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SOLVABILITY AND BOUNDEDNESS FOR GENERAL VARIATIONAL INEQUALITY PROBLEMS

Gui-Mei Luo

Abstract. In this paper, we propose a sufficient condition for the existence of solutions to general variational inequality problems ($GVI(K, F, g)$). The condition is also necessary when F is a g-P_*^M function. We also investigate the boundedness of the solution set of ($GVI(K, F, g)$). Furthermore, we show that when F is norm-coercive, the general complementarity problems ($GCP(K, F, g)$) has a nonempty compact solution set. Finally, we establish some existence theorems for ($GNCP(K, F, g)$).

1. Introduction

The finite-dimensional general variational inequality problem ($GVI(K, F, g)$), introduced by Noor [14], is to find a vector $x^* \in \Re^n$, such that $g(x^*) \in K$ and

$$F(x^*)^T(y - g(x^*)) \geq 0, \quad \forall y \in K, \tag{1.1}$$

where K is a closed convex subset of $\Re^n$, F and g are mappings from K into $\Re^n$. When K is a cone, problem (1.1) is called a general complementarity problem, denoted by $GCP(K, F, g)$, which is equivalent to finding a vector $x^* \in \Re^n$ such that

$$g(x^*) \in K, \quad F(x^*) \in K^* \quad \text{and} \quad g(x^*)^T F(x^*) = 0, \tag{1.2}$$

where $K^* \equiv \{z \in \Re^n : z^T x \geq 0, \forall x \in K\}$ is the dual cone of K. If $K = \Re^n_+$ (the nonnegative orthant of $\Re^n$), then $GCP(K, F, g)$ is called a general nonlinear complementarity problem, denoted by $GNCP(K, F, g)$:

$$g(x^*) \geq 0, \quad F(x^*) \geq 0 \quad \text{and} \quad g(x^*)^T F(x^*) = 0. \tag{1.3}$$

In particular, when $g \equiv I$, the identity operator, problems (1.1)-(1.3) reduce to the standard variational inequality problem ($VI(K, F)$), complementarity problem ($CP(K, F)$) and nonlinear complementarity problem ($NCP(K, F)$),

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respectively. $VI(K,F)$ and $CP(K,F)$ have been extensively studied since the mid-1960s and have been developed into a very fruitful discipline with rich theoretical results and numerical algorithms (see e.g., [1, 3, 5, 6, 7, 8, 9, 10, 12] and references therein). We refer to [6] for a good review in the progress of $VI(K,F)$ and $CP(K,F)$.

Since Noor [14] introduced the general variational inequality problem of odd-order obstacle problems, the general variational inequality problem theory has been extended to nonsymmetric, odd-order free, moving, unilateral and equilibrium problems arising in elasticity, transportation, circuit analysis, finance, economics and operations research etc. (see [2, 4, 8, 24, 11, 19, 21, 22, 23] and references therein). They have been developed many numerical methods for general variational inequality problems. We refer to [16, 18] for a good review. The study on the theory of general variational inequality problems has also taken good progress. Under the assumption that functions F and g are locally Lipschitz continuous, Pang and Yao [20] provided some sufficient conditions for the existence of solution to $GVI(K,F,g)$ by means of Fréchet approximate Jacobian matrix. Luc and Noor [13] proved the local uniqueness of the solution of $GVI(K,F,g)$, in which the local Lipschitz restriction was removed. Noor [15] established the equivalence between problem $GVI(K,F,g)$ and the Wiener-Hopf equation. Xiu [21] showed the equivalence between $GVI(K,F,g)$ and the tangent projection equation. Recently, using a projection technique, Noor [17] established the equivalence between the extended general variational inequalities and the general nonlinear projection equation. In this paper, we further study the existence and boundedness of the solution of $GVI(K,F,g)$ and $GCP(K,F,g)$. The results obtained in the paper can be regarded as extensions of those obtained in [3, 25, 26] to general variational inequality problems. The major contributions of the paper are listed as follows.

We give a necessary and sufficient condition for a general variational inequality problem with g-P_*^M function to have a solution. Since g-P_*^M function contains pseudo-monotone function and P_0 function as special cases, the necessary and sufficient condition given in this paper is extensions of Theorems 2.3.4 and 3.5.11 in [3].

Under the condition that F is strictly feasible and quasi-g-P_*^M, we obtain the nonemptiness of the solution set for $GNCP(K,F,g)$. The solution set for $GNCP(K,F,g)$ is nonempty and compact if F is strictly feasible and strictly quasi-g-P_*^M.

The paper is organized as follows. Section 2 establishes sufficient and necessary conditions for the existence of the solution to the $GVI(K,F,g)$ by using a g-P_*^M function. Section 3 focuses on the boundedness of the solution set of the $GCP(K,F,g)$ and $GNCP(K,F,g)$.

2. Existence and boundedness of the solution set to $GVI(K, F, g)$

In this section, we first derive a necessary and sufficient condition for the existence of the solution to $GVI(K, F, g)$. To this end, we need some useful lemmas.

The following lemma comes from [14].

Lemma 2.1. *Let K be a closed convex subset in $\Re^n$. Let F and g be continuous mappings from K into $\Re^n$. Then x^* is a solution to the $GVI(K, F, g)$ if and only if $g(x^*) = P_K(g(x^*) - F(x^*))$, where $P_K(\cdot)$ denotes the orthogonal projection operator on the convex set K.*

Remark 2.1. If we define $F^{nat}_{K,g}(x) = g(x) - P_K(g(x) - F(x))$ as the natural map of the triple (K, F, g), then Lemma 2.1 is equivalent to saying that x solves the $GVI(K, F, g)$ if and only if $F^{nat}_{K,g}(x) = 0$. If $g \equiv I$, Lemma 2.1 reduces to Proposition 1.5.8 in [3].

The following lemma comes from [3].

Lemma 2.2. *Let D be an open bounded subset in $\Re^n$. Assume that $F : \overline{D} \to \Re^n$ is continuous. If $y \in \Re^n$, $y \notin F(\partial D)$ and $\deg(F, D, y) \neq 0$, then $F(x) = y$ has a solution in D.*

By the use of Lemmas 2.1 and 2.2, it is not difficult to prove the following lemma.

Lemma 2.3. *Let $K \subseteq \Re^n$ be closed and convex and $F : U \supseteq K \to \Re^n$ be continuous on an open set U. Let $F^{nat}_{K,g}$ denote the natural mapping of the triple (K, F, g). If there exists a bounded open set D such that $\overline{D} \subseteq U$ and $\deg(F^{nat}_{K,g}, D)$ is well defined and nonzero, then the $GVI(K, F, g)$ has a solution in D.*

We introduce another lemma from [3].

Lemma 2.4. *Let D be an open bounded subset in $\Re^n$, F and G be two continuous functions from D into $\Re^n$. Define the homotopy $H(x, t)$ by*

$$H(x, t) = tG(x) + (1 - t)F(x), \qquad 0 \leq t \leq 1.$$

Let y be an arbitrary point in $\Re^n$. If $y \notin \{H(x, t) : x \in \partial D, t \in [0, 1]\}$, then

$$\deg(G, D, y) = \deg(F, D, y).$$

In this section, we consider the problem (1.1) with $K \subseteq \Re^n$ given by

$$K = \Pi_{\nu=1}^{N} K_\nu, \tag{2.1}$$

where N is a positive integer and each K_ν is a subset of $\Re^{n_\nu}$ with $\sum_{\nu=1}^{N} n_\nu = n$. Consistent with this structure of K, we write $\Re^n = \Pi_{\nu=1}^{N} \Re^{n_\nu}$ and $g = (g_1, g_2, \ldots, g_N)$, where g_ν is a mapping from K_ν into $\Re^{n_\nu}$, $\nu = 1, \ldots, N$.

Theorem 2.1. *Let K be a closed convex subset of $\Re^n$ defined by* (2.1)*. Let F be a continuous mapping from K into $\Re^n$ and g be a continuous injective mapping from K into $\Re^n$. Consider the following three statements.*

(a) *There exists an $x^* \in \Re^n$ such that $g(x^*) \in K$ and the set*

$$L'_< \equiv \{x \in \Re^n : g(x) \in K, (g_\nu(x) - g_\nu(x^*))^T F_\nu(x) < 0, \ \forall \nu \text{ s.t. } g_\nu(x) \neq g_\nu(x^*)\}$$

is bounded.

(b) *There exist a bounded open subset $D \subseteq \Re^n$ and a vector $x^* \in \Re^n$ with $g(x^*) \in D \cap K$ such that for every $x \in \Re^n$ with $g(x) \in K \cap \partial D$, there exists a ν satisfying $g_\nu(x^*) \neq g_\nu(x)$ and $(g_\nu(x) - g_\nu(x^*))^T F_\nu(x) \geq 0$.*

(c) *The $GVI(K, F, g)$ has a solution.*

It holds that (a)$\Rightarrow$(b)$\Rightarrow$(c)*. Moreover, if the set*

$$L'_\leq \equiv \{x \in \Re^n : g(x) \in K, \max_{1\leq\nu\leq N} (g_\nu(x) - g_\nu(x^*))^T F_\nu(x) \leq 0\},$$

which contains $L'_<$, is bounded, then the $GVI(K, F, g)$ has a nonempty and compact solution set $SOL(K, F, g)$.

Proof. The proof of the relations among (a), (b) and (c) and the existence of the solution is similar to the proof of Proposition 3.5.1 in [3]. We only show the compactness of the solution set $SOL(K, F, g)$.

For any $x \in SOL(K, F, g)$, it is clear that $g(x) \in K$ and $(y - g(x))^T F(x) \geq 0, \forall y \in K$. Define $y = (y_{\nu'}) \in K$ by

$$y_{\nu'} = \begin{cases} g_\nu(x^*), & \text{if } \nu = \nu', \\ g_\nu(x), & \text{if } \nu \neq \nu'. \end{cases}$$

Then we obtain

$$(g_\nu(x) - g_\nu(x^*))^T F_\nu(x) \leq 0, \quad \forall \nu.$$

This implies $x \in L'_\leq$. It is easy to show the closedness of the set $SOL(K, F, g)$. □

Theorem 2.1 extends Proposition 3.5.1 in [3] (where $g \equiv I$). In what follows, we investigate the equivalence among (a), (b) and (c) under mild conditions. To this end, we introduce the following definition.

Definition 2.1. Let K be a closed convex subset of $\Re^n$, and F and g be mappings from K into $\Re^n$. F is said to be

(a) a g-P_0 function on K if for any $x, y \in \Re^n$ satisfying $g(x), g(y) \in K, g(x) \neq g(y)$, there exists $\nu \in \{1, 2, \ldots, N\}$ such that $g_\nu(x) \neq g_\nu(y)$ and

$$(g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \geq 0;$$

(b) a g-pseudo-monotone function on K if for any $x, y \in \Re^n$ satisfying $g(x)$, $g(y) \in K$, $g(x) \neq g(y)$,

$$(g(x) - g(y))^T F(y) \geq 0 \Longrightarrow (g(x) - g(y))^T F(x) \geq 0;$$

(c) a g-P_*^M function on K if there exists a constant $\tau \geq 0$ such that for any $x, y \in \Re^n$ with $g(x)$, $g(y) \in K$, $g(x) \neq g(y)$, there exists $\nu \in \{1, 2, \ldots, N\}$ satisfying $g_\nu(x) \neq g_\nu(y)$ and

$$(g_\nu(x) - g_\nu(y))^T F_\nu(y) - \tau \max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \geq 0 \tag{2.2}$$

implies $(g_\nu(x) - g_\nu(y))^T F_\nu(x) \geq 0$.

(d) a quasi-g-P_*^M function on K if (2.2) is replaced by

$$(g(x) - g(y))^T F(y) - \tau \max_{1 \leq i \leq n} (g_i(x) - g_i(y))^T (F_i(x) - F_i(y)) > 0. \tag{2.3}$$

(e) a strictly quasi-g-P_*^M function on K if

$$(g(x) - g(y))^T F(y) - \tau \max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) > 0$$

implies

$$(g(x) - g(y))^T F(x) > 0.$$

(f) a g-P function on K if for any $x, y \in \Re^n$ satisfies $g(x)$, $g(y) \in K$, $g(x) \neq g(y)$,

$$\max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) > 0.$$

(g) a g-ξ-P function on K for some $\xi > 1$ if there exists a constant $\mu > 0$ such that for any $x, y \in \Re^n$ satisfying $g(x)$, $g(y) \in K$, $g(x) \neq g(y)$,

$$\max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \geq \mu \|g(x) - g(y)\|^\xi.$$

A particular case of the g-ξ-P function is the g-2-P function. It is called a uniformly g-P function.

Remark 2.2. (1) If $g \equiv I$, then (a)-(e) reduce to the definitions of P_0, pseudo-monotone, P_*^M (quasi-P_*^M, strictly quasi-P_*^M), P and ξ-P functions, respectively (see [3, 25, 26]).

(2) It is easy to see that a g-P_0 function must be a g-P_*^M function. In fact, for a g-P_0 function F, there exists $\nu \in \{1, 2, \ldots, N\}$ such that

$$(g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \geq 0.$$

This implies

$$\max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \geq 0.$$

Therefore, we get

$$\begin{aligned}(g_\nu(x) - g_\nu(y))^T F_\nu(x) &\geq (g_\nu(x) - g_\nu(y))^T F_\nu(y) \\ &\geq \tau \max_{1 \leq \nu \leq N} (g_\nu(x) - g_\nu(y))^T (F_\nu(x) - F_\nu(y)) \\ &\geq 0.\end{aligned}$$

It is obvious that a g-pseudo-monotone function must be a g-P_*^M function.

The following theorem establishes the equivalence among (a), (b) and (c). It can be proved in a way similar to the proof of Theorem 3.5.11 in [3].

Theorem 2.2. *Let the conditions in Theorem 2.1 be satisfied. If F is a g-P_*^M function, then the conditions* (a), (b) *and* (c) *in Theorem 2.1 are equivalent.*

Remark 2.3. Since a g-P_0 function must be a g-P_*^M function and g can be any continuous injective function, Theorem 2.2 is an extension of Theorem 3.5.11 in [3]. In fact, if we let $g \equiv I$ and F be a P_0 function, Theorem 2.2 reduces to Theorem 3.5.11 in [3]. Similarly, Theorem 2.2 can also be viewed as an extension of Theorem 2.3.4 in [3], since a pseudo-monotone function must be a g-P_*^M function.

Theorem 2.3. *Let the conditions in Theorem 2.1 be satisfied. If F is a g-ξ-P function and g satisfies $\|g(x)\| \to \infty$ as $\|x\| \to \infty$, then the $GVI(K, F, G)$ has a unique solution.*

Proof. We first show that $L'_{\leq}$ must be bounded for all $x^* \in \Re^n$ with $g(x^*) \in K$. Suppose on the contrary that there exists an $x^* \in \Re^n$ such that $g(x^*) \in K$ and

$$L'_{\leq} \equiv \{x \in \Re^n : g(x) \in K, \max_{1\leq\nu\leq N} (g_\nu(x) - g_\nu(x^*))^T F_\nu(x) \leq 0\}$$

is unbounded. This means that $L'_{\leq}$ contains an unbounded sequence $\{x^k\}$. Since $x^k \in L'_{\leq}$, $\forall k$, we have

$$g(x^k) \in K \quad \text{and} \quad (g_\nu(x^k) - g_\nu(x^*))^T F_\nu(x^k) \leq 0, \quad \forall 1 \leq \nu \leq N, \quad \forall k. \tag{2.4}$$

Taking into account that F is a g-ξ-P function, there exists a $\bar\nu \in \{1, 2, \ldots, N\}$ such that

$$(g_{\bar\nu}(x^k) - g_{\bar\nu}(x^*))^T (F_{\bar\nu}(x^k) - F_{\bar\nu}(x^*)) \geq \mu \|g(x) - g(x^*)\|^\xi. \tag{2.5}$$

Combining (2.4) with (2.5), we obtain

$$\begin{aligned} 0 &\geq F_{\bar\nu}(x^k)^T (g_{\bar\nu}(x^k) - g_{\bar\nu}(x^*)) \\ &= (F_{\bar\nu}(x^k) - F_{\bar\nu}(x^*))^T (g_{\bar\nu}(x^k) - g_{\bar\nu}(x^*)) + F_{\bar\nu}(x^*)^T (g_{\bar\nu}(x^k) - g_{\bar\nu}(x^*)) \\ &\geq \mu \|g(x^k) - g(x^*)\|^\xi - \|g_{\bar\nu}(x^k) - g_{\bar\nu}(x^*)\| \cdot \|F_{\bar\nu}(x^*)\| \\ &\geq \mu \|g(x^k) - g(x^*)\|^\xi - \|g(x^k) - g(x^*)\| \cdot \|F_{\bar\nu}(x^*)\| \\ &= \|g(x^k) - g(x^*)\| (\mu \|g(x^k) - g(x^*)\|^{\xi - 1} - \|F_{\bar\nu}(x^*)\|). \end{aligned}$$

Letting $k \to \infty$, then the right-hand side tends to $+\infty$ since $\xi > 1$. This yields a contradiction. Consequently, $L'_{\leq}$ is bounded. It then follows from Theorem 2.1 that the $GVI(K, F, g)$ has a nonempty and compact solution set.

We are going to show the uniqueness of the solution set. Suppose that there exist two distinct solutions x_1 and x_2. We define for each index $\nu \in \{1, 2, \ldots, N\}$, the vector $y = (y_{\nu'}) \in K$ by

$$y_{\nu'} = \begin{cases} g_\nu(x_1), & \text{if } \nu = \nu', \\ g_\nu(x_2), & \text{if } \nu \neq \nu'. \end{cases}$$

It is clear that

$$0 \leq (y - g(x_2))^T F(x_2) = (g_\nu(x_1) - g_\nu(x_2))^T F_\nu(x_2).$$

Similarly, we have

$$0 \leq (g_\nu(x_2) - g_\nu(x_1))^T F_\nu(x_1).$$

Adding these two inequalities, we obtain

$$(g_\nu(x_1) - g_\nu(x_2))^T (F_\nu(x_1) - F_\nu(x_2)) \leq 0, \forall \nu = 1, \dots, N.$$

This contradicts the g-ξ-P property of F. The complete is complete. □

Remark 2.4. Since a g-ξ-P function includes a uniform g-P function, the above result is an extension of Proposition 3.5.10 in [3] (where $g \equiv I$).

3. Boundedness of the solution set to $GCP(K, F, g)$

In this section, we investigate the boundedness of the solution set to $GCP(K, F, g)$, where K is a closed convex cone in $\Re^n$. First, in a way similar to the proofs of Theorem 2.6.1 and Corollary 2.6.2 in [3], it is not difficult to establish the following proposition.

Proposition 3.1. *Let K be a closed convex cone in $\Re^n$. Let F be a continuous mapping from K into $\Re^n$ and g be an injective continuous mapping from K into $\Re^n$. Then either the $GCP(K, F, g)$ has a solution or there exist an unbounded sequence $\{x^k\}$ and a positive sequence $\{\tau_k\}$ such that*

$$K \ni g(x^k) \perp F(x^k) + \tau_k g(x^k) \in K^*, \quad \forall k.$$

By the use of Proposition 3.1, we can establish the following theorem.

Theorem 3.1. *Let the conditions in Proposition 3.1 be satisfied. Suppose that there exists a vector $d \in \Re^n$ such that*

$$g(x)^T (F(x) - d) \geq 0, \quad \forall g(x) \in K, \tag{3.1}$$

and that the natural map $F^{nat}_{K,g}(x)$ is norm-coercive on K, that is,

$$\lim_{g(x) \in K, \|x\| \to \infty} \|F^{nat}_{K,g}(x)\| = \infty. \tag{3.2}$$

Then the $GCP(K, q+F, g)$ has a nonempty compact solution set for all $q \in \Re^n$.

Proof. Since the solution set coincides with the set of zeros of $F^{nat}_{K,g}(x)$, the boundedness of the solution set follows from (3.2) immediately. The closedness of the solution set is also obvious. In what follows, we verify the nonemptiness of the solution set. Consider the case $q = 0$. For the sake of contradiction, we suppose that the solution set $SOL(K, F, g)$ is empty. By Proposition 3.1, there exist an unbounded sequence $\{x^k\}$ and a positive sequence $\{\tau_k\}$ such that

$$K \ni g(x^k) \perp F(x^k) + \tau_k g(x^k) \in K^*, \quad \forall k. \tag{3.3}$$

This implies

$$P_K(g(x^k) - (F(x^k) + \tau_k g(x^k))) = g(x^k), \quad \forall k. \tag{3.4}$$

It follows from (3.1) and (3.3) that

$$0 \leq g(x^k)^T (F(x^k) - d)$$

$$= g(x^k)^T(F(x^k) + \tau_k g(x^k) - \tau_k g(x^k) - d)$$
$$= -\tau_k \|g(x^k)\|^2 - g(x^k)^T d.$$

So, we get

$$\tau_k \|g(x^k)\|^2 \leq -g(x^k)^T d \leq \|g(x^k)\| \cdot \|d\|,$$

which implies

$$\tau_k \|g(x^k)\| \leq \|d\|, \quad \forall k. \tag{3.5}$$

On the other hand, by (3.2) and (3.4), we have

$$\begin{aligned}
\infty &= \lim_{k\to+\infty} \|F^{nat}_{K,g}(x^k)\| \\
&= \lim_{k\to+\infty} \|g(x^k) - P_K(g(x^k) - F(x^k))\| \\
&= \lim_{k\to+\infty} \|P_K(g(x^k) - (F(x^k) + \tau_k g(x^k))) - P_K(g(x^k) - F(x^k))\| \\
&\leq \lim_{k\to+\infty} \|\tau_k g(x^k)\| \\
&\leq \|d\|,
\end{aligned}$$

where the first inequality follows from the nonexpansive property of the projection operator. The last inequality yields a contradiction. Consequently, the set $SOL(K, F, g)$ is not empty.

Consider the case $q \neq 0$. It suffices to verify that the function $g(x) - P_K(g(x) - F(x) - q)$ is norm-coercive on K. Indeed, we have

$$\begin{aligned}
&\lim_{g(x)\in K, \|x\|\to\infty} \|g(x) - P_K(g(x) - F(x) - q)\| \\
= &\lim_{g(x)\in K, \|x\|\to\infty} \|g(x) - P_K(g(x) - F(x)) + P_K(g(x) - F(x)) \\
&\qquad - P_K(g(x) - F(x) - q)\| \\
\geq &\lim_{g(x)\in K, \|x\|\to\infty} \|g(x) - P_K(g(x) - F(x))\| \\
&- \lim_{g(x)\in K, \|x\|\to\infty} \|P_K(g(x) - F(x)) - P_K(g(x) - F(x) - q)\| \\
\geq &\lim_{g(x)\in K, \|x\|\to\infty} \|F^{nat}_{K,g}(x)\| - \lim_{g(x)\in K, \|x\|\to\infty} \|q\| \\
= &\lim_{g(x)\in K, \|x\|\to\infty} \|F^{nat}_{K,g}(x)\| - \|q\| \\
= &\infty.
\end{aligned}$$

The proof is complete. □

Remark 3.1. Theorem 3.1 extends Corollary 2.6.4 in [3] in two-folds. First the problem here is $GCP(K, F, g)$ which is an extension of the problem $CP(K, F)$. Second the condition (3.1) is weaker than the condition in [3].

The following theorem gives another condition to guarantee the boundedness of the solution set for $GCP(K, F, g)$. It is an improvement of Theorem 3.1.

Theorem 3.2. *Let the conditions in Proposition 3.1 be satisfied. Suppose that there is a vector* $d \in \Re^n$ *such that inequality* (3.1) *and*

$$\liminf_{g(x)\in K, \|x\|\to\infty} \|F_{K,g}^{nat}(x)\| > \|d\| \tag{3.6}$$

hold. Then the $GCP(K, F, g)$ *has a nonempty compact solution set.*

Proof. Obviously, the solution set must be bounded and closed by (3.6). We only need to show that the solution set of $GCP(K, F, g)$ is nonempty. Suppose that the $GCP(K, F, g)$ has no solution. Similarly as in the proof in Theorem 3.1, we can obtain (3.4) and (3.5). Therefore, we get from (3.6), (3.4) and (3.5),

$$\begin{aligned}\|d\| &< \liminf_{k\to+\infty} \|F_{K,g}^{nat}(x)\| \\ &= \liminf_{k\to+\infty} \|g(x^k) - P_K(g(x^k) - F(x^k))\| \\ &= \liminf_{k\to+\infty} \|P_K(g(x^k) - (F(x^k) + \tau_k g(x^k))) - P_K(g(x^k) - F(x^k))\| \\ &\le \liminf_{k\to+\infty} \|\tau_k g(x^k)\| \\ &\le \|d\|.\end{aligned}$$

The last inequality yields a contradiction. Consequently, the solution set of $GCP(K, F, G)$ is not empty. □

The remainder of the paper is devoted to the existence of the solution to the general nonlinear complementary problem $GNCP(K, F, g)$. We first introduce the following definition.

Definition 3.1. A mapping F is said to be g-proper at some point x^* with $g(x^*) \in K$ if the set

$$\{x \in \Re^n : g(x) \in K, (g(x) - g(x^*))^T F(x^*) \le 0\}$$

is bounded.

Theorem 3.3. *Let the conditions in Proposition 3.1 be satisfied. Suppose that* g *satisfies* $\|g(x)\| \to \infty$ *as* $\|x\| \to \infty$*, and that* F *is a quasi-*g*-*P_*^M *function. If* $GNCP(K, F, g)$ *is strictly feasible, then it has a solution.*

Proof. Let x^* be a strictly feasible point. That is, it satisfies $g(x^*) \ge 0$ and $F(x^*) > 0$. It suffices to show that $L_<$ is bounded at x^*.

Suppose on the contrary that $L_<$ is nonempty and unbounded. Then there exists a sequence $\{x^k\} \subseteq L_<$ with $\|x^k\| \to \infty$. By the definition of $L'_<$, we obviously have $g(x^k) \ge 0$ for all k. Since $F(x^*) > 0$ and $\|g(x_k)\| \to \infty$ as $k \to \infty$, we claim

$$(g(x^k) - g(x^*))^T F(x^*) \to +\infty \quad \text{as} \quad k \to \infty.$$

Therefore, inequality (2.3) holds with $\tau = 0$ for k sufficiently large. Notice that F is a quasi-g-P_*^M function, we have

$$(g(x^k) - g(x^*))^T F(x^k) \ge 0 \tag{3.7}$$

for k sufficiently large. This yields a contraction with $x^k \notin L_<$. Consequently, the $GNCP(K, F, g)$ has a solution. □

Remark 3.2. Theorem 3.3 is an extension of Theorem 3.1 in [26] where $g \equiv I$.

If the quasi-g-P_*^M function is replaced by the strictly quasi-g-P_*^M function in the last theorem, it is not difficult to show that the solution set of $GNCP(K, F, g)$ is nonempty and compact. We state the related theorem as follows but omit the proof.

Theorem 3.4. *Let the conditions in Proposition 3.1 be satisfied. Suppose that g satisfies $\|g(x)\| \to \infty$ as $\|x\| \to \infty$ and that F is a strictly quasi-g-P_*^M function. If $GNCP(K, F, g)$ is strictly feasible, then it has a nonempty and compact solution set.*

4. Conclusion

In our work, we provided some necessary or sufficient conditions for a general variational inequality problem to have a solution. At the same time, we considered the structure of the solution set under different conditions. In our future work, we may study the sensitivity analysis of the associated dynamical system related to the general variational inequalities, or extend the general variational inequality problem to multi-valued and system of extended general variational inequalities, or investigate the applications of general variational inequality problem such as equilibrium theory and engineering etc..

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Department of Applied Mathematics
Guangdong University of Finance
Guangzhou, Guangdong 510521, P. R. China
E-mail address: lgm2008@163.com