Bull. Korean Math. Soc. **50** (2013), No. 2, pp. 537–542
http://dx.doi.org/10.4134/BKMS.2013.50.2.537

ADDENDUM AND ERRATUM TO "ON THE STRUCTURE OF THE FUNDAMENTAL GROUP OF MANIFOLDS WITH POSITIVE SCALAR CURVATURE"

Jin Hong Kim

After the publication of the paper [6], it has been noticed that there is an incomplete argument in Lemma 3.2. So we would like to first correct one of the main results of [6] (Theorem 1.2) and then have an opportunity to add some more related results (Theorem 1 below).

As remarked in Remark 2.2 of [6], if the dimension of the manifold M is four, then the scalar curvature of N_α can be made positive after a suitable conformal change. Since N_α is compact, this implies that the scalar curvature of N_α is bounded from below by the constant $k > 0$. Thus the scalar curvature of the universal cover $\bar{N}_\alpha$ with respect to the pullback metric is again bounded from below by k. But then a result of Gromov-Lawson or Schoen-Yau says that the homotopy fill radius of $\bar{N}_\alpha$ is bounded from above. So Theorem 1.2 in Section 4 holds to be true in this case without any further condition. On the other hand, if the dimension of M is greater than 4 and less than or equal to 7, then at the moment we need to add one of the following extra conditions to Theorem 1.2: $\bar{N}_\alpha$ has the bounded homotopy fill radius or the self-adjoint elliptic operator $\bar{\mathcal{L}} = -\Delta_{\bar{N}_\alpha} + \frac{n-3}{4(n-2)} R_{\bar{N}_\alpha}$ is positive-definite on the universal cover $\bar{N}_\alpha$ of N_α. Actually the latter implies the former, as shown in Theorem 3.1 of the paper [6].

Recently we have also obtained an interesting result which is closely related to the main results of the paper [6] (refer to the paper [5] for more detailed accounts). To be precise, our result is stated as follows.

Theorem 1. *Let M be a closed oriented Riemannian manifold of dimension 4 with positive isotropic curvature. Then the fundamental group of M does not contain a subgroup isomorphic to the fundamental group of a compact Riemann surface of genus ≥ 2.*

This extends previous results of Fraser [3], Fraser and Wolfson [4], and Brendle-Schoen [1] to the case of a closed oriented Riemannian manifold of

Received November 14, 2011.
2010 *Mathematics Subject Classification.* 53C20.
Key words and phrases. positive scalar curvature, positive isotropic curvature, fundamental group.

dimension 4 with positive isotropic curvature and a compact Riemann surface with genus ≥ 2. The proof of Theorem 1 is achieved by essentially adapting their proofs to the case of a compact Riemann surface with genus ≥ 2, and we provide it here for the sake of reader's convenience. We also remark that in their paper [2] B.-L. Chen, S.-H. Tang, and X.-P. Zhu announced a complete classification of closed oriented Riemannian manifolds of dimension 4 with positive isotropic curvature which would affirmatively answer the conjecture of Gromov and so Theorem 1 (refer to Conjecture 1.1 of the paper [6]). However, as far as we know, their result has not been published anywhere, yet, and the method of the proof of Theorem 1 is completely different from theirs.

Proof of Theorem 1. For the proof, we suppose that $\pi_1(M)$ contains a subgroup G which is isomorphic to π_1 of a compact Riemann surface Σ_0 of genus $g_0 \geq 2$. Then we will derive a contradiction. The following lemma plays a crucial role.

Lemma 2 (Theorem 1.1 in [4]). *Given any $C > 0$, there is an integer k and a normal subgroup $N(k, C)$ of G with index k such that*

(1) *there is a smooth map $h_{k,C} : \Sigma \to M$ of a compact Riemann surface Σ into M satisfying the property that*
$$(h_{k,C})_* : \pi_1(\Sigma) \to \pi_1(M)$$
is injective onto $N(k, C)$,

(2) *for any such $h_{k,C}$, every closed non-trivial geodesic γ on Σ has length $> C$ with respect to the induced metric by $h_{k,C}$.*

If we carefully look at the proof of Lemma 2 (or Theorem 1.1 in [4]), then one can easily see that as C goes to infinity, so does k. For the proof of Theorem 1, we will also need the following lemma.

Lemma 3 (Theorem 1.2 in [4]). *If every closed non-trivial geodesic γ on a compact Riemann surface Σ has length $> C$, then there is a Lipschitz distance decreasing degree-one map $f : \Sigma \to S^2$ such that $C|df| \leq D$, where D is some constant independent of C.*

Now, we may assume that M has positive isotropic curvature $\geq \kappa > 0$, since M is compact. Then fix a positive constant $C > 0$ which will be chosen explicitly later. Let $h_0 : \Sigma_0 \to M$ be a smooth map such that $(h_0)_* : \pi_1(\Sigma_0) \to \pi_1(M)$ is an isomorphism onto G. Then it follows from Lemma 2 that there are a compact Riemann surface Σ of genus g, a regular k-covering $p : \Sigma \to \Sigma_0$, and a smooth map $h_{k,C} : \Sigma \to M$ given by $h_{k,C} = h_0 \circ p$ such that $(h_{k,C})_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective with $(h_{k,C})_*(\pi_1(M)) =: N(k, C)$ a normal subgroup of G of index k. For simplicity, let $h = h_{k,C}$. Note that the Euler characteristic $\chi(\Sigma)$ is equal to k times of the Euler characteristic $\chi(\Sigma_0)$ of Σ_0, so we obtain $2 - 2g = k(2 - 2g_0)$ and thus $g = k(g_0 - 1) + 1 \geq 1$. Note also from the proof of Lemma 2 that for every map $\tilde{h} : \Sigma \to M$ whose induced map $\tilde{h}_*$ on $\pi_1(\Sigma)$ equals h_*, every closed non-trivial geodesic γ has length $> C$ with respect to the induced metric by $\tilde{h}$. Since h is incompressible in the sense that $h_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective, it follows from a theorem of Schoen and Yau that there is a stable conformal branched minimal immersion $u : \Sigma \to M$ such that $u_* = h_*$ on $\pi_1(\Sigma)$.

Let F denote the pull-back of the normal bundle $\mathcal{N}$ of the minimal surface $u(\Sigma)$ equipped with the pull-back metric and normal connection $\nabla^\perp$. Then it is known that F is a smooth vector bundle of real rank 2 over Σ, even across the branch points (see p. 8 of [1]). Let E be the complexification of F, so $E = F \otimes \mathbf{C}$. Since E is a complexification of a real bundle F which is isomorphic to its dual bundle F, E is isomorphic to its dual bundle E^* so that we have $c_1(E) = 0$. The metric on F extends as a complex bilinear form $(\ ,\ )$ on E or as a Hermitian metric $\langle\ ,\ \rangle$ on E. So the connection $\nabla^\perp$ and the curvature form extend complex linearly to sections s of E. Then there is a unique holomorphic structure on E such that the $\bar\partial$ operator is given by

$$\bar\partial s = \left(\nabla^\perp_{\frac{\partial}{\partial \bar z}} s\right) d\bar z,$$

where x and y are local coordinates on Σ and $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Moreover, E splits as a direct sum of a holomorphic line bundle $E^{(1,0)}$ and an anti-holomorphic line bundle $E^{(0,1)}$. Hence we have

$$0 = c_1(E) = c_1(E^{(1,0)}) + c_1(E^{(0,1)}).$$

This implies that we may assume without loss of generality that $c_1(E^{(1,0)}) \geq 0$.

Recall that, by using the complexified formula for the second variation of area, the stability condition can be stated as

$$\int_\Sigma \left(|\nabla^\perp_{\frac{\partial}{\partial z}} s|^2 - |\nabla^T_{\frac{\partial}{\partial z}} s|^2\right) dxdy \geq \int_\Sigma \langle R(s, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar z}, s\rangle dxdy \tag{0.1}$$

for all s in the space of sections $\Gamma(E)$ (see [3] and [4]). Notice that every section $s \in \Gamma(E^{(1,0)})$ is isotropic in the sense that $(s, s) = 0$. Since M has positive isotropic curvature, it follows from (0.1) that we obtain

$$\int_\Sigma |\nabla^\perp_{\frac{\partial}{\partial \bar z}} s|^2 dxdy \geq \int_\Sigma \kappa |\frac{\partial u}{\partial z}|^2 |s|^2 dxdy \tag{0.2}$$

for all $s \in \Gamma(E^{(1,0)})$. By using the inequality (0.2) and the fact that u is non-constant, one can also show that there is a positive constant $\varepsilon = \varepsilon(C)$ such that

$$\int_\Sigma |\nabla^\perp_{\frac{\partial}{\partial \bar z}} s|^2 dxdy + \frac{1}{2}\int_\Sigma \kappa |\frac{\partial u}{\partial z}|^2 |s|^2 dxdy \geq \frac{\kappa\varepsilon}{2}\int_\Sigma |s|^2 dxdy \tag{0.3}$$

for all $s \in \Gamma(E^{(1,0)})$. Indeed, suppose that for every $\varepsilon > 0$ we have

$$\int_\Sigma |\nabla^\perp_{\frac{\partial}{\partial \bar z}} s|^2 dxdy + \frac{1}{2}\int_\Sigma \kappa |\frac{\partial u}{\partial z}|^2 |s|^2 dxdy < \frac{\kappa\varepsilon}{2}\int_\Sigma |s|^2 dxdy. \tag{0.4}$$

Then it follows from (0.2) and (0.4) that we have

$$\begin{aligned}\frac{3}{2}\kappa\int_\Sigma |\frac{\partial u}{\partial z}|^2|s|^2dxdy &\leq \int_\Sigma |\nabla^\perp_{\frac{\partial}{\partial\bar z}} s|^2dxdy + \frac{1}{2}\kappa\int_\Sigma |\frac{\partial u}{\partial z}|^2|s|^2dxdy\\ &< \frac{\kappa\varepsilon}{2}\int_\Sigma |s|^2dxdy.\end{aligned}$$

Let $t = \frac{s}{\left(\int_\Sigma |s|^2dxdy\right)^{1/2}}$. Thus we obtain

$$\int_\Sigma |\frac{\partial u}{\partial z}|^2|t|^2dxdy < \frac{\varepsilon}{3}$$

for any $\varepsilon > 0$. Since $|t|^2$ is not zero everywhere and $|\frac{\partial u}{\partial z}| = |\frac{\partial u}{\partial \bar z}|$, we have $|\frac{\partial u}{\partial z}|^2 = 0$ and so $\frac{\partial u}{\partial z} = 0$ and $\frac{\partial u}{\partial \bar z} = 0$. This implies that u is constant, which is clearly a contradiction.

Now, taking the arithmetic mean of (0.2) and (0.3), we easily obtain

$$\int_\Sigma |\nabla^\perp_{\frac{\partial}{\partial\bar z}} s|^2dxdy \geq \frac{\kappa}{4}\int_\Sigma \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon\right)|s|^2dxdy \tag{0.5}$$

for all $s \in \Gamma(E^{(1,0)})$.

Next, we define a new Riemannian metric $\tilde g$ on Σ by

$$\tilde g = u^*g + 2\varepsilon(dx\otimes dx + dy\otimes dy) = u^*g + \varepsilon(dz\otimes d\bar z + d\bar z\otimes dz).$$

Then every closed non-trivial geodesic on Σ has length $> C$ with respect to $\tilde g$. So it follows from Lemma 3 that there is a Lipschitz distance decreasing map $f : \Sigma \to S^2$ of degree one with

$$C|df| \leq D,$$

where D is a constant independent of f. Hence we have

$$C^2|\frac{\partial f}{\partial z}|^2 \leq D|\frac{\partial}{\partial z}|^2_{\tilde g} = D\left(|\frac{\partial u}{\partial z}|^2 + \varepsilon\right). \tag{0.6}$$

Let ξ be a holomorphic line bundle over S^2 with $c_1(\xi) > g - 1$, where g is the genus of Σ. Fix a metric and a connection on ξ, and choose sections α_1 and α_2 in $\Gamma(\xi^*)$ such that $|\alpha_1| + |\alpha_2| \geq 1$ on S^2. Let $\tilde\xi = f^*\xi$. Then we have $c_1(\tilde\xi) > g-1$. Since $c_1(E^{(1,0)}) \geq 0$, we also have $c_1(E^{(1,0)}\otimes\tilde\xi) \geq c_1(\tilde\xi) > g-1$. By the Riemann-Roch theorem, we have

$$\begin{aligned}h^0(\Sigma, E^{(1,0)}\otimes\tilde\xi) &= h^1(\Sigma, E^{(1,0)}\otimes\tilde\xi) + c_1(E^{(1,0)}\otimes\tilde\xi) - g + 1\\ &\geq c_1(\tilde\xi) - g + 1 > 0,\end{aligned}$$

where $h^j(\Sigma, E^{(1,0)}\otimes\tilde\xi)$ $(j = 0, 1)$ denotes the complex dimension of the Dolbeaut cohomology $H^j(\Sigma, E^{(1,0)}\otimes\tilde\xi)$. This implies that there is a non-vanishing holomorphic section σ on $E^{(1,0)}\otimes\tilde\xi$.

For each $j = 1, 2$, set $\tau_j = f^*\alpha_j \in \Gamma(\tilde\xi^*)$ and $s_j = \sigma\otimes\tau_j \in \Gamma(E^{(1,0)})$. Since σ is holomorphic, we obtain $\nabla^\perp_{\frac{\partial}{\partial\bar z}} s_j = \sigma\otimes\nabla_{\frac{\partial}{\partial\bar z}}\tau_j$ and $|\nabla_{\frac{\partial}{\partial\bar z}}\tau_j|^2 = |\nabla_{\frac{\partial f}{\partial\bar z}}\alpha_j|^2 \leq$

$C'|\frac{\partial f}{\partial z}|^2$, where C' is a positive constant independent of k and C. Hence we have

$$|\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_j|^2 = |\sigma|^2 |\nabla_{\frac{\partial}{\partial \bar z}} \tau_j|^2 \leq C'|\sigma|^2 |\frac{\partial f}{\partial z}|^2 \tag{0.7}$$

for $j = 1, 2$. If we combine the inequality (0.6) with (0.7), we obtain

$$C^2 |\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_j|^2 \leq C^2 C' |\sigma|^2 |\frac{\partial f}{\partial z}|^2 \leq C' D |\sigma|^2 \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right).$$

On the other hand, it is easy to see

$$C^2 \int_\Sigma \left(|\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_1|^2 + |\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_2|^2 \right) dxdy \leq 2C'D \int_\Sigma |\sigma|^2 \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right) dxdy. \tag{0.8}$$

Since $|s_1| + |s_2| = |\sigma|(|\tau_1| + |\tau_2|) \geq |\sigma|$ on Σ, by (0.5) we also have

$$\begin{aligned} \int_\Sigma \left(|\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_1|^2 + |\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_2|^2 \right) dxdy &\geq \frac{\kappa}{4} \int_\Sigma (|s_1|^2 + |s_2|^2) \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right) dxdy \\ &\geq \frac{\kappa}{8} \int_\Sigma |\sigma|^2 \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right) dxdy. \end{aligned} \tag{0.9}$$

By comparing two inequalities (0.8) and (0.9), we have

$$\begin{aligned} 2C'D \int_\Sigma |\sigma|^2 \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right) dxdy &\geq C^2 \int_\Sigma \left(|\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_1|^2 + |\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} s_2|^2 \right) dxdy \\ &\geq \frac{\kappa}{8} C^2 \int_\Sigma |\sigma|^2 \left(|\frac{\partial u}{\partial z}|^2 + \varepsilon \right) dxdy. \end{aligned}$$

Thus it is easy to obtain $2C'D \geq \frac{\kappa}{8} C^2$, and so C should be less than or equal to $\left(\frac{16C'D}{\kappa}\right)^{1/2}$. Finally, if we take $C > \left(\frac{16C'D}{\kappa}\right)^{1/2}$, then we would have a contradiction. This completes the proof of Theorem 1. □

Note added in proof: The paper [2] is now published in Journal of Differential Geometry **91** (2012), no. 1, 41–80.

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0007404).

References

[1] S. Brendle and R. Schoen, *Sphere theorems in geometry*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, 49–84, Surv. Differ. Geom., 13, Int. Press, Somerville, MA, 2009.

[2] B.-L Chen, S.-H. Tang, and X.-P. Zhu, *Complete classification of compact four-manifolds with positive isotropic curvature*, preprint, 2008.; arXiv:0810.1999v1.

[3] A. Fraser, *Fundamental groups of manifolds with positive isotropic curvature*, Ann. Math. **158** (2003), no. 1, 345–354.

[4] A. Fraser and J. Wolfson, *The fundamental group of manifolds of positive isotropic curvature and surface groups*, Duke Math. J. **133** (2006), no. 2, 325–334.

[5] J. H. Kim, *Surface groups and four-manifolds with positive isotropic curvature*, preprint, 2011.

[6] J. H. Kim and H. C. Park, *On the structure of the fundamental group of manifolds with positive scalar curvature*, Bull. Korean Math. Soc. **48** (2011), no. 1, 129–140.

Department of Mathematical Sciences
KAIST
Daejeon 305-701, Korea
E-mail address: jinhkim11@gmail.com