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SCHUR POWER CONVEXITY OF GINI MEANS

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Abstract. In this paper, the Schur convexity is generalized to Schur f-convexity, which contains the Schur geometrical convexity, harmonic convexity and so on. When $f : \mathbb{R}_+ \to \mathbb{R}$ is defined as $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$, the necessary and sufficient conditions for f-convexity (is called Schur m-power convexity) of Gini means are given, which generalize and unify certain known results.

1. Introduction

Let $p, q \in \mathbb{R}$ and $a, b \in \mathbb{R}_+ := (0, \infty)$. The Gini means [13] are defined as

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{1/(p-q)}, & p \neq q, \\ \exp\left(\dfrac{a^p \ln a + b^p \ln b}{a^p + b^p}\right), & p = q. \end{cases} \tag{1.1}$$

It is easy to see that the Gini means $G_{p,q}(a,b)$ are continuous on the domain $\{(a,b;p,q) : a,b \in \mathbb{R}_+; p,q \in \mathbb{R}\}$ and differentiable with respect to $(a,b) \in \mathbb{R}_+^2$ for fixed $p, q \in \mathbb{R}$. Also, Gini means are symmetric with respect to a, b and p, q.

Gini means $G_{p,q}(a,b)$ contain many classical two variable means, for example, $G_{1,0} = A$ is the arithmetic mean, $G_{0,0} = G$ is the geometric mean, $G_{-1,0} = H$ is the harmonic mean, and more generally, the p-th power mean is equal to $G_{p,0}$, $G_{p,p-1}$ is the Lehmer mean. The basic properties of Gini means, as well as their comparison theorems, log-convexities, and inequalities are studied in papers [8, 9, 10, 11, 20, 21, 25, 26, 27, 30, 36, 43, 44, 45, 48].

Schur convexity was introduced by Schur in 1923 [22], and it has many important applications in analytic inequalities [2, 15, 49], linear regression [35], graphs and matrices [7], combinatorial optimization [16], information-theoretic topics [12], Gamma functions [23], stochastic orderings [32], reliability [17], and other related fields.

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In recent years, the Schur convexity and Schur geometrical convexity of $G_{p,q}(a,b)$ have attracted the attention of a considerable number of mathematicians [4, 5, 19, 29, 28, 31, 33]. Sándor [31] proved that the Gini means $G_{p,q}(a,b)$ are Schur convex on $(-\infty,0]\times(-\infty,0]$ and Schur concave on $[0,\infty)\times[0,\infty)$ with respect to (p,q) for fixed $a,b>0$ with $a\neq b$. Yang [47] improved Sándor's result and proved that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a,b>0$ with $a\neq b$ if and only if $p+q<0$ and Schur concave if and only if $p+q>0$. Wang and Zhang [38, 39] showed that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\geq 1$, $p,q\geq 0$ and Schur concave if and only if $p+q\leq 1$, $p\leq 0$ or $p+q\leq 1$, $q\leq 0$. Gu and Shi [14, 34] also discussed the Schur convexity. Recently, Chu and Xia [6] also proved the same result as Wang and Zhang's.

The Schur geometrical convexity was introduced by Zhang [50]. Wang and Zhang [39] proved Gini means $G_{p,q}(a,b)$ are Schur geometrically convex with respect to $(a,b)\in\mathbb{R}_+^2$ if $p+q\geq 0$ and Schur geometrically concave if $p+q\leq 0$. Gu and Shi [14, 34] also investigated the Schur geometrical convexities of Lehmer mean $G_{p,1-p}(a,b)$ and Gini means $G_{p,q}(a,b)$, respectively.

Recently, Anderson et al. [1] discussed an attractive class of inequalities, which arise from the notion of harmonic convexity. And then it was started to research for *Schur harmonic convexity.* Chu et al. [3] showed that the Hamy symmetric function is Schur harmonic convex and obtained some analytic inequalities including the well-known Weierstrass inequalities. Xia [40] proved that the Lehmer mean $G_{p,p-1}(a,b)$ is Schur harmonic convex (Schur harmonic concave) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p\geq(\leq)0$.

The purpose of this paper is to generalize the notion of Schur convexity and to investigate the so-called *Schur power convexity* of Gini means $G_{p,q}(a,b)$.

Our main results are as follows.

Theorem 1.1. *For $m>0$ and fixed $(p,q)\in\mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\geq m$ and $\min(p,q)\geq 0$.*

Theorem 1.2. *For $m>0$ and fixed $(p,q)\in\mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\leq m$ and $\min(p,q)\leq 0$.*

Theorem 1.3. *For $m<0$ and fixed $(p,q)\in\mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\geq m$ and $\max(p,q)\geq 0$.*

Theorem 1.4. *For $m<0$ and fixed $(p,q)\in\mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power concave with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\leq m$ and $\max(p,q)\leq 0$.*

Theorem 1.5. *For $m=0$ and fixed $(p,q)\in\mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p+q\geq(\leq)0$.*

The organization of the paper is as follows. In Section 2, based on the notions and lemmas of Schur convexity, we introduce the definition of Schur f-convex and Schur f-concave function, and prove the decision theorem for Schur f-convexity. As special case, the definition and decision theorem of Schur power convexity are deduced. In Section 3, some lemmas are given. In Section 4, our main results are proved.

2. Schur f-convexity and Schur power convexity

For convenience of readers, we recall some definitions as follows.

Definition 2.1 ([22, 37])**.** Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n (n \geq 2)$.

(i) $\mathbf{x}$ is said to by majorized by $\mathbf{y}$ (in symbol $\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \text{ for } 1 \leq k \leq n-1, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{2.1}$$

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a decreasing order.

(ii) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$. Let $\Omega \subseteq \mathbb{R}^n (n \geq 2)$. The function $\phi : \Omega \to \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$. ϕ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for all $\mathbf{x}$, $\mathbf{y}$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iv) Let $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ be a set with nonempty interior. Then $\phi : \Omega \to \mathbb{R}$ is said to be Schur convex if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. ϕ is said to be Schur concave if $-\phi$ is Schur convex.

Definition 2.2 ([22])**.** (i) $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ is called a symmetric set, if $\mathbf{x} \in \Omega$ implies $\mathbf{xP} \in \Omega$ for every $n \times n$ permutation matrix $\mathbf{P}$.

(ii) The function $\phi : \Omega \to \mathbb{R}^n$ is called symmetric if for every permutation matrix $\mathbf{P}$, $\phi(\mathbf{xP}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

For the Schur convexity, there is the following well-known result.

Lemma 2.1 ([22, 37])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric set with nonempty interior Ω^0 and $\phi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur convex* (*Schur concave*) *on Ω if and only if ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0. \tag{2.2}$$

Next, let us define the Schur f-convexity as follows.

Definition 2.3. Let $\Omega = \mathbb{U}^n (\mathbb{U} \subseteq \mathbb{R})$ and f be a strictly monotone function defined on $\mathbb{U}$. Assume that

$$f(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_n)) \text{ and } f(\mathbf{y}) = (f(y_1), f(y_2), \dots, f(y_n)).$$

(i) Ω is called a f-convex set if $(f^{-1}(\alpha f(x_1)+\beta f(y_1)),\ldots,f^{-1}(\alpha f(x_n)+\beta f(y_n)))\in\Omega$ for all $\mathbf{x},\mathbf{y}\in\Omega$ and all $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

(ii) Let Ω be a set with nonempty interior. Then function $\phi:\Omega\to\mathbb{R}$ is said to be Schur f-convex on Ω if $f(\mathbf{x})\prec f(\mathbf{y})$ on Ω implies $\phi(\mathbf{x})\le\phi(\mathbf{y})$.

ϕ is said to be Schur f-concave if $-\phi$ is Schur f-convex.

Remark 2.1. Let $\Omega=\mathbb{U}^n(\mathbb{U}\subseteq\mathbb{R})$ and f be a strictly monotone function defined on $\mathbb{U}$ and $f(\Omega)=\{f(\mathbf{x}):\mathbf{x}\in\Omega\}$. Then function $\phi:\Omega\to\mathbb{R}$ is Schur f-convex (Schur f-concave) if and only if $\phi\circ f^{-1}$ is Schur convex (Schur concave) on $f(\Omega)$.

Indeed, if function $\phi:\Omega\to\mathbb{R}$ is Schur f-convex, then $\forall\mathbf{x}',\mathbf{y}'\in f(\Omega)$, there are $\mathbf{x},\mathbf{y}\in\mathbf{\Omega}$ such that $\mathbf{x}'=f(\mathbf{x}),\mathbf{y}'=f(\mathbf{y})$. If $f(\mathbf{x})\prec f(\mathbf{y})$, that is, $\mathbf{x}'\prec\mathbf{y}'$, then $\phi(\mathbf{x})\le\phi(\mathbf{y})$, that is, $\phi((f^{-1}(\mathbf{x}'))\le\phi((f^{-1}(\mathbf{y}'))$. This shows that $\phi\circ f^{-1}$ is Schur convex on $f(\Omega)$. Conversely, if $\phi\circ f^{-1}$ is Schur convex on $f(\Omega)$, then $\forall\mathbf{x},\mathbf{y}\in\Omega$ such that $f(\mathbf{x})\prec f(\mathbf{y})$, we have $\phi((f^{-1}(f(\mathbf{x})))\le\phi((f^{-1}(f(\mathbf{y})))$, that is, $\phi(\mathbf{x})\le\phi(\mathbf{y})$. This indicates ϕ is Schur f-convex on Ω.

In the same way, we can show that ϕ is Schur f-concave on Ω if and only if $\phi\circ f^{-1}$ is Schur concave on $f(\Omega)$.

Remark 2.2. Let $\Omega\subseteq\mathbb{R}^n(n\ge2)$ be a symmetric set and the function $\phi:\Omega\to\mathbb{R}$ be Schur f-convex (Schur f-concave). Then ϕ is symmetric on Ω.

In fact, for any $\mathbf{x}\in\Omega$ and every permutation matrix P, we have $\mathbf{xP}\in\Omega$. Note $\mathbf{xP}$ is another permutation of $\mathbf{x}$, hence $f(\mathbf{x})\prec f(\mathbf{xP})\prec f(\mathbf{x})$. Since ϕ is Schur f-convex (Schur f-concave), we have $\phi(\mathbf{x})\le(\ge)\phi(\mathbf{xP})\le(\ge)\phi(\mathbf{x})$, that is, $\phi(\mathbf{xP})=\phi(\mathbf{x})$ for all $\mathbf{x}\in\Omega$. This shows that ϕ is symmetric on Ω.

By Lemma 2.1 and Remarks 2.1, 2.2, we have the following:

Theorem 2.1. *Assume that* $\Omega=\mathbb{U}^n(\mathbb{U}\subseteq\mathbb{R})$ *is a symmetric set with nonempty interior* Ω^0, f *is a strictly monotone and derivable function defined on* $\mathbb{U}$, *and* $\phi:\Omega\to\mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω^0. *Then* ϕ *is Schur* f*-convex* (*Schur* f*-concave*) *on* Ω *if and only if* ϕ *is symmetric on* Ω *and*

$$(f(x_1)-f(x_2))\left(\frac{1}{f'(x_1)}\frac{\partial\phi}{\partial x_1}-\frac{1}{f'(x_2)}\frac{\partial\phi}{\partial x_2}\right)\ge(\le)0 \tag{2.3}$$

holds for any $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\Omega^0$ *with* $x_1\ne x_2$.

Proof. We easily check that $\phi\circ f^{-1}$ is symmetric on $f(\Omega)$ if and only if ϕ is symmetric on Ω.

By Remark 2.1 and Lemma 2.1, $\phi\circ f^{-1}$ is Schur convex (Schur concave) if and only if $\phi\circ f^{-1}$ is symmetric on $f(\Omega)$ and

$$(y_1-y_2)\left(\frac{\partial(\phi\circ f^{-1})}{\partial y_1}-\frac{\partial(\phi\circ f^{-1})}{\partial y_2}\right)\ge(\le)0$$

holds for any $\mathbf{y}\in f(\Omega)^0$ with $y_1\ne y_2$. Substituting $f^{-1}(\mathbf{y})=\mathbf{x}$ yields (2.3), where $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\Omega^0$ with $x_1\ne x_2$.

This proof is finished. □

Putting $f(x) = 1, \ln x, x^{-1}$ in Definition 2.3 yield the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have:

Definition 2.4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$. Then function $\phi : \Omega(\subseteq \mathbb{R}^n_+) \to \mathbb{R}$ is said to be Schur m-power convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$.

ϕ is said to be Schur m-power concave if $-\phi$ is Schur m-power convex.

For the Schur power convexity, by Theorem 2.1 we have:

Corollary 2.1. *Let $\Omega \subseteq \mathbb{R}^n_+$ be a symmetric set with nonempty interior Ω^0 and $\phi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur m-power convex (Schur m-power concave) on Ω if and only if ϕ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial \phi}{\partial x_1} - x_2^{1-m}\frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \text{ if } m \neq 0, \tag{2.4}$$

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \phi}{\partial x_1} - x_2\frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \text{ if } m = 0 \tag{2.5}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$ with $x_1 \neq x_2$.

3. Lemmas

To prove the main results, we need the following useful lemmas.

Lemma 3.1. *For fixed $(p, q) \in \mathbb{R}^2$, Gini means $G_{p,q}(a, b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $g(t) \geq (\leq) 0$ for all $t > 0$, where*

$$g(t) := g_{p,q}(t) = \begin{cases} \frac{(p-q)\sinh At + p\sinh Bt + q\sinh Ct}{p-q} & \text{if } p \neq q, \\ \sinh(2p - m)t - \sinh mt + 2pt\cosh mt & \text{if } p = q, \end{cases} \tag{3.1}$$

and

$$A = p + q - m, \quad B = p - q - m, \quad C = p - q + m. \tag{3.2}$$

Proof. Let $m \neq 0$ and $G = G_{p,q} := G_{p,q}(a, b)$ defined by (1.1).

For $p \neq q$, some simple partial derivative calculations yield

$$\frac{\partial \ln G}{\partial a} = \frac{1}{G}\frac{\partial G}{\partial a} = \frac{1}{p-q}\left(\frac{pa^{p-1}}{a^p + b^p} - \frac{qa^{q-1}}{a^q + b^q}\right),$$
$$\frac{\partial \ln G}{\partial b} = \frac{1}{G}\frac{\partial G}{\partial b} = \frac{1}{p-q}\left(\frac{pb^{p-1}}{a^p + b^p} - \frac{qb^{q-1}}{a^q + b^q}\right).$$

Therefore, we have

$$a^{1-m}\frac{\partial \phi}{\partial a} - b^{1-m}\frac{\partial \phi}{\partial b} = \frac{G}{p-q}\left(p\frac{a^{p-m} - b^{p-m}}{a^p + b^p} - q\frac{a^{q-m} - b^{q-m}}{a^q + b^q}\right).$$

Substituting $\ln\sqrt{a/b}=t$ and using $\sinh x=\frac{1}{2}(e^{x}-e^{-x}), \cosh x=\frac{1}{2}(e^{x}+e^{-x})$, the right hand side above can be written as

$$\begin{aligned} &a^{1-m}\frac{\partial\phi}{\partial a}-b^{1-m}\frac{\partial\phi}{\partial b}\\ =&\ \frac{G\,(ab)^{-m/2}}{p-q}\left(p\frac{\sinh(p-m)t}{\cosh pt}-q\frac{\sinh(q-m)t}{\cosh qt}\right)\\ =&\ \frac{G\,(ab)^{-m/2}}{2\cosh pt\cosh qt}\frac{2p\sinh(p-m)t\cosh qt-2q\sinh(q-m)t\cosh pt}{p-q}. \end{aligned}$$

Using the "product into sum" formula for hyperbolic functions and (3.1), we have

$$\begin{aligned} \Delta :=&\ \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial G_{p,q}}{\partial a}-b^{1-m}\frac{\partial G_{p,q}}{\partial b}\right)\\ =&\ \frac{a^m-b^m}{m(a-b)}\frac{(a-b)G_{p,q}}{2\,(ab)^{m/2}\cosh pt\cosh qt}\frac{(p-q)\sinh At+p\sinh Bt+q\sinh Ct}{p-q}\\ =&\ d_{p,q}(t)\cdot g_{p,q}(t), \end{aligned}$$

where

$$d_{p,q}(t)=\frac{a^m-b^m}{m(a-b)}\frac{(a-b)G_{p,q}}{2\,(ab)^{m/2}\cosh pt\cosh qt}\quad (p\neq q)$$

and $g_{p,q}(t)$ is defined by (3.1).

In the case of $p=q$, since $G_{p,q}(a,b)\in C^1$ we have

$$\frac{\partial G_{p,p}}{\partial a}=\lim_{q\to p}\frac{\partial G_{p,q}}{\partial a},\qquad \frac{\partial G_{p,p}}{\partial b}=\lim_{q\to p}\frac{\partial G_{p,q}}{\partial b}.$$

It follows that

$$\begin{aligned} \Delta =&\ \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial G_{p,p}}{\partial a}-b^{1-m}\frac{\partial G_{p,p}}{\partial b}\right)\\ =&\ \lim_{q\to p}\left(\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial G_{p,q}}{\partial a}-b^{1-m}\frac{\partial G_{p,q}}{\partial b}\right)\right)\\ =&\ \lim_{q\to p}\left(d_{p,q}(t)g_{p,q}(t)\right)=g_{p,p}(t)\lim_{q\to p}d_{p,q}(t). \end{aligned}$$

Summarizing two cases above yield

$$\begin{aligned} \Delta =&\ \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial\phi}{\partial a}-b^{1-m}\frac{\partial\phi}{\partial b}\right)\\ =&\ \begin{cases} g_{p,q}(t)\cdot d_{p,q}(t) & \text{if } p\neq q,\\ g_{p,p}(t)\lim_{q\to p}d_{p,q}(t) & \text{if } p=q. \end{cases} \end{aligned}$$

Since Δ is symmetric with respect to a and b, without loss of generality we assume $a>b$. It is easy to verify that $\frac{a^m-b^m}{m(a-b)}>0$, $\frac{(a-b)G_{p,q}}{2(ab)^{m/2}}>0$, and $\frac{1}{\cosh pt\cosh qt}>0$ for $t=\ln\sqrt{a/b}>0$, which implies that $d_{p,q}(t)$ and its limit at

$p = q$ are both positive. Thus by Corollary 2.1 Gini mean $G_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $\Delta \geq (\leq) 0$ if and only if $g(t) = g_{p,q}(t) \geq (\leq) 0$ for all $t > 0$.

It is easy to check that for $m = 0$ this lemma is also true.

This lemma is proved. □

Lemma 3.2. *Let* $g(t) = g_{p,q}(t)$ *be defined by* (3.1). *Then*

$$\lim_{t\to 0, t>0} \frac{g_{p,q}(t)}{2t} = p + q - m. \tag{3.3}$$

Proof. It is easy to check that $g(0) = 0$.

In the case of $p \neq q$, applying L'Hospital's rule yields

$$\begin{aligned}\lim_{t\to 0, t>0} \frac{g_{p,q}(t)}{2t} &= \lim_{t\to 0, t>0} \frac{\partial g_{p,q}(t)}{2\partial t} \\ &= \frac{(p-q)A + pB + qC}{2(p-q)} = p + q - m.\end{aligned}$$

In the case of $p = q$, we have

$$\lim_{t\to 0, t>0} \frac{g_{p,p}(t)}{2t} = 2p - m.$$

This completes the proof. □

Lemma 3.3. *Let* $m > 0$ *and* $\beta = \max(|A|, |B|, |C|)$ *where* A, B, C *are defined by* (3.1). *Then*

(i) *if* $p > q$, *then*

$$\lim_{t\to\infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} = \begin{cases} p + q - m & \text{if } p > q > m \text{ or } 0 > p > q, \\ \frac{p^2}{p-m} & \text{if } p > q = m, \\ 2(q-m) & \text{if } p = 0 > q, \\ \frac{q(p-q+m)}{p-q} & \text{if } p > 0, q < m, p > q; \end{cases} \tag{3.4}$$

(ii) *if* $p = q$, *then*

$$\lim_{t\to\infty} \frac{2\beta g_{p,p}(t)}{e^{\beta t}} = \begin{cases} 2p - m & \text{if } p > m \text{ or } p < 0, \\ -2m & \text{if } p = 0, \\ \infty & \text{if } 0 < p \leq m. \end{cases} \tag{3.5}$$

Proof. (3.4)-(3.5) easily follows from the following limit relations:

$$\begin{aligned}\lim_{t\to\infty} \frac{2\cosh \alpha t}{e^{\beta t}} &= \begin{cases} 1 & \text{if } \beta = |\alpha|, \\ 0 & \text{if } \beta > |\alpha|, \end{cases} \\ \lim_{t\to\infty} \frac{2\alpha t \sinh \alpha t}{e^{\beta t}} &= \begin{cases} \infty & \text{if } \beta = |\alpha|, \\ 0 & \text{if } \beta > |\alpha|. \end{cases}\end{aligned}$$

(i) If $p > q$, then $\beta = \max(|A|, |B|, |C|) = \max(|A|, |C|)$ because $|C|^2 - |B|^2 = 4m(p-q) > 0$. We have

$$(p-q) \lim_{t\to\infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} = (p-q) \lim_{t\to\infty} \frac{2}{e^{\beta t}} \frac{\partial g_{p,q}(t)}{\partial t}$$

$$= \lim_{t\to\infty} 2\frac{(p-q)A\cosh At + pB\cosh Bt + qC\cosh Ct}{e^{\beta t}}$$

$$= \begin{cases} (p-q)A & \text{if } |A| > |C|, \text{ i.e., } p(q-m) > 0, \\ (p-q)A + qC & \text{if } |A| = |C|, \text{ i.e., } p(q-m) = 0, \\ qC & \text{if } |A| < |C|, \text{ i.e., } p(q-m) < 0. \end{cases}$$

$$= \begin{cases} (p-q)(p+q-m) & \text{if } p > q > m \text{ or } 0 > p > q, \\ p^2 & \text{if } p > q = m, \\ -2q(q-m) & \text{if } p = 0 > q, \\ q(p-q+m) & \text{if } p > 0, q < m, p > q. \end{cases}$$

Dividing by $(p-q)$ in the above limit relation yields (3.4).

(ii) If $p = q$, then $\beta = \max(|A|, |B|, |C|) = \max(|2p-m|, m)$. We have

$$\lim_{t\to\infty} \frac{2\beta g_{p,p}(t)}{e^{\beta t}} = \lim_{t\to\infty} \frac{2}{e^{\beta t}} \frac{\partial g_{p,p}(t)}{\partial t}$$

$$= \lim_{t\to\infty} 2\frac{(2p-m)\cosh(2p-m)t + (2p-m)\cosh mt + 2mp\sinh mt}{e^{\beta t}}$$

$$= \begin{cases} 2p-m & \text{if } |2p-m| > m, \text{ i.e., } p > m \text{ or } p < 0, \\ \infty & \text{if } |2p-m| = m,\ p \neq 0, \text{ i.e., } p = m, \\ -2m & \text{if } |2p-m| = m,\ p = 0, \text{ i.e. } p = 0, \\ \infty & \text{if } |2p-m| < m, \text{ i.e., } 0 < p < m, \end{cases}$$

which implies (3.5).

This completes the proof. □

4. Proof of main results

Proof of Theorem 1.1. Assume that

$$E_1 = \{(p,q) : p+q-m \geq 0, \min(p,q) \geq 0\} \quad (m > 0).$$

By Lemma 3.1, to prove Theorem 1.1, it suffices to prove that $g_{p,q}(t) \geq 0$ for all $t > 0$ if and only if $(p,q) \in E_1$.

Necessity. We prove that $(p,q) \in E_1$ is the necessary conditions for $g(t) = g_{p,q}(t) \geq 0$ for all $t > 0$. It is obvious that

$$\lim_{t\to 0, t>0} \frac{g_{p,q}(t)}{2t} \geq 0 \text{ and } \lim_{t\to\infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} \geq 0. \tag{4.1}$$

Now, we get the necessary conditions from (4.1) together with (3.4) and (3.5). To this aim, we distinguish three cases.

(i) Case 1: $p > q$. By (4.1) together with (3.3) and (3.4), we have

Subcase 1:

$$\begin{cases} p+q-m \geq 0, \\ p+q-m \geq 0, \\ p > q > m \text{ or } 0 > p > q \end{cases} \Longrightarrow p > q > m,$$

which implies $(p,q) \in \{(p,q) : p > q > m\} := E_{11}$.

Subcase 2:

$$\begin{cases} p+q-m \geq 0, \\ \frac{p^2}{p-m} \geq 0, \\ p > q = m \end{cases} \Longrightarrow p > q = m,$$

which implies $(p,q) \in \{(p,q) : p > q = m\} := E_{12}$.

Subcase 3:

$$\begin{cases} p+q-m \geq 0, \\ 2(q-m) \geq 0, \\ p = 0 > q \end{cases} \Longrightarrow \text{which is impossible.}$$

Subcase 4:

$$\begin{cases} p+q-m \geq 0, \\ \frac{q(p-q+m)}{p-q} \geq 0, \\ p > 0, \\ q < m, \\ p > q \end{cases} \Longrightarrow \begin{cases} p+q-m \geq 0, \\ p > 0, \\ 0 < q < m, \\ p > q, \end{cases}$$

which implies $(p,q) \in \{(p,q) : p+q-m \geq 0, p > 0, 0 < q < m, p > q\} := E_{14}$.

(i′) Case 1′: $p < q$. Since $g_{p,q}(t)$ is symmetric with respect to p and q, we get $(p,q) \in E'_{111} \cup E'_{112} \cup E'_{114}$, where

$$\begin{aligned} E'_{11} &= \{(p,q) : q > p > m\},\ E'_{12} = \{(p,q) : q > p = m\}, \\ E'_{14} &= \{(p,q) : p+q-m \geq 0, q > 0, 0 < p < m, q > p\}. \end{aligned}$$

(ii) Case 2: $p = q$. By (4.1) together with (3.3) and (3.5), we have

Subcase 1:

$$\begin{cases} p+q-m \geq 0, \\ 2p-m \geq 0, \\ p > m \text{ or } p < 0 \end{cases} \Longrightarrow p = q > m.$$

Subcase 2:

$$\begin{cases} p+q-m \geq 0, \\ -2m \geq 0, \\ p = 0 \end{cases} \Longrightarrow \text{which is impossible.}$$

Subcase 3:

$$\begin{cases} p+q-m \geq 0, \\ \infty \geq 0, \\ 0 < p \leq m \end{cases} \Longrightarrow \frac{m}{2} \leq p = q < m.$$

The above three subcases imply $(p,q) \in \{(p,q) : p = q \geq \frac{m}{2}\} := E_{10}$.

Summarizing all the cases (i), (i′) and (ii) yields

$$(p,q) \in (E_{11} \cup E_{12} \cup E_{14}) \cup (E'_{11} \cup E'_{12} \cup E'_{14}) \cup E_{10} = E_1.$$

Sufficiency. We prove the condition $(p,q) \in E_1$ is sufficient for $g(t) = g_{p,q}(t) \geq 0$ for all $t > 0$. Since $g(0) = 0$, it is enough to prove $g'(t) \geq 0$ if $(p,q) \in E_1$. For symmetry, we may assume again that $p \geq q$.

Noting

$$(p-q)A = pB + qC \quad \text{or} \quad pB = (p-q)A - qC,$$

we have

$$\begin{aligned}(p-q)g'(t) &= (p-q)A\cosh At + pB\cosh Bt + qC\cosh Ct \\ &= (p-q)A(\cosh At + \cosh Bt) + qC(\cosh Ct - \cosh Bt) \\ &= (p-q)A(\cosh At + \cosh Bt) + 2qC\sinh(p-q)t\sinh mt. \end{aligned} \tag{4.2}$$

If $p > q$ and $(p,q) \in E_1$, then $A = p+q-m \geq 0$, $q = \min(p,q) \geq 0$, $C = p - q + m > 0$. It follows that $(p-q)g'(t) \geq 0$ for $(p,q) \in E_1$.

If $p = q$ and $(p,q) \in E_1$, then $2p - m \geq 0$, $p = \min(p,q) \geq 0$. Therefore,

$$g'(t) = (2p-m)\cosh(2p-m)t + (2p-m)\cosh mt + 2mp\sinh mt \geq 0. \tag{4.3}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Assume that

$$E_2 = \{(p,q\} : p+q-m \leq 0, p \geq q, q \leq 0\} \quad (m > 0),$$
$$E_2' = \{(p,q\} : p+q-m \leq 0, q \geq p, p \leq 0\} \quad (m > 0),$$

then

$$E_2 \cup E_2' = \{(p,q\} : p+q-m \leq 0 \text{ and } \min(p,q) \leq 0\} \quad (m > 0).$$

By Lemma 3.1, to prove Theorem 1.2, it suffices to show that $g_{p,q}(t) \leq 0$ for all $t > 0$ if and only if $(p,q) \in E_2 \cup E_2'$.

Necessity. If $g_{p,q}(t) \leq 0$ for all $t > 0$, then

$$\lim_{t\to 0, t>0} \frac{g_{p,q}(t)}{2t} \leq 0 \text{ and } \lim_{t\to\infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} \leq 0. \tag{4.4}$$

Similarly, we divide the proof of necessity into three cases.

(i) Case 1: $p > q$. By (4.4) together with (3.3) and (3.4), we have

Subcase 1:

$$\begin{cases} p+q-m \leq 0, \\ p+q-m \leq 0, \\ p > q > m \text{ or } 0 > p > q \end{cases} \Longrightarrow 0 > p > q,$$

which implies $(p,q) \in \{(p,q) : 0 > p > q\} := E_{21}$.

Subcase 2:

$$\begin{cases} p+q-m \leq 0, \\ \frac{p^2}{p-m} \leq 0, \\ p > q = m \end{cases} \Longrightarrow \text{which is impossible.}$$

Subcase 3:

$$\begin{cases} p+q-m \leq 0, \\ 2(q-m) \leq 0, \\ p = 0 > q \end{cases} \Longrightarrow p = 0 > q,$$

which implies $(p,q) \in \{(p,q) : p = 0 > q\} := E_{23}$.

Subcase 4:

$$\begin{cases} p+q-m \le 0, \\ \frac{q(p-q+m)}{p-q} \le 0, \\ p > 0, \\ q < m, \\ p > q \end{cases} \implies \begin{cases} p+q-m \le 0, \\ p > 0 \ge q, \end{cases}$$

which implies $(p,q) \in \{(p,q) : p+q-m \le 0, p > 0 \ge q\} := E_{24}$.

(i′) Case 1′: $p < q$. Since $g_{p,q}(t)$ is symmetric with respect to p and q, so $(p,q) \in E'_{21} \cup E'_{23} \cup E'_{24}$, where

$$E'_{21} = \{(p,q) : 0 > q > p\},$$
$$E'_{23} = \{(p,q) : q = 0 > p\},$$
$$E'_{24} = \{(p,q) : p+q-m \le 0, q > 0 \ge p\}.$$

(ii) Case 2: $p = q$. By (4.4) together with (3.3) and (3.5), we have

Subcase 1:

$$\begin{cases} p+q-m \le 0, \\ 2p-m \le 0, \\ p > m \text{ or } p < 0 \end{cases} \implies p = q < 0.$$

Subcase 2:

$$\begin{cases} p+q-m \le 0, \\ -2m \le 0, \\ p = 0 \end{cases} \implies p = q = 0.$$

Subcase 3:

$$\begin{cases} p+q-m \le 0, \\ \infty \le 0, \\ 0 < p \le m \end{cases} \implies \text{which is impossible.}$$

The above three subcases imply $(p,q) \in \{(p,q) : p = q \le 0\} := E_{20}$.

Summarizing all the cases (i), (i′) and (ii) yields

$$(p,q) \in (E_{21} \cup E_{23} \cup E_{24}) \cup (E'_{21} \cup E'_{23} \cup E'_{24}) \cup E_{20} = E_2 \cup E'_2.$$

Sufficiency. Similarly to proof of sufficiency of Theorem 1.1, by (4.2) and (4.3) we easily prove $g'(t) \le 0$ if $(p,q) \in E_2 \cup E'_2$. Hence $g_{p,q}(t) = g(t) \le g(0) = 0$ for all $t > 0$.

The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Let $g_{p,q,m}(t) := g_{p,q}(t)$ be defined by (3.1) and

$$p' = -p, \quad q' = -q, \quad m' = -m.$$

We easily verify that, for $p, q, p', q', m, m' \in \mathbb{R}$,

$$g_{p,q,m}(t) = -g_{p',q',m'}(t).$$

From this and Lemma 3.1, for $m < 0$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex if and only if $G_{p\prime,q\prime}(a,b)$ is Schur m'-power concave with respect to $(a,b) \in \mathbb{R}^2_+$, which, by Theorem 1.2, if and only if

$$p' + q' \le m' \quad \text{and} \quad \min(p', q') \le 0,$$

that is,

$$p + q \ge m \quad \text{and} \quad \max(p, q) \ge 0.$$

Theorem 1.3 follows. □

Proof of Theorem 1.4. Similarly as in the proof of Theorem 1.3, for $m < 0$, Gini mean $G_{p,q}(a,b)$ is Schur m-power concave if and only if $G_{p\prime,q\prime}(a,b)$ is Schur m'-power convex with respect to $(a,b) \in \mathbb{R}^2_+$, which, by Theorem 1.1, if and only if

$$p' + q' \ge m' \quad \text{and} \quad \min(p', q') \ge 0,$$

that is,

$$p + q \le m \text{ and } \max(p, q) \le 0,$$

The proof of Theorem 1.4 ends. □

Proof of Theorem 1.5. By Lemma 3.1, to prove Theorem 1.5, it is enough to prove that $g_{p,q}(t) \ge (\le) 0$ for all $t > 0$ if and only if $p + q \ge (\le) 0$ for $m = 0$. To this end, we divide the proof into two cases.

(i) **Case 1**: $p \ne q$. By (3.1), we have

$$\begin{aligned} g_{p,q}(t) &= \frac{(p-q)\sinh(p+q)t + (p+q)\sinh(p-q)t}{p-q} \\ &= \begin{cases} t(p+q)\left(\frac{\sinh(p+q)t}{(p+q)t} + \frac{\sinh(p-q)t}{(p-q)t}\right) & \text{if } p+q \ne 0, \\ 0 & \text{if } p+q = 0. \end{cases} \end{aligned}$$

Since $\frac{\sinh u}{u} > 0$ for all $u \ne 0$ and $t > 0$, we obtain $\operatorname{sgn}(g_{p,q}(t)) = \operatorname{sgn}(p+q)$.

(ii) **Case 2**: $p = q$. By (3.1), we have

$$g_{p,p}(t) = \begin{cases} 2pt\left(\frac{\sinh(2pt)}{2pt} + 1\right) & \text{if } p \ne 0, \\ 0 & \text{if } p = 0. \end{cases}$$

It is obvious that $\operatorname{sgn}(g_{p,p}(t)) = \operatorname{sgn}(p)$.

In brief, $g_{p,q}(t) \ge (\le) 0$ for all $t > 0$ if and only if $p + q \ge (\le) 0$.

The proof of Theorem 1.5 is finished. □

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