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n-ARY HYPERGROUPS ASSOCIATED WITH n-ARY RELATIONS

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Abstract. The notion of n-ary algebraic hyperstructures is a generalization of ordinary algebraic hyperstructures. In this paper, we associate an n-ary hypergroupoid (H, f) with an $(n+1)$-ary relation ρ_{n+1} defined on a non-empty set H. Then, we obtain some basic results in this respect. In particular, we investigate when it is an n-ary H_v-group, an n-ary hypergroup or a join n-ary space.

1. Introduction and basic definitions

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced by Marty [14]. The connections between hyperstructures and binary relations have been analyzed by many researchers, such as Corsini [1], Corsini and Leoreanu [2], De Salvo and Lo Faro [7, 8], Leoreanu and Leoreanu [13], Rosenberg [16], Rasouli and Davvaz [15], Spartalis [17], Spartalis and Mamaloukas [18] and so on. n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. In [6], Davvaz and Vougiouklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. Also, we can consider n-ary hypergroups as a nice generalization of n-ary groups. In [11], Leoreanu-Fotea and Davvaz introduced and studied the notion of a partial n-ary hypergroupoid associated with a binary relation. Some important results concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-ary hypergroupoids. Then, n-ary hypergroups associated with union, intersection, products of relations and also mutually associative n-ary hypergroupoids are analyzed. Also, in [5], they investigated binary relations on ternary semihypergroups and studied some basic properties of binary relations on them. Davvaz and et al. in [4] considered a class of algebraic hypersystems which represent a generalization of semigroups, semihypergroups and n-ary semigroups. In

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[12], Leoreanu-Fotea and Davvaz studied the rough sets within the context of the commutative n-ary hypergroups. In [3], Cristea and Stefanescu extended some results on the hypergroups connected with binary relations to the case of n-ary relations. In particular, they established some connections between hypergroupoids associated with n-ary relations and hypergroupoids associated with binary or ternary relations.

Let H be a non-empty set and f a mapping $f : H^n \longrightarrow \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then, f is called an n-ary hyperoperation on H. We denoted by H^n the Cartesian product $H \times \cdots \times H$, where H appears n times and an element of H^n will be denoted by $(x_1, \dots, x_n)$, such that $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $f : H^n \longrightarrow \wp^*(H)$ is called an n-ary hyperoperation and n is called the arity of hyperoperation. Let f be an n-ary hyperoperation on H and $A_1, \dots, A_n$ be non-empty subsets of H. We define $f(A_1, \dots, A_n) = \cup\{f(x_1, \dots, x_n)|x_i \in A_i, i = 1, \dots, n\}$. We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \dots, x_j$ will be denoted by x_i^j. Also, for every $a \in H$, we write $f(\underbrace{a, \dots, a}_{n}) = f(\overset{(n)}{a})$ and for $j < i$, x_i^j is the empty set. In this convention $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, x_{j+1}, \dots, x_n)$ will be written $f(x_1^i, y_{i+1}^j, x_{j+1}^n)$. A non-empty set H with an n-ary hyperoperation $f : H^n \longrightarrow \wp^*(H)$ will be called an n-ary hypergroupoid and will be denote by (H, f). An n-ary hypergroupoid (H, f) is commutative if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1^n) = f(a_{\sigma(1)}^{\sigma(n)})$. An n-ary hypergroupoid (H, f) is called an n-ary semihypergroup if for any $i, j \in \{1, 2, \dots, n\}$ and $a_1^{2n-1} \in H$, we have

$$f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}) \quad \text{(associative law)}.$$

An n-ary hypergroupoid (H, f), in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in H$ and $1 \leq i \leq n$, is called a quasi n-ary hypergroup. A quasi n-ary hypergroup (H, f) with the associative law is called an n-ary hypergroup. An n-ary hypergroupoid (H, f) is called an n-ary H_v-semigroup if the following week associative axiom holds:

$$\bigcap_{i=1}^{2n-1} f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) \neq \emptyset$$

for any $x_1, x_2, \dots, x_{2n-1} \in H$. An n-ary H_v-semigroup (H, f) in which is a quasi n-ary hypergroup is called an n-ary H_v-group. Note that the notion of n-ary H_v-group is a generalization of H_v-group [20, 21].

2. n-ary relations

In this section, we present some basic results about the n-ary relations. Suppose that H is a non-empty set and $\rho \subseteq H^n$ is an n-ary relation on H. We recall the following definition from [3].

Definition 2.1. The relation ρ is said to be

(1) reflexive, if for any $x \in H$, the n-tuple $(x, \dots, x) \in \rho$;

(2) n-transitive if it has the following property: if $(x_1, \dots x_n) \in \rho$, $(y_1, \dots y_n) \in \rho$ hold and if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1}, \dots, x_{i_k}, y_{j_{k+1}}, \dots, y_{j_n}) \in \rho$, for any natural number $1 \leq k < n$ and $i_1, \dots, i_k, j_{k+1}, \dots, j_n$ such that $1 \leq i_1 < \dots < i_k < i_0, j_0 < j_{k+1} < \dots < j_n \leq n$;

(3) symmetric if $(x_1, x_2, \dots, x_n) \in \rho$ implies $(x_n, x_{n-1}, \dots, x_1) \in \rho$;

(4) strongly symmetric if $(x_1, x_2, \dots, x_n) \in \rho$ implies $(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \rho$ for any permutation σ of the set $\{1, \dots, n\}$;

(5) n-ary preordering on H if it is reflexive and n-transitive;

(6) n-equivalence on H if it is reflexive, strongly symmetric and n-transitive.

Example 1. Let $H = \mathbb{C}$ (complex numbers) and $(x_1, \dots, x_n) \in \rho$ when $|x_1| = |x_2| = \dots = |x_n|$. Then, ρ is reflexive, strongly symmetric and n-transitive.

Example 2. Let $H = \mathbb{N}$ (natural numbers) and $(x_1, \dots, x_n) \in \rho$ when $x_1 < x_2 < \dots < x_n$. It is easily to see that ρ is n-transitive but it is not reflexive and strongly symmetric.

Definition 2.2. Let ρ be an n-ary relation on a set H. For any $x \in H$ and any $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, n-(i+1)\}$, we define:

$$\begin{aligned} L_i(x) = \{y \in H \mid \exists u_1, \dots, u_{n-2} \in H : (y, u_1, \dots, u_{i-1}, x, u_i, \dots, u_{n-2}) \in \rho \\ \vee (u_1, \dots, u_k, y, u_{k+1}, \dots, u_{k+i-1}, x, u_{k+i}, \dots, u_{n-2}) \in \rho\}, \end{aligned}$$

and

$$\begin{aligned} R_i(x) = \{y \in H \mid \exists u_1, \dots, u_{n-2} \in H : (x, u_1, \dots, u_{i-1}, y, u_i, \dots, u_{n-2}) \in \rho \\ \vee (u_1, \dots, u_k, x, u_{k+1}, \dots, u_{k+i-1}, y, u_{k+i}, \dots, u_{n-2}) \in \rho\}. \end{aligned}$$

Example 3. In Example 1, for any $x \in H$ and $i \in \{2, \dots, n-1\}$, we have

$$L_i(x) = R_i(x) = \{z \in \mathbb{C} \mid \ |z| = |x|\}.$$

Example 4. In Example 2, for any $x \in H$ and $i \in \{1, \dots, n\}$, we have

$$L_i(x) = \{y \in \mathbb{N} | y < x + i\},$$
$$R_i(x) = \{y \in \mathbb{N} | y > x + i\}.$$

Remark 1. Let ρ be an n-ary relation on a set H. Then, it is obvious that

(1) $y \in L_i(x)$ if and only if $x \in R_i(y)$ for any $(x, y) \in H^2$ and any $i \in \{1, \dots, n\}$.

(2) $L_i(H) = \bigcup_{x \in H} L_i(x) \neq H$ if and only if there exists $y \in H$ such that $R_i(y) = \emptyset$,

(3) $R_i(H) = \bigcup_{x \in H} R_i(x) \neq H$ if and only if there exists $y \in H$ such that $L_i(y) = \emptyset$,

(4) $x \notin L_i(H)$ if and only if $R_i(x) = \emptyset$,

(5) $x \notin R_i(H)$ if and only if $L_i(x) = \emptyset$.

Indeed, $\bigcup_{x\in H} L_i(x) \neq H$ if there exists $y \in H$ such that $y \notin \bigcup_{x\in H} L_i(x)$, which is equivalent to the fact there exists $y \in H$ such that $y \notin L_i(x)$ for any $x \in H$, equivalent to the fact that there exists $y \in H$ such that $R_i(y) = \emptyset$.

Definition 2.3. Let ρ be an n-ary relation on the non-empty set H. Set $m = [\frac{n+1}{2}]$. We define on H the following n-ary hyperoperation:

$$f_\rho(x_1,\dots,x_n) = \bigcup_{i=1}^{m} L_i(x_i) \cup \bigcup_{i=1}^{m} R_i(x_{n-i+1}).$$

We notice that if (H, f_ρ) is an n-ary hypergroupoid, then $L_i(x) \neq \emptyset$ or $R_i(x) \neq \emptyset$ for some $x \in H$ and $i \in \{1,\dots,m\}$.

Theorem 2.4. *Let ρ be an n-ary relation on the non-empty set H. The n-ary hypergroupoid (H, f_ρ) is a quasi n-ary hypergroup if and only if for any $x \in H$ and any $1 \le i \le m$, $L_i(x) \neq \emptyset$ and $R_i(x) \neq \emptyset$.*

Proof. Let for any $x \in H$ and for any $1 \le i \le m$, $L_i(x) \neq \emptyset$ and $R_i(x) \neq \emptyset$. Then, $L_i(H) = H$ and $R_i(H) = H$. So, for every $x_1,\dots,x_n \in H$, we have

$$\begin{aligned}
&f_\rho(H, x_2,\dots,x_n)\\
&= L_1(H) \cup L_2(x_2) \cup \dots \cup L_m(x_m) \cup R_1(x_n) \cup \dots \cup R_m(x_{n-m+1}) = H,\\
&f_\rho(x_1, H,\dots,x_n)\\
&= L_1(x_1) \cup L_2(H) \cup \dots \cup L_m(x_m) \cup R_1(x_n) \cup \dots \cup R_m(x_{n-m+1}) = H,\\
&\quad\vdots\\
&f_\rho(x_1,\dots,H,x_n)\\
&= L_1(x_1) \cup \dots \cup L_m(x_m) \cup R_1(x_n) \cup R_2(H) \cup \dots \cup R_m(x_{n-m+1}) = H,\\
&f_\rho(x_1,\dots,x_{n-1},H)\\
&= L_1(x_1) \cup \dots \cup L_m(x_m) \cup R_1(H) \cup R_2(x_{n-1}) \cup \cdots\\
&\quad \cup R_{m-1}(x_{n-m+2}) \cup R_m(x_{n-m+1}) = H.
\end{aligned}$$

Thus, (H, f_ρ) is reproductive, so it is a quasi n-ary hypergroup.

Conversely, suppose that (H, f_ρ) is a quasi n-ary hypergroup and for some $i \in \{1,\dots,m\}$, there exists $x \in H$ such that $L_i(x) = \emptyset$ or $R_i(x) = \emptyset$.

If $L_i(x) = \emptyset$, then $x \notin R_i(H)$. Also, it easy to see that for any $j \in \{1,\dots,m\}$, $x \notin L_j(x)$ (also $x \notin R_j(x)$). Therefore,

$$\begin{aligned}
x \notin \quad & L_1(x) \cup \dots \cup L_m(x) \cup R_1(x) \cup \dots \cup R_i(H) \cup \dots \cup R_m(x)\\
&= f_\rho(x,\dots,H,\dots,x) = H,
\end{aligned}$$

where H is in the i-place and this contradicts the reproducibility low. If $R_i(x) = \emptyset$, the similar argument implies a contradiction. □

Example 5. Let $H = \{1,2,3,4\}$ and

$$\rho = \{(\underbrace{1,1,\dots,1}_{n-1},2), (3,\underbrace{1,1,\dots,1}_{n-2},3), (2,\underbrace{3,\dots,3}_{n-2},1),$$

$$(\underbrace{2,\ldots,2}_{n-1},3),(3,\underbrace{4,\ldots,4}_{n-1}),(\underbrace{4,\ldots,4}_{n-1},1)\}.$$

Now, for any $x \in H$ and $1 \le i \le m$, we have

	L_1	L_2	L_3	$\ldots$	L_m	R_1	R_2	$\ldots$	R_m
1	{1,3,4}	{1,3,4}	{1,3,4}	$\ldots$	{1,3,4}	{1,2,3}	{1,2,3}	$\ldots$	{1,2,3}
2	{1,2}	{1,2}	{1,2}	$\ldots$	{1,2}	{2,3}	{2,3}	$\ldots$	{2,3}
3	{1,2}	{1,2}	{1,2}	$\ldots$	{1,2}	{1,4}	{1,4}	$\ldots$	{1,4}
4	{3,4}	{3,4}	{3,4}	$\ldots$	{3,4}	{1,4}	{1,4}	$\ldots$	{1,4}

$$L_i(H) = L_i(1) \cup L_i(2) \cup L_i(3) \cup L_i(4) = \{1,2,3,4\},$$
$$R_i(H) = R_i(1) \cup R_i(2) \cup R_i(3) \cup R_i(4) = \{1,2,3,4\}.$$

Also,

$$\begin{aligned}
&f_\rho(H, x_2, \ldots, x_n)\\
&= (L_1(H) = \bigcup_{x\in H} L_1(x)) \cup L_2(x_2) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup\\
&\quad \cdots \cup R_m(x_{n-m+1})\\
&= \{1,2,3,4\} \cup L_2(x_2) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \ldots \cup R_m(x_{n-m+1}) = H,\\
&f_\rho(x_1, H, \ldots, x_n)\\
&= L_1(x_1) \cup (L_2(H) = \bigcup_{x\in H} L_2(x)) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup\\
&\quad \cdots \cup R_m(x_{n-m+1})\\
&= L_1(x_1) \cup \{1,2,3,4\} \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \cdots \cup R_m(x_{n-m+1}) = H,\\
&\qquad \vdots\\
&f_\rho(x_1, \ldots, H, x_n)\\
&= L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup (R_2(H) = \bigcup_{x\in H} R_2(x)) \cup\\
&\quad \cdots \cup R_m(x_{n-m+1})\\
&= L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \{1,2,3,4\} \cup \cdots \cup R_m(x_{n-m+1}) = H,\\
&f_\rho(x_1, \ldots, x_{n-1}, H)\\
&= L_1(x_1) \cup \cdots \cup L_m(x_m) \cup (R_1(H) = \bigcup_{x\in H} R_1(x)) \cup R_2(x_{n-1}) \cup\\
&\quad \cdots \cup R_m(x_{n-m+1})\\
&= L_1(x_1) \cup \cdots \cup L_m(x_m) \cup \{1,2,3,4\} \cup R_2(x_{n-1}) \cup \cdots \cup R_m(x_{n-m+1}) = H.
\end{aligned}$$

Therefore, the n-ary hypergroupoid (H, f_ρ) is a quasi n-ary hypergroup.

Theorem 2.5. *Let ρ be an n-ary relation on the non-empty set H. The n-ary hypergroupoid (H, f_ρ) is an n-ary H_v-group if and only if, for any $x \in H$ and $i \in \{1, \ldots, m\}$, $L_i(x) \neq \emptyset$ and $R_i(x) \neq \emptyset$.*

Proof. If (H, f_ρ) is an n-ary H_v-group, then it is a quasi n-ary hypergroup and by Theorem 2.4, it follows that for any $x \in H$ and $i \in \{1, \ldots, m\}$, $L_i(x) \neq \emptyset$ and $R_i(x) \neq \emptyset$.

Conversely, suppose that for any $x \in H$ and $i \in \{1, \ldots, m\}$, $L_i(x) \neq \emptyset$ and $R_i(x) \neq \emptyset$. By Theorem 2.4, it follows that (H, f_ρ) is a quasi n-ary hypergroup. It remains to prove that the n-ary hyperoperation f_ρ is weakly associative. For this, we show that, for any $x_1^{2n-1} \in H$,

$$\bigcap_{i=1}^{2n-1} f_\rho(x_1^{i-1}, f_\rho(x_i^{n+i-1}), x_{n+i}^{2n-1}) \neq \emptyset.$$

We have

(i_1)

$$\begin{aligned}
&f_\rho(f_\rho(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) \\
&= \{L_1(u) \cup L_2(x_{n+1}) \cup \cdots \cup L_m(x_{n+m-1}) \cup R_1(x_{2n-1}) \cup \cdots \cup R_m(x_{2n-m}) \mid \\
&\quad u \in L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \cdots \cup R_m(x_{n-m+1})\} \\
&\supseteq \{L_1(u) \mid u \in L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \cdots \cup R_m(x_{n-m+1})\} \\
&\supseteq \{L_1(u) \mid u \in R_1(x_n)\} = \{L_1(u) | x_n \in L_1(u)\} \ni x_n,
\end{aligned}$$

(i_2)

$$\begin{aligned}
&f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \\
&= \{L_1(x_1) \cup L_2(u) \cup \ldots \cup L_m(x_{n+m-1}) \cup R_1(x_{2n-1}) \cup \cdots \cup R_m(x_{2n-m}) \mid \\
&\quad u \in L_1(x_2) \cup \cdots \cup L_m(x_{m+1}) \cup R_1(x_{n+1}) \cup \cdots \cup R_m(x_{n-m+2})\} \\
&\supseteq \{L_1(u) \mid u \in L_1(x_2) \cup \cdots \cup L_m(x_{m+1}) \cup R_1(x_{n+1}) \cup \cdots \cup R_m(x_{n-m+2})\} \\
&\supseteq \{L_2(u) \mid u \in R_2(x_n)\} = \{L_2(u) | x_n \in L_2(u)\} \ni x_n,
\end{aligned}$$

$\vdots$

(i_{n-1})

$$\begin{aligned}
&f_\rho(x_1, \ldots, x_{n-2}, f_\rho(x_{n-1}, \ldots, x_{2n-2}), x_{2n-1}) \\
&= \{L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_{2n-1}) \cup R_2(u) \cup \cdots \cup R_m(x_{n-m+1}) \mid \\
&\quad u \in L_1(x_{n-1}) \cup \cdots \cup L_m(x_{n+m-2}) \cup R_1(x_{2n-2}) \cup \cdots \cup R_m(x_{2n-m-1})\} \\
&\supseteq \{R_2(u) \mid u \in L_1(x_{n-1}) \cup \cdots \cup L_m(x_{n+m-2}) \cup R_1(x_{2n-2}) \cup \\
&\qquad \cdots \cup R_m(x_{2n-m-1})\} \\
&\supseteq \{R_2(u) \mid u \in L_2(x_n)\} = \{R_2(u) | x_n \in R_2(u)\} \ni x_n,
\end{aligned}$$

(i_n)

$$f_\rho(x_1, \ldots, x_{n-1}, f_\rho(x_n, \ldots, x_{2n-1}))$$

$$= \{L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(u) \cup R_2(x_{n-1}) \cup \cdots \cup R_m(x_{n-m+1}) \mid$$
$$u \in L_1(x_n) \cup \cdots \cup L_m(x_{n+m-1}) \cup R_1(x_{2n-1}) \cup \cdots \cup R_m(x_{2n-m})\}$$
$$\supseteq \{R_1(u) \mid u \in L_1(x_n) \cup \cdots \cup L_m(x_{n+m-1}) \cup R_1(x_{2n-1}) \cup \cdots \cup R_m(x_{2n-m})\}$$
$$\supseteq \{R_1(u) \mid u \in L_1(x_n)\} = \{R_1(u) | x_n \in R_1(u)\} \ni x_n.$$

It follows that (H, f_ρ) is an n-ary H_v-group. □

Example 6. Let $H = \{1, 2, 3\}$ and $\rho = \{(\underbrace{1, \ldots, 1}_{n-2}, 2, 1), (2, \underbrace{3, \ldots, 3}_{n-1}), (\underbrace{2, \ldots, 2}_{n})\}$ be an n-ary relation on H. Now, we have:

	L_1	L_2	L_3	$\ldots$	L_m	R_1	R_2	$\ldots$	R_m
1	$\{1,2\}$	$\{1\}$	$\{1\}$	$\ldots$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$
2	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$	$\{1,2,3\}$	$\{2,3\}$	$\ldots$	$\{2,3\}$
3	$\{2\}$	$\{2,3\}$	$\{2,3\}$	$\ldots$	$\{2,3\}$	$\{3\}$	$\{3\}$	$\ldots$	$\{3\}$

Also,

$$f_\rho(f_\rho(\underbrace{1, \ldots, 1}_{n}), \underbrace{1, \ldots, 1}_{n-1}) = f_\rho(\{1, 2\}, 1, \ldots, 1) = \{1, 2\},$$
$$f_\rho(\underbrace{1, \ldots, 1}_{n-1}, f_\rho(\underbrace{1, \ldots, 1}_{n})) = f_\rho(\underbrace{1, \ldots, 1}_{n-1}, \{1, 2\}) = \{1, 2, 3\}.$$

This example shows that for every $x \in H$ and for any $i \in \{1, \ldots, m\}$, $L_i(x) \neq \emptyset$, $R_i(x) \neq \emptyset$ and (H, f_ρ) is an n-ary H_v-group but it is not an n-ary hypergroup.

Corollary 2.6. *Let ρ be an n-ary relation on a set H. The n-ary hypergroupoid (H, f_ρ) is an n-ary H_v-group if and only if it is a quasi n-ary hypergroup.*

Lemma 2.7. *Let ρ be an n-ary preordering on a set H. Then, for any $a, x, u \in H$ and $i \in \{1, \ldots, n-1\}$, such that $a \in L_i(u)$ $[a \in R_i(u)]$ and $u \in L_i(x)$ $[u \in R_i(x)]$, it follows that $a \in L_i(x)$ $[a \in R_i(x)]$.*

Proof. Let $a, x, u \in H$ such that $a \in L_i(u)$ and $u \in L_i(x)$. Then, there exist $a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2} \in H$ such that $(a, a_1, \ldots, a_{i-1}, u, a_i, \ldots, a_{n-2}) \in \rho$ or $(a_1, \ldots, a_k, a, a_{k+1}, \ldots, a_{k+i-1}, u, a_{k+i} \ldots, a_{n-2}) \in \rho$ for any $k \in \{1, \ldots, n-i-1\}$. Also, we have $(u, b_1, \ldots, b_{i-1}, x, b_i, \ldots, b_{n-2}) \in \rho$ or $(b_1, \ldots, b_h, u, b_{h+1}, \ldots, b_{h+i-1}, x, b_{h+i} \ldots, b_{n-2}) \in \rho$ for any $h \in \{1, \ldots, n-i-1\}$. In the all of the situations, by n-transitivity, we have $a \in L_i(x)$. In the similar way from $a \in R_i(u)$ and $u \in R_i(x)$ implies $a \in R_i(x)$. □

Definition 2.8 ([11])**.** Let (H, f_ρ) be a commutative n-ary hypergroup. For $a, b_1, \ldots, b_{n-1} \in H$, we denote $a/b_1^{n-1} = \{x \mid a \in f_\rho(x, b_1, \ldots, b_{n-1})\}$. We say that the commutative n-ary hypergroup (H, f_ρ) is a join n-ary space, if for any $a, c, b_1, b_2, \ldots, b_{n-1}, d_1, d_2, \ldots, d_{n-1} \in H$, the following implication holds:

$$a/b_1^{n-1} \cap c/d_1^{n-1} \neq \emptyset \Rightarrow f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c) \neq \emptyset.$$

Example 7. Let $\rho = \{(x, x, \ldots, x) | x \in H\}$ be the diagonal n-ary relation on a set H. Then, (H, f_ρ) is a join n-ary space. In fact, for any $i \in \{1, \ldots, m\}$ and $x \in H$, we obtain $L_i(x) = R_i(x) = \{x\}$ and thus, for any $x_1^n \in H$, it follows that $f_\rho(x_1, \ldots, x_n) = f_\rho(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \{x_1, \ldots, x_n\}$. Also, for any $x_1^n \in H$, $f_\rho(H, x_2, \ldots, x_n) = f_\rho(x_1, H, x_3, \ldots, x_n) = \ldots = f_\rho(x_1, \ldots, x_{n-1}, H) = H$. Moreover, for any $x_1^{2n-1} \in H$,

$$\begin{aligned}
(i_1) \quad & f_\rho(f_\rho(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) \\
& = f_\rho(\{x_1, \ldots, x_n\}, x_{n+1}, \ldots, x_{2n-1}) = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1}\}, \\
(i_2) \quad & f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \\
& = f_\rho(x_1, \{x_2 \ldots, x_{n+1}\}, x_{n+2}, \ldots, x_{2n-1}) = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1}\}, \\
& \vdots \\
(i_n) \quad & f_\rho(x_1, , x_{n-1}, f_\rho(x_n, \ldots, x_{2n-1})) \\
& = f_\rho(x_1, \ldots, x_{n-1}, \{x_n, \ldots, x_{2n-1}\}) = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1}\}.
\end{aligned}$$

So, (H, f_ρ) is a commutative n-ary hypergroup. It remains to prove that, for any $a, c, b_1, b_2, \ldots, b_{n-1}, d_1, d_2, \ldots, d_{n-1} \in H$,

$$a/b_1^{n-1} \cap c/d_1^{n-1} \neq \emptyset \Rightarrow f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c) \neq \emptyset.$$

We obtain that

$$\begin{aligned}
& a/a, b_1, \ldots, b_{n-2} \\
= & \{x \in H | a \in f_\rho(x, a, b_1, \ldots, b_{n-2})\} \\
= & \{x \in H | a \in L_1(x) \cup L_2(a) \cup L_3(b_1) \cup \cdots \cup L_m(b_{m-2}) \cup R_1(b_{n-2}) \cup \\
& \qquad \cdots \cup R_m(b_{n-m-1})\} \\
= & H, \\
& a/b_1, a, b_2, \ldots, b_{n-2} \\
= & \{x \in H | a \in f_\rho(x, b_1, a, b_2, \ldots, b_{n-2})\} \\
= & \{x \in H | a \in L_1(x) \cup L_2(b_1) \cup L_3(a) \cup L_4(b_2) \cup \cdots \cup L_m(b_{m-2}) \cup R_1(b_{n-2}) \cup \\
& \qquad \cdots \cup R_m(b_{n-m-1})\} \\
= & H, \\
& \vdots \\
& a/b_1, b_2, \ldots, b_{n-2}, a \\
= & \{x \in H | a \in f_\rho(x, b_1, \ldots, b_{n-2}, a)\} \\
= & \{x \in H | a \in L_1(x) \cup L_2(b_1) \cup \ldots \cup L_m(b_{m-1}) \cup R_1(a) \cup \cdots \cup R_m(b_{n-m})\} \\
= & H.
\end{aligned}$$

If $a \neq b_1, \ldots, b_{n-1}$, then $a/b_1, \ldots, b_{n-1} = \{x \in H \mid a \in f_\rho(x, b_1, \ldots, b_{n-1})\}$ $= \{x \in H \mid a \in \{x, b_1, \ldots, b_{n-1}\}\} = \{a\}$. Let $a, c, b_1, b_2, \ldots, b_{n-1}, d_1, d_2, \ldots,$

$d_{n-1} \in H$, $a/b_1^{n-1} \cap c/d_1^{n-1} \neq \emptyset$. If there exist $i, j \in \{1, 2, \ldots, n-1\}$ such that $a = b_i$ or $c = d_j$, then $a \in f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c)$ or $a \in f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c)$. If $a \neq b_1, \ldots, b_{n-1}$ and $c \neq d_1, \ldots, d_{n-1}$, then $f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c) \neq \emptyset$ if and only if $a = c$ and thus $a \in f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c)$. In both cases $f_\rho(a, d_1, \ldots, d_{n-1}) \cap f_\rho(b_1, \ldots, b_{n-1}, c) \neq \emptyset$. Therefore, n-ary hypergroup (H, f_ρ) is a join n-ary space.

Theorem 2.9. *If ρ is an n-ary preordering on a set H such that $L_i(x) = R_j(x)$ for any $x \in H$ and $1 \leq i, j \leq m$, then (H, f_ρ) is a join n-ary space.*

Proof. Set $L_i(x) = R_j(x) = L(x)$. Since ρ is reflexive, it follows by Theorem 2.4, that (H, f_ρ) is a quasi n-ary hypergroup. Moreover, since $L_i(x) = R_j(x)$ for any $x \in H$ and $1 \leq i, j \leq m$, it follows that

$$\begin{aligned} f_\rho(x_1, \ldots, x_n) &= L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \cdots \cup R_m(x_{n-m+1}) \\ &= L(x_1) \cup \cdots \cup L(x_m) \cup L(x_{m+1}) \cup \cdots \cup L(x_n) \\ &= \bigcup_{i=1}^{n} L(x_i) \end{aligned}$$

and this implies that $f_\rho(x_1, \ldots, x_n) = f_\rho(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any $x_1^n \in H$ and for any permutation $\sigma \in \{1, \ldots, n\}$. Therefore, (H, f_ρ) is commutative. Now, we prove that the n-ary hyperoperation f_ρ is associative, that means for any $i, j \in \{1, 2, \ldots, n\}$ and $a_1^{2n-1} \in H$, we have

$$f_\rho(a_1^{i-1}, f_\rho(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f_\rho(a_1^{j-1}, f_\rho(a_j^{n+j-1}), a_{n+j}^{2n-1}).$$

For any $a \in f_\rho(f_\rho(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1})$, there exists

$$\begin{aligned} u \in \quad & L_1(x_1) \cup \cdots \cup L_m(x_m) \cup R_1(x_n) \cup \cdots \cup R_m(x_{n-m+1}) \\ = \; & L(x_1) \cup \cdots \cup L(x_m) \cup L(x_n) \cup \cdots \cup L(x_{n-m+1}), \end{aligned}$$

such that $a \in L(u) \cup L(x_{n+1}) \cup \cdots \cup L(x_{n+m-1}) \cup L(x_{2n-1}) \cup \cdots \cup L(x_{2n-m})$. Moreover,

(∗)

$$\begin{aligned} & f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \\ = \; & \{L(x_1) \cup L(v) \cup L(x_{n+2}) \cup \ldots \cup L(x_{2n-1}) \mid v \in L(x_2) \cup \ldots \cup L(x_{n+1})\}. \end{aligned}$$

We distinguish the following cases:

(i_1) If $a \in L(u)$ and $u \in L(x_1)$, by Lemma 2.7, $a \in L(x_1)$. Therefore, we have $a \in f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1})$.

(i_2) If $a \in L(u)$ and $u \in L(x_2)$, then by (∗), $a \in f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1})$.

⋮

(i_n) If $a \in L(u)$ and $u \in L(x_n)$, then $a \in f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1})$.

(i_{n+1}) If $a \in L_2(x_{n+1}) = L(x_{n+1})$, then there exist $b_1, \dots, b_{n-2} \in H$ such that $(a, b_1, x_{n+1}, b_2, \dots, b_{n-2}) \in \rho$ or $(b_1, \dots, b_k, a, b_{k+1}, x_{n+1}, b_{k+2}, \dots, b_{n-2}) \in \rho$ for any $k \in \{1, 2, \dots, n-3\}$. For example, if $(a, b_1, x_{n+1}, b_2, \dots, b_{n-2}) \in \rho$, then $b_2 \in R_1(x_{n+1}) = L(x_{n+1})$, so $x_{n+1} \in R(b_2) = L(b_2)$. $a \in L(x_{n+1})$ and $x_{n+1} \in L(b_2)$, by Lemma 2.7, $a \in L(b_2)$. $a \in L(b_2)$ and $b_2 \in L(x_{n+1})$ from $(*)$ implies $a \in f_\rho(x_1, f_\rho(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1})$.

(i_{n+2}) If $a \in L(x_{n+2})$, then by $(*)$, we have $a \in f_\rho(x_1, f_\rho(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1})$.

$\vdots$

(i_{2n-1}) If $a \in L(x_{2n-1})$, then by $(*)$ we have $a \in f_\rho(x_1, f_\rho(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1})$.

The proofs of the other inclusions are similar and with long computations. It remains to check the condition of the join n-ary space. Set $a, b_1, \dots, b_{n-1}, c, d_1, \dots, d_{n-1} \in H$ such that $a/b_1^{n-1} \cap c/d_1^{n-1} \neq \emptyset$. Then, there exists $x \in a/b_1^{n-1} \cap c/d_1^{n-1}$. Hence,

$$x \in a/b_1^{n-1} \Rightarrow a \in f_\rho(x, b_1, \dots, b_{n-1}) = L(x) \cup L(b_1) \cup \dots \cup L(b_{n-1}),$$
$$x \in c/d_1^{n-1} \Rightarrow c \in f_\rho(x, d_1, \dots, d_{n-1}) = L(x) \cup L(d_1) \cup \dots \cup L(d_{n-1}).$$

Now, we consider the following situations:

1 $a \in L(x), c \in L(x) \Rightarrow x \in L(a), x \in L(c) \Rightarrow x \in [L(a) \cup L(d_1) \cup \dots \cup L(d_{n-1})] \cap [L(b_1) \cup \dots \cup L(b_{n-1}) \cup L(c)] = f_\rho(a, d_1, \dots, d_{n-1}) \cap f_\rho(b_1, \dots, b_{n-1}, c)$.

2 If $a \in L(x)$ and $c \in L(d_i)$, $i \in \{1, \dots, n-1\}$. Since $c \in L(c)$ (by reflexivity), it follows that

$$c \in [L(a) \cup L(d_1) \cup \ldots \cup L(d_{n-1})] \cap [L(b_1) \cup \ldots \cup L(b_{n-1}) \cup L(c)].$$

3 If $a \in L(b_i)$ and $c \in L(x)$, $i \in \{1, \dots, n-1\}$, then $b_i \in R(a) = L(a)$. Since $b_i \in L(b_i)$ (by reflexivity), it follows that

$$b_i \in [L(a) \cup L(d_1) \cup \ldots \cup L(d_{n-1})] \cap [L(b_1) \cup \ldots \cup L(b_{n-1}) \cup L(c)].$$

4 If $a \in L(b_i)$ and $c \in L(d_j)$, $i, j \in \{1, \dots, n-1\}$, then $b_i \in R(a) = L(a)$, since $b_i \in L(b_i)$ (by reflexivity), it follows that

$$b_i \in [L(a) \cup L(d_1) \cup \ldots \cup L(d_{n-1})] \cap [L(b_1) \cup \ldots \cup L(b_{n-1}) \cup L(c)].$$

Since for $a, b_1, \dots, b_{n-1}, c, d_1, \dots, d_{n-1} \in H$ such that $a/b_1^{n-1} \cap c/d_1^{n-1} \neq \emptyset$, we have $f_\rho(a, d_1, \dots, d_{n-1}) \cap f_\rho(b_1, \dots, b_{n-1}, c) \neq \emptyset$, so (H, f_ρ) is a join n-ary space. □

Remark 2. Let ρ be an n-ary reflexive relation on a set H. By Lemma 2.7, if ρ is n-transitive, then ρ satisfies the following property:

(T)

$$\text{for any } a, x, u \in H \text{ and } i \in \{1, \dots, n-1\} \text{ such that } a \in L_i(u)\ [a \in R_i(u)]$$
$$\text{and } u \in L_i(x)\ [u \in R_i(x)], \text{ it follows that } a \in L_i(x)\ [a \in R_i(x)].$$

Theorem 2.10. *Let ρ be an n-ary relation on H such that $x \in L_i(x) = R_j(x)$ for any $x \in H$ and $1 \leq i, j \leq m$. If ρ satisfies the properties* (**T**)*, then f_ρ is associative and so (H, f_ρ) is a join n-ary spaces.*

Proof. The reproducibility follows from Theorem 2.4. Suppose that f_ρ is not associative. Then, there exists x_1^{2n-1} such that $f_\rho(f_\rho(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1})$ is not equal to

$$f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \text{ or}$$
$$f_\rho(x_1, x_2, f_\rho(x_3, \ldots, x_{n+2}), x_{n+3}, \ldots, x_{2n-1}) \ldots \text{or}$$
$$f_\rho(x_1, \ldots, x_{n-1}, f_\rho(x_n, \ldots, x_{2n-1})).$$

Suppose that there exists $u \in f_\rho(f_\rho(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1})$ such that

$$u \notin f_\rho(x_1, f_\rho(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1})$$

or vice versa. We consider the first situation: it follows that there exists $t \in f_\rho(x_1, \ldots, x_n)$ such that $u \in L(t) \cup L(x_{n+1}) \cup \cdots \cup L(x_{2n-1})$ and for any $s \in f_\rho(x_2, \ldots, x_{n+1})$, $u \notin L(x_1) \cup L(s) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$. Now, we distinguish the following situation:

(1) If $u \in L(t)$ and $t \in L(x_1)$, then $u \in L(x_1)$, so $u \in L(x_1) \cup \cdots \cup L(x_{2n-1})$.

(2) If $u \in L(t)$ and $t \in L(x_2)$, then $u \in L(x_2)$. Since $x_2 \in f_\rho(x_2, \ldots, x_{n+1})$. Thus, $u \in L(x_1) \cup L(x_2) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$.

$\vdots$

(n) If $u \in L(t)$ and $t \in L(x_n)$, then $u \in L(x_n)$. Since $x_n \in f_\rho(x_2, \ldots, x_{n+1})$, $u \in L(x_1) \cup L(x_n) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$.

($n+1$) If $u \in L(x_{n+1})$, then $u \in L(x_1) \cup L(x_{n+1}) \cup \cdots \cup L(x_{2n-1})$, since $x_{n+1} \in f_\rho(x_2, \ldots, x_{n+1})$,

($n+2$) If $u \in L(x_{n+2})$, then $u \in L(x_1) \cup L(s) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$.

$\vdots$

($2n-1$) If $u \in L(x_{2n-1})$, then $u \in L(x_1) \cup L(s) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$.

For the all cases, we obtain a contradiction with the fact

$$u \notin L(x_1) \cup L(s) \cup L(x_{n+2}) \cup \cdots \cup L(x_{2n-1})$$

for any $s \in f_\rho(x_2, \ldots, x_{n+1})$. The proofs of the other inclusions are similar. Therefore, f_ρ is associative. □

Example 8. Let $H = \{0, 1, 2\}$ and

$$\rho = \{(\underbrace{0, \ldots, 0}_{n}), (1, 2, \ldots, 1, 2), (2, 1, \ldots, 2, 1), (\underbrace{1, \ldots, 1}_{n-1}, 2), (\underbrace{2, \ldots, 2}_{n-1}, 1).$$

Then, we have

	L_1	L_2	L_3	$\ldots$	L_m	R_1	R_2	$\ldots$	R_m
0	$\{0\}$	$\{0\}$	$\{0\}$	$\ldots$	$\{0\}$	$\{0\}$	$\{0\}$	$\ldots$	$\{0\}$
1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$
2	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\ldots$	$\{1,2\}$

for every $x \in H$ and $1 \leq i, j \leq m$, we have $L_i(x) = R_j(x)$. So,

$$f_\rho(x_1, \dots, x_n) = \begin{cases} \{0\} & \text{if } \{x_1, \dots, x_n\} = \{0\}, \\ \{1, 2\} & \text{if } \{x_1, \dots, x_n\} \subseteq \{1, 2\}, \\ \{0, 1, 2\} & \text{otherwise.} \end{cases}$$

It is not difficult to see that (H, f_ρ) is a join n-ary space.

3. n-ary H_v-groups associated with n-ary relations

Given an n-ary hypergroupoid (H, f), we may consider the $(n+1)$-ary relation ρ_k on H associated with the n-ary hyperoperation f as follows

$$(x_1, \dots, x_{n+1}) \in \rho_k \Leftrightarrow x_k \in f(x_1^{k-1}, x_{k+1}^{n+1}).$$

This is the most natural way to define an $(n+1)$-ary relation associated with an n-ary hyperoperation. If (H, f) is an n-ary hypergroup, then ρ_k satisfies the following conditions:

(1) For all $x_1, \dots, x_n \in H$, there exists at least one element $x \in H$ such that $(x_1^{k-1}, x, x_k^n) \in \rho_k$.

(2) If, for $x_1, \dots, x_{2n+1}, z \in H$, there exists $x \in H$ such that for any $k \leq i$ and $k \leq j$, we have $(x_1^{k-1}, z, x_k^{i-1}, x, x_{n+i}^{2n-1}) \in \rho_k$ and $(x_i^{i+k-2}, x, x_{i+k-1}^{i+n-1}) \in \rho_k$, then there exists $y \in H$ such that $(x_1^{k-1}, z, x_k^{j-1}, y, x_{n+j}^{2n-1}) \in \rho_k$ and $(x_j^{j+k-2}, x, x_{j+k-1}^{j+n-1}) \in \rho_k$, and conversely.

(3) If, for $x_1, \dots, x_{2n+1}, z \in H$, there exists $x \in H$ such that for any $k \leq i$ and $k > j$, we have $(x_1^{k-1}, z, x_k^{i-1}, x, x_{n+i}^{2n-1}) \in \rho_k$ and $(x_i^{i+k-2}, x, x_{i+k-1}^{i+n-1}) \in \rho_k$, then there exists $y \in H$ such that $(x_1^{j-1}, y, x_{n+j}^{n+k-2}, z, x_{n+k-1}^{2n-1}) \in \rho_k$ and $(x_j^{j+k-2}, x, x_{j+k-1}^{j+n-1}) \in \rho_k$, and conversely.

(4) If, for $x_1, \dots, x_{2n+1}, z \in H$, there exists $x \in H$ such that for any $k > i$ and $k > j$, we have $(x_1^{i-1}, y, x_{n+i}^{n+k-2}, z, x_{n+k-1}^{2n-1}) \in \rho_k$ and $(x_i^{i+k-2}, x, x_{i+k-1}^{i+n-1}) \in \rho_k$, then there exists $y \in H$ such that $(x_1^{j-1}, y, x_{n+j}^{n+k-2}, z, x_{n+k-1}^{2n-1}) \in \rho_k$ and $(x_j^{j+k-2}, x, x_{j+k-1}^{j+n-1}) \in \rho_k$, and conversely.

(5) For all $x_1^n \in H$ and $1 \leq i \leq n$, there exists $x \in H$ such that $(x_1^{i-1}, x, x_i^n) \in \rho_k$.

Conversely, if ρ is an $(n+1)$-ary relation on a set H such that the conditions (1)-(5) are satisfied, then we take the n-ary hyperoperation

$$f_k(x_1, \dots, x_n) = \{z \in H \mid (x_1^{k-1}, z, x_k^n) \in \rho\}.$$

Hence, (H, f_k) is an n-ary hypergroup. Let σ_n be an n-ary relation on a set H. We associate an $(n+1)$-ary relation denoted by $\sigma_{n+1} \subseteq H^{n+1}$ as follows:

$$(1) \qquad (x_1, \dots, x_{n+1}) \in \sigma_{n+1} \iff \forall 1 \leq i \leq n+1,\ (x_1^{i-1}, x_{i+1}^{n+1}) \in \sigma_n.$$

Proposition 3.1. *The unique $(n+1)$-ary relation σ_{n+1} obtained from an n-ary relation σ_n using the method* (1) *and such that*

$$(2) \qquad (x_1^{i-1}, x_{i+1}^{n+1}) \in H^n, \quad \exists x_i \in H : (x_1, \dots, x_{n+1}) \in \sigma_{n+1}$$

is the total relation $\sigma_{n+1} = \underbrace{H \times \cdots \times H}_{n+1}$.

Proof. The condition (2) is equivalent to the following one: for any $(x_1, \ldots, x_n) \in H^n$, $(x_1, \ldots, x_n) \in \sigma_n$, so the n-ary relation σ_n is the total relation $\underbrace{H \times \cdots \times H}_{n}$. Thus, for any $(x_1, \ldots, x_{n+1}) \in H^{n+1}$ and for any $1 \leq i \leq n+1$, we have $(x_1^{i-1}, x_{i+1}^{n+1}) \in H^n = \sigma_n$. Therefore, by using the method (1), $(x_1, \ldots, x_{n+1}) \in \sigma_{n+1}$. So, $H^{n+1} \subseteq \sigma_{n+1}$. Therefore, $\sigma_{n+1} = H^{n+1}$. □

Moreover, the n-ary hypergroupoid obtained from σ_{n+1} taking

$$f(x_1^{i-1}, x_{i+1}^{n+1}) = \{x_i \in H | (x_1, \ldots, x_{n+1}) \in \sigma_{n+1}\}$$

is the total n-ary hypergroup on H. Conversely, with any $(n+1)$-ary relation ρ_{n+1} on H, we associate an n-ary relation $\rho_n \subseteq H^n$ as follows:

$$(3) \qquad (x_1^{i-1}, x_{i+1}^{n+1}) \in \rho_n \Longleftrightarrow \exists x_i \in H : (x_1, \ldots, x_{n+1}) \in \rho_{n+1}.$$

Let (H, f) be an arbitrary n-ary hypergroupoid which determines the $(n+1)$-ary relation ρ_{n+1} defined by

$$(x_1, \ldots, x_{n+1}) \in \rho_{n+1} \Leftrightarrow x_i \in f(x_1^{i-1}, x_{i+1}^{n+1}) \text{ for some } 1 \leq i \leq n.$$

Proposition 3.2. *The unique n-ary relation ρ_n obtained from an $(n+1)$-ary relation ρ_{n+1} using the method* (3) *and such that*

$$(x_1, \ldots, x_{n+1}) \in \rho_{n+1} \quad \Leftrightarrow \quad x_i \in f(x_1^{i-1}, x_{i+1}^{n+1}),$$

is the total relation $\rho_n = \underbrace{H \times \cdots \times H}_{n}$.

Proof. Since (H, f) is an n-ary hypergroupoid, it follows that, for any $(x_1^{i-1}, x_{i+1}^{n+1}) \in H^n$, there exists $x_i \in H$ such that $x_i \in f(x_1^{i-1}, x_{i+1}^{n+1})$, that is $(x_1, \ldots, x_{n+1}) \in \rho_{n+1}$. Therefore, for any $(x_1^{i-1}, x_{i+1}^{n+1}) \in H^n$, we obtain $(x_1^{i-1}, x_{i+1}^{n+1}) \in \rho_n$, that is $\rho_n = \underbrace{H \times \cdots \times H}_{n}$. □

Definition 3.3. Let (H, f) be an n-ary hypergroup, such that the n-ary hyperoperation f is constructed by the $(n+1)$-ary relation ρ, which satisfy the conditions (1)-(3). We define an $(n+1)$-ary hyperoperation:

$$h(x_1, \ldots, x_{n+1}) = \bigcup_{i=1}^{n+1} f(x_1^{i-1}, x_{i+1}^{n+1}) \text{ for all } x_1, \ldots, x_{n+1} \in H.$$

Theorem 3.4. *Let h be the $(n+1)$-ary hyperoperation in Definition* 3.3. *Then, (H, h) is an $(n+1)$-ary H_v-group.*

Proof. Since (H, f) is an n-ary hypergroup, (H, h) is an $(n+1)$-ary hypergroupoid. Let $x_1, \ldots, x_{n+1} \in H$. Then, producibility of (H, f) implies that

(i_1)

$h(H, x_2, \ldots, x_{n+1})$

$$= f(x_2, \ldots, x_{n+1}) \cup f(H, x_3, \ldots, x_{n+1}) \cup \ldots \cup f(H, x_2, \ldots, x_n) = H,$$

(i_2)

$$h(x_1, H, x_3, \ldots, x_{n+1})$$
$$= f(H, x_3, \ldots, x_{n+1}) \cup f(x_1, x_3, \ldots, x_{n+1}) \cup \ldots \cup f(x_1, H, \ldots, x_n) = H,$$

$$\vdots$$

(i_{n+1})

$$h(x_1, x_2, \ldots, x_n, H)$$
$$= f(x_2, x_3, \ldots, x_n, H) \cup f(x_1, x_3, \ldots, x_n, H) \cup \ldots \cup f(x_1, \ldots, x_n) = H.$$

Thus, (H, h) is productive. Now, we prove that h is weakly associative. Suppose that $x_1^{2n+1} \in H$. Then,

(i_1)

$$\begin{aligned} h(h(x_1^{n+1}), x_{n+2}, \ldots, x_{2n+1}) &= h(\bigcup_{i=1}^{n+1} f(x_1^{i-1}, x_{i+1}^{n+1}), x_{n+2}, \ldots, x_{2n+1}) \\ &= h(g, x_{n+2}, \ldots, x_{2n+1}) \\ &\supseteq f(g, x_{n+3}, \ldots, x_{2n+1}) \\ &\supseteq f(f(x_2, \ldots, x_{n+1}), x_{n+3}, \ldots, x_{2n+1}), \end{aligned}$$

(i_2)

$$\begin{aligned} h(x_1, h(x_2^{n+2}), x_{n+3}, \ldots, x_{2n+1}) &= h(x_1, \bigcup_{i=1}^{n+1} f(x_2^{i}, x_{i+2}^{n+2}), x_{n+3}, \ldots, x_{2n+1}) \\ &= h(x_1, g, x_{n+3}, \ldots, x_{2n+1}) \\ &\supseteq f(g, x_{n+3}, \ldots, x_{2n+1}) \\ &\supseteq f(f(x_2, \ldots, x_{n+1}), x_{n+3}, \ldots, x_{2n+1}), \\ &\vdots \end{aligned}$$

(i_{n+1})

$$\begin{aligned} h(x_1, \ldots x_n, h(x_{n+1}, \ldots, x_{2n+1})) &= h(x_1, \ldots x_n, \bigcup_{i=1}^{n+1} f(x_{n+1}^{n+i-1}, x_{n+i+1}^{2n+1})) \\ &= h(x_1, \ldots x_n, g) \\ &\supseteq f(x_2, \ldots, x_n, g) \\ &\supseteq f(x_2, \ldots, x_n, f(x_{n+1}, x_{n+3}, \ldots, x_{2n+1})). \end{aligned}$$

Therefore,

$$\bigcap_{i=1}^{2n+1} h(x_1^{i-1}, h(x_i^{n+i-1}), x_{n+i}^{2n+1}) \neq \emptyset.$$

This implies that $(n+1)$-ary hypergroupoid (H,h) is weakly associative. $\square$

Example 9. Let $H=\{a_1,\dots,a_n\}$ and f be an n-ary relation on H such that

$$f(\underbrace{a_1,\dots,a_1}_{n})=\{a_n\}, f(\underbrace{a_2,\dots,a_2}_{n})=\{a_{n-1}\}$$
$$\vdots$$
$$f(\underbrace{a_{n-1},\dots,a_{n-1}}_{n})=\{a_2\}, f(\underbrace{a_n,\dots,a_n}_{n})=\{a_1\}$$
$$f(a_1,\dots,a_n)=f(\underbrace{a_1,\dots,a_1}_{n})\cup\dots\cup f(\underbrace{a_n,\dots,a_n}_{n}).$$

Then, (H,f) is an n-ary hypergroupoid. Now, if

$$h(a_1,\dots,a_{n+1})=\bigcup_{i=1}^{n+1} f(a_1^{i-1},a_{i+1}^{n+1}),$$

then the $(n+1)$-ary hypergroupoid (H,h) is an $(n+1)$-ary H_v-group, but it is not an $(n+1)$-ary hypergroup. For instance,

$$\begin{aligned}
&h(\underbrace{a_1,\dots,a_1}_{n},h(\underbrace{a_1,\dots,a_1}_{n},a_2))\\
&=h(\underbrace{a_1,\dots,a_1}_{n},\{a_n,a_{n-1}\})\\
&=h(\underbrace{a_1,\dots,a_1}_{n+1})\cup h(\underbrace{a_{n-1},\dots,a_{n-1}}_{n+1})\cup h(\underbrace{a_n,\dots,a_n}_{n+1})=\{a_1,a_2,a_n\},\\
&h(\underbrace{a_1,\dots,a_1}_{n-1},h(\underbrace{a_1,\dots,a_1}_{n+1}),a_2)\\
&=h(\underbrace{a_1,\dots,a_1}_{n-1},\{a_n\},a_2)\\
&=f(\underbrace{a_1,\dots,a_1}_{n+1})\cup f(\underbrace{a_n,\dots,a_n}_{n+1})\cup f(\underbrace{a_2,\dots,a_2}_{n+1})=\{a_1,a_{n-1},a_n\}.
\end{aligned}$$

Let ρ be a binary relation on a non-empty set H. For any $a\in H$, we denote $f_\rho(\underbrace{a,\dots,a}_{n})=\{y\mid (a,y)\in\rho\}$, and for any $a_1,\dots,a_n\in H$,

$$f_\rho(a_1,a_2,\dots,a_n)=f_\rho(\underbrace{a_1,\dots,a_1}_{n})\cup f_\rho(\underbrace{a_2,\dots,a_2}_{n})\cup\dots\cup f_\rho(\underbrace{a_n,\dots,a_n}_{n}).$$

Definition 3.5. Let ρ be a binary relation on a set H. We define the $(n+1)$-ary hyperoperation $\mathbb{F}_\rho$ as follows:

$$\mathbb{F}_\rho(x_1,\dots,x_{n+1})=\bigcup_{i=1}^{n+1} f_\rho(\underbrace{x_i,\dots,x_i}_{n})=\bigcup_{i=1}^{n+1} U_{x_i},$$

when $U_{x_i} = f_\rho(\underbrace{x_i, \ldots, x_i}_{n})$.

Theorem 3.6. *Let ρ be a binary relation on a set H, with full domain and full range. Let $\mathbb{F}_\rho$ be the $(n+1)$-ary hyperoperation in Definition 3.5. Then, $(H, \mathbb{F}_\rho)$ is an $(n+1)$-ary H_v-group.*

Proof. Since $D(\rho) = H$ and (H, f_ρ) is an n-ary hypergroupoid, $(H, \mathbb{F}_\rho)$ is an $(n+1)$-ary hypergroupoid. Let $x_1, \ldots, x_n \in H$. Then,

$$U_H = f_\rho(\underbrace{H, \ldots, H}_{n}) = \{y \in H | (H, y) \in \rho\} = \{y \in H | \exists x \in H, (x, y) \in \rho\}$$
$$= D(\rho) = H.$$

So, for $x_1, \ldots, x_n \in H$ we have

$$\mathbb{F}_\rho(H, x_2, \ldots, x_{n+1}) = U_H \cup \bigcup_{i=2}^{n+1} U_{x_i} = H.$$

By the similar way, we have $\mathbb{F}_\rho(x_1, H, x_3, \ldots, x_{n+1}) = \cdots = \mathbb{F}_\rho(x_1, \ldots, x_n, H)$ $= H$. Now, we prove that $\mathbb{F}_\rho$ is weakly associative. If $x_1^{2n+1} \in H$, then,

(i_1)

$$\begin{aligned}\mathbb{F}_\rho(\mathbb{F}_\rho(x_1^{n+1}), x_{n+2}, \ldots, x_{2n+1}) &= \mathbb{F}_\rho(U_{x_1} \cup \cdots \cup U_{x_{n+1}}, x_{n+2}, \ldots, x_{2n+1}) \\ &= \bigcup_{g \in U_{x_1} \cup \cdots \cup U_{x_{n+1}}} \mathbb{F}_\rho(g, x_{n+2}, \ldots, x_{2n+1}) \\ &\supseteq \bigcup_{g \in U_{x_{n+1}}} \mathbb{F}_\rho(g, x_{n+2}, \ldots, x_{2n+1}) \\ &= \mathbb{F}_\rho(U_{x_{n+1}}, x_{n+2}, \ldots, x_{2n+1}) \\ &\supseteq \bigcup_{g \in U_{x_{n+1}}} U_g,\end{aligned}$$

(i_2)

$$\begin{aligned}\mathbb{F}_\rho(x_1, \mathbb{F}_\rho(x_2^{n+2}), x_{n+3}, \ldots, x_{2n+1}) &= \mathbb{F}_\rho(x_1, U_{x_2} \cup \cdots \cup U_{x_{n+2}}, x_{n+3}, \ldots, x_{2n+1}) \\ &= \bigcup_{g \in U_{x_2} \cup \cdots \cup U_{x_{n+2}}} \mathbb{F}_\rho(x_1, g, x_{n+3}, \ldots, x_{2n+1}) \\ &\supseteq \bigcup_{g \in U_{x_{n+1}}} \mathbb{F}_\rho(x_1, g, x_{n+3}, \ldots, x_{2n+1}) \\ &= \mathbb{F}_\rho(x_1, U_{x_{n+1}}, x_{n+3}, \ldots, x_{2n+1}) \\ &\supseteq \bigcup_{g \in U_{x_{n+1}}} U_g,\end{aligned}$$

$$\vdots$$

(i_{n+1})

$$\begin{aligned}\mathbb{F}_\rho(x_1,\dots,x_n,\mathbb{F}_\rho(x_{n+1}^{2n+1})) &= \mathbb{F}_\rho(x_1,\dots,x_n,U_{x_{n+1}}\cup\dots\cup U_{x_{2n+1}})\\ &= \bigcup_{g\in U_{x_{n+1}}\cup\dots\cup U_{x_{2n+1}}} \mathbb{F}_\rho(x_1,\dots,x_n,g)\\ &\supseteq \bigcup_{g\in U_{x_{n+1}}} \mathbb{F}_\rho(x_1,\dots,x_n,g)\\ &= \mathbb{F}_\rho(x_1,\dots,x_n,U_{x_{n+1}})\\ &\supseteq \bigcup_{g\in U_{x_{n+1}}} U_g.\end{aligned}$$

It follows that $(H,\mathbb{F}_\rho)$ is an $(n+1)$-ary H_v-group. □

Example 10. Let $H=\{1,\dots,n\}$, $n\geq 4$ and $\rho=\{(1,n),\dots,(i,n-i+1),\dots,(n,1)\}$ be a binary relation on a set H, with full domain and full range. Then,

$$\begin{aligned}&U_1=f_\rho(\underbrace{1,\dots,1}_{n})=\{n\},\quad U_2=f_\rho(\underbrace{2,\dots,2}_{n})=\{n-1\},\dots,\\ &U_{n-1}=f_\rho(\underbrace{n-1,\dots,n-1}_{n})=\{2\},\quad U_n=f_\rho(\underbrace{n,\dots,n}_{n})=\{1\}.\end{aligned}$$

The properties of (H,f_ρ), where $f_\rho(x_1,\dots,x_n)=\bigcup_{i=1}^n U_{x_i}$, as n-ary H_v-group, whit rarely computations guarantee that the $(H,\mathbb{F}_\rho)$ is an $(n+1)$-ary H_v-group properties. But $(H,\mathbb{F}_\rho)$ is not an $(n+1)$-ary hypergroup.

For instance

$$\begin{aligned}\mathbb{F}_\rho(\underbrace{1,\dots,1}_{n},\mathbb{F}_\rho(\underbrace{1,\dots,1}_{n},2)) &= \mathbb{F}_\rho(\underbrace{1,\dots,1}_{n},U_1\cup U_2)\\ &= \mathbb{F}_\rho(\underbrace{1,\dots,1}_{n},\{n,n-1\})\\ &= U_1\cup U_{n-1}\cup U_n=\{1,2,n\},\\ \mathbb{F}_\rho(\underbrace{1,\dots,1}_{n-1},\mathbb{F}_\rho(\underbrace{1,\dots,1}_{n+1}),2) &= \mathbb{F}_\rho(\underbrace{1,\dots,1}_{n-1},U_1,2)\\ &= \mathbb{F}_\rho(\underbrace{1,\dots,1}_{n-1},\{n\},2)\\ &= U_1\cup U_2\cup U_n=\{1,n-1,n\}.\end{aligned}$$

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