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A CORRECTION TO A PAPER ON ROMAN k-DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and k be a positive integer. A k-dominating set of G is a subset $S \subseteq V$ such that each vertex in $V \setminus S$ has at least k neighbors in S. A Roman k-dominating function on G is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. In the paper titled "Roman k-domination in graphs" (J. Korean Math. Soc. **46** (2009), no. 6, 1309–1318) K. Kammerling and L. Volkmann showed that for any graph G with n vertices, $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq \min \{2n, 4k + 1\}$, and the equality holds if and only if $n \leq 2k$ or $k \geq 2$ and $n = 2k + 1$ or $k = 1$ and G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$. In this paper we find a counterexample of Kammerling and Volkmann's result and then give a correction to the result.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A *k-dominating set* of G is a subset $S \subseteq V$ such that every vertex in $V \setminus S$ has at least k neighbors in S. The *k-domination number* $\gamma_k(G)$ of G is the minimum cardinality among the k-dominating sets of G. A 1-domination number $\gamma_1(G)$ is identified with the usual domination number $\gamma(G)$ (see [1, 3, 5]). A *Roman k-dominating function* on a graph G is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The weight of a Roman k-dominating function f is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman k-dominating function on a graph G is said to be the *Roman k-domination number* $\gamma_{kR}(G)$ of G. A Roman k-dominating function on a graph G of minimum weight is called a γ_{kR}-function of G. A Roman 1-domination number $\gamma_{1R}(G)$ of a graph G is identified with the usual Roman domination number $\gamma_R(G)$ (see [2, 4]). The

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order of a graph $G = (V, E)$ is the cardinality of V denoted by $|V|$ or $n(G)$ and the induced subgraph of G generated by subset $U \subseteq V$ is denoted by $G[U]$.

In 2009, K. Kammerling and L. Volkmann [2] studied Roman k-domination number of graphs and they showed the following.

Theorem 1 ([2], Theorem 2.8). *If G is a graph of order n, then*

$$\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq \min\{2n, 4k+1\}. \tag{1}$$

Furthermore the equality holds in (1) *if and only if $n \leq 2k$ or $k \geq 2$ and $n = 2k+1$ or $k = 1$ and G or $\overline{G}$ has a vertex of degree $n-1$ and its complement has a vertex of degree $n-2$.*

In this paper, we find a counterexample of the equality part of the above result and then give a correction to this result.

2. Main results

In this section we improve Theorem 1. The following results from [2] are useful.

Theorem 2 ([2], Proposition 2.6). *If G is a graph of order n, then $\gamma_{kR}(G) \geq \min\{n,\ \gamma_k(G) + k\}$.*

Theorem 3 ([2], Proposition 2.7). *Let G be a graph of order n.*
(i) *If $n \leq 2k$, then $\gamma_{kR}(G) = n$.*
(ii) *If $n \geq 2k+1$, then $\gamma_{kR}(G) \geq 2k$.*
(iii) *If $n \geq 2k+1$ and $\gamma_k(G) = k$, then $\gamma_{kR}(G) = \gamma_k(G) + k = 2k$.*

The following has a straightforward proof, so its proof is left to the reader.

Observation 4. *Let G be a graph with t component $H_1, H_2, \ldots, H_t$. Then*

$$\gamma_{kR}(G) = \sum_{i=1}^{t} \gamma_{kR}(H_i).$$

First we present a counterexample.

A counterexample to Theorem 1. Let k be a positive integer $k \geq 2$, and let G be a graph such that $V(G) = \{a_0, a_1, a_2, \ldots, a_{2k}\}$, $E(G) = \{a_0a_i \mid 1 \leq i \leq k\} \cup \{a_{2i-1}a_{2i} \mid 1 \leq i \leq k\}$ (see Figure 1 for an illustration).

It is easy to see that $\gamma_{kR}(G) \leq 2k+1$ and $\gamma_{kR}(\overline{G}) \leq 2k+1$, since the function defined by $f(v) = 1$ for all v is a Roman k-dominating function on both G and $\overline{G}$. We will show that $\gamma_k(G) > k$. Suppose that there exists a k-dominating set D of G such that $|D| = k$. Then any vertex in D is adjacent to any vertex in $V(G) \setminus D$. Since $|V(G)| = 2k+1$ and $|D| = k$, G has k vertices whose degrees are at least $k+1$. However, the vertex a_0 is the only one vertex which has degree at least $k+1$, a contradiction. Therefore, $\gamma_k(G) > k$ and so $\gamma_{kR}(G) \geq 2k+1$ by Theorem 2. We can conclude that $\gamma_{kR}(G) = 2k+1$.

Now consider the complement $\overline{G}$ of G. Then $\overline{G}$ is the disjoint union of an isolated vertex a_0 and the complete k-partite graph with partite sets of equal

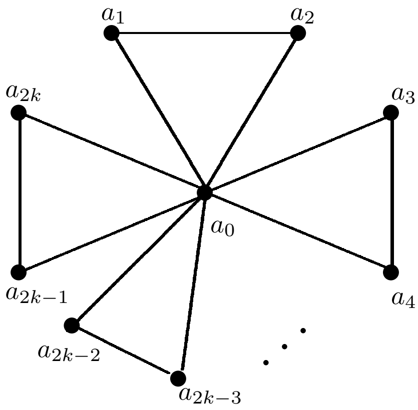

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Figure 1.

size 2, and call those two connected components H_1 and H_2, respectively. By (i) of Theorem 3, $\gamma_{kR}(H_1) = 1$ and $\gamma_{kR}(H_2) = 2k$. By Observation 4, $\gamma_{kR}(\overline{G}) = \gamma_{kR}(H_1) + \gamma_{kR}(H_2)$, and therefore $\gamma_{kR}(\overline{G}) = 2k + 1$.

As we shown that $\gamma_{kR}(G) = \gamma_{kR}(\overline{G}) = 2k + 1$, we obtain that

$$\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = 4k + 2 > \min\{2|V(G)|, 4k + 1\} = 4k + 1,$$

which violates the equality part of Theorem 1.

Now we give a correction of Theorem 1. If $f : V \to \{0, 1, 2\}$ is a Roman k-dominating function on a graph G, then $\{V_0, V_1, V_2\}$ is a partition of V where for $i = 0, 1, 2$, $V_i = \{v \in V(G) \mid f(v) = i\}$. In the rest of the paper, we denote the function f by (V_0, V_1, V_2) for simplicity.

Theorem 5. *If G is a graph of order n, then $\gamma_{kR}(G)+\gamma_{kR}(\overline{G}) \geq \min\{2n, 4k+1\}$ and the equality holds if and only if one of the following holds*:

(i) $n \leq 2k$;

(ii) $n = 2k + 1$, *and either* $\gamma_k(G) = k$ *or* $\gamma_k(\overline{G}) = k$;

(iii) $k = 1$, $n \geq 4$ *and G or $\overline{G}$ has a vertex of degree $n-1$ and its complement has a vertex of degree $n - 2$.*

Proof. The proof of inequality part is identified with the correspondence proof of Theorem 1 ([2] Theorem 2.8).

If (i) holds, then $\gamma_{kR}(G) = n = \gamma_{kR}(\overline{G})$ and $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = 2n$ and therefore $2n = \min\{2n, 4k + 1\}$.

Suppose that (ii) holds. Without loss of generality, we assume that $\gamma_k(G) = k$. By (iii) of Theorem 2, $\gamma_{kR}(G) = 2k$. Since $f(\emptyset, V(\overline{G}), \emptyset)$ is a γ_{kR}-function of $\overline{G}$, $\gamma_{kR}(\overline{G}) \leq n = 2k + 1$. Therefore $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \leq 4k + 1$. From the inequality part and the fact that $\min\{2n, 4k + 1\} = 4k + 1$, it holds that $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq 4k + 1$. Thus $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = 4k + 1$.

Let $k = 1$, $n \geq 4$ and G or $\overline{G}$ has a vertex of degree $n-1$ and its complement has a vertex of degree $n-2$. We can assume that G has a vertex of degree $n-1$. Therefore there exists a vertex in G that dominates G and hence $\gamma_{kR}(G) = 2$. The vertex of degree $n-1$ is an isolated vertex in $\overline{G}$. Thus the isolated vertex and the vertex of degree $n-2$ in $\overline{G}$ dominate $\overline{G}$. So $\gamma_{kR}(\overline{G}) = 2+1 = 3$ and $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = \gamma_R(G) + \gamma_R(\overline{G}) = 2+3 = 5 = \min\{2n, 4k+1\}$.

Conversely, let $\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) = \min\{2n, 4k+1\}$. If $n \leq 2k$, then (i) immediately follows. Suppose that $n \geq 2k+1$. Then $\min\{2n, 4k+1\} = 4k+1$. By (ii) of Theorem 2, $\gamma_{kR}(G) \geq 2k$ and $\gamma_{kR}(\overline{G}) \geq 2k$. Without loss of generality, we may assume that $\gamma_{kR}(G) = 2k$ and $\gamma_{kR}(\overline{G}) = 2k+1$. Since $\gamma_{kR}(G) = 2k$, it follows that there exists a γ_{kR}-function $f(V_0, V_1, V_2)$ on G such that $|V_0| = n-k$, $V_1 = \emptyset$, $|V_2| = k$, and V_2 is a k-dominating set of G. Note that $\gamma_k(G) = k$.

Since any vertex of V_0 and any vertex of V_2 are adjacent in G and $V_1 = \emptyset$, $\overline{G}$ is the union of $\overline{G}[V_0]$ and $\overline{G}[V_2]$. Therefore, by Observation 4,

$$\gamma_{kR}(\overline{G}) = \gamma_{kR}(\overline{G}[V_0]) + \gamma_{kR}(\overline{G}[V_2]).$$

Since $\gamma_{kR}(\overline{G}[V_2]) = k$ by (i) of Theorem 2 and $\gamma_{kR}(\overline{G}) = 2k+1$ by the assumption, it follows that $\gamma_{kR}(\overline{G}[V_0]) = k+1$.

On the other hand, since $\overline{G}[V_0]$ has $n-k$ vertices, by (i) and (ii) of Theorem 2, one of the following holds:

(a) $n-k \leq 2k$ and $\gamma_{kR}(\overline{G}[V_0]) = n-k$;

(b) $n-k \geq 2k+1$ and $\gamma_{kR}(\overline{G}[V_0]) \geq 2k$.

Suppose that (a) holds. Then $k+1 = n-k$ and so $n = 2k+1$. Since we already have $\gamma_k(G) = k$, (ii) immediately follows. Suppose that (b) holds. Then $k+1 \geq 2k$ and so $k = 1$. In addition, $n-k \geq 2k+1$ implies $n \geq 4$. Since we already showed that any vertex in V_2 has degree $n-k$, G has a vertex of degree $n-1$. Since $k = 1$, $\gamma_{kR}(\overline{G}[V_0]) = k+1 = 2k$, which implies that $\overline{G}[V_0]$ has a k-dominating set of size k. Then $\overline{G}[V_0]$ has a vertex which is adjacent to the other vertices of $\overline{G}[V_0]$. Since $|V_0| = n-k = n-1$, we can conclude that $\overline{G}$ has a vertex of degree $n-2$. Thus (iii) holds. □

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