

$$\mathcal{E} \ll_\delta T^2 \|f\|_\infty q^{1/2+\delta} gcd(h, q)^{1/2} + T \|\nabla f\|_\infty q^{3/2+\delta} gcd(h, q)^{1/2} + \frac{\|\nabla f\|_\infty |\mathcal{I}||\mathcal{J}|}{T},$$

where $\|f\|_\infty$ and $\|\nabla f\|_\infty$ denote the sup-norm of f and respectively $|\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$ on $\mathcal{I} \times \mathcal{J}$.

Proof of Theorem 1.2. Let $Q = \lfloor X \rfloor$. Since $r \in \mathcal{F}_Q$ we have

$$\sum_{\substack{r \in \mathbb{Q}_+ \cap [0,1] \\ h(r) \leq X}} s(r) = \sum_{\substack{1 \leq q \leq Q \\ 1 \leq p < q \\ (p,q)=1}} s(p/q).$$

We use (7) and Lemma 3.2 with $T = q^{\frac{1}{6} - \frac{\delta}{3}}$ to get that the right-hand sum is equal to

$$\sum_{1 \leq q \leq Q} \sum_{\substack{1 \leq p < q \\ 1 \leq \bar{p} < q \\ p\bar{p} \equiv 1 \pmod{q} \\ (p,q)=1}} \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)} = \sum_{1 \leq q \leq Q} \frac{\phi(q)}{q^2} \iint_{[1,q)^2} \frac{vu - 1 + uq}{q(v + q)} du dv + \mathcal{E}$$

$$= \sum_{1 \leq q \leq Q} \phi(q) \iint_{[1/q,1]^2} \frac{xy - \frac{1}{q^2} + x}{y + 1} dx dy + \mathcal{E}.$$

where $\mathcal{E} \ll_\delta q^{5/6+\delta}$. The integral is equal to

$$\frac{1}{2}\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)-\frac{q-1}{q^3}\left(\log 2-\log\left(1+\frac{1}{q}\right)\right)=\frac{1}{2}+O\left(\frac{1}{q}\right)$$

so

$$\sum_{\substack{r\in\mathbb{Q}_+\cap[0,1]\\ h(r)\le X}} s(r) = \frac{1}{2}\sum_{1\le q\le Q}\phi(q)+O\left(\sum_{1\le q\le Q}\frac{\phi(q)}{q}\right)+O\left(\sum_{1\le q\le Q} q^{5/6+\delta}\right).$$

One can use the methods of Section 2 to estimate the sums over $\phi(q)$, or use partial summation along with standard estimates (see for example [13] or Chapter 18 of [5]). This gives the main term of our theorem; the first error term above is $O(X)$, and the second is $O_\delta(X^{11/6+\delta})$. □

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