Bull. Korean Math. Soc. **50** (2013), No. 2, pp. 459–467
http://dx.doi.org/10.4134/BKMS.2013.50.2.459

MATLIS INJECTIVE MODULES

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Abstract. In this paper, Matlis injective modules are introduced and studied. It is shown that every R-module has a (special) Matlis injective preenvelope over any ring R and every right R-module has a Matlis injective envelope when R is a right Noetherian ring. Moreover, it is shown that every right R-module has an $\mathcal{F}^{\perp_1}$-envelope when R is a right Noetherian ring and $\mathcal{F}$ is a class of injective right R-modules.

1. Introduction

Throughout this paper, R will denote an associative ring with identity and all modules will be unitary right R-modules.

The motivation of this paper is from [4], where the notion of Whitehead modules was studied. Recall that an R-module M is called a *Whitehead module* or *W-module* if $\mathrm{Ext}^1_R(M,R)=0$. We introduce the notion of Matlis injective modules as a dual notion of Whitehead modules in some sense. An R-module M is called *Matlis injective* if $\mathrm{Ext}^1_R(E(R),M)=0$, where $E(R)$ denotes the injective envelope of R. Let R be an integral domain and Q its field of quotients, an R-module C is called *Matlis cotorsion* or *weakly cotorson* if $\mathrm{Ext}^1_R(Q,C)=0$. Then, it is easy to see that the notion of Matlis injective R-modules coincides with the notion of Matlis cotorsion R-modules when R is an integral domain. Following [7], an R-module M is called *copure injective* if $\mathrm{Ext}^1_R(E,M)=0$ for any injective R-module E. Clearly, every copure injective R-module is Matlis injective, but it is easy to see that the converse is not true in general. Thus Matlis injective R-modules can be seen as a generalization of copure injective R-modules.

Let $\mathcal{C}$ be a class of R-modules. Enochs defined a $\mathcal{C}$-(pre)cover ($\mathcal{C}$-(pre)envelope) of an R-module in [6]. Therefore, it is natural to study the existence of Matlis injective (pre)covers and Matlis injective (pre)envelopes. Obviously, the class of Matlis injective R-modules is closed under direct summands, but we show that it is not closed under direct sums in general. So there exist a ring

Received October 15, 2011; Revised January 20, 2012.
2010 *Mathematics Subject Classification.* 16D10, 16E30.
Key words and phrases. Matlis injective module, (pre)envelope, Σ-pure injective.
This research was partially supported by Research Fund of China Pharmaceutical University (No. 211148).

R and an R-module M such that M doesn't have a Matlis injective precover. Then, we are only interested in the existence of Matlis injective (pre)-envelopes in this paper. Let $\mathcal{F}$ be a class of R-modules, we denote by $\mathcal{F}^{\perp_1}$ the class of R-modules N such that $\mathrm{Ext}^1_R(F, N) = 0$ for every $F \in \mathcal{F}$. In [5, Theorem 10], Eklof and Trlifaj proved that if there is a set $\mathcal{S}$ of R-modules such that $\mathcal{F}^{\perp_1} = \mathcal{S}^{\perp_1}$, then every R-module has an $\mathcal{F}^{\perp_1}$-preenvelope. Using this result, we show that every R-module has a Matlis injective preenvelope. If R is a right Noetherian ring, we show that every R-module has an $\mathcal{F}^{\perp_1}$-envelope, where $\mathcal{F}$ is any subclass of the class of injective R-modules. As a byproduct, we show that every R-module has a Matlis injective envelope when R is a right Noetherian ring.

2. Preliminaries

In this section we briefly recall some definitions and results required in this paper.

For a ring R, Mod-R will denote the category of all right R-modules and $pd(M)$ will denote the projective dimension of M. For an R-module M, we denote by $E(M)$ the injective envelope of M. We frequently identify M with its image in $E(M)$ and think of M as a submodule of $E(M)$.

Let $\mathcal{C} \subseteq$Mod-R. Define

$$\mathcal{C}^{\perp_1} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\},$$
$$^{\perp_1}\mathcal{C} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(X, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

Add($\mathcal{C}$)=$\{X \in$ Mod-R | X is a direct summand of $\bigoplus_{i\in I} C_i$, where I is a set and where for any $i \in I, C_i$ is isomorphic to an element of $\mathcal{C}\}$.

For $\mathcal{C} = \{C\}$, we write $C^{\perp_1}$, $^{\perp_1}C$ and Add(C) in place of $\{C\}^{\perp_1}$, $^{\perp_1}\{C\}$ and Add($\{C\}$), respectively.

Let $M \in$ Mod-R. A homomorphism $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope* of M provided that the abelian group homomorphism $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. The $\mathcal{C}$-preenvelope f is called a *$\mathcal{C}$-envelope* of M provided that $f = gf$ implies g is an automorphism for each $g \in \mathrm{End}_R(C)$. Moreover, a $\mathcal{C}$-preenvelope $f : M \to C$ of M is called *special* provided that f is injective and Coker $f \in {}^{\perp_1}\mathcal{C}$. $\mathcal{C}$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. If $\mathcal{C}$ is the class of injective modules, then we get the usual injective envelopes.

$\mathcal{C}$-precovers and $\mathcal{C}$-covers are defined dually. These generalize the projective covers introduced by Bass in the 1960's.

A pair $(\mathcal{A}, \mathcal{B})$ of R-module classes is called a *cotorsion theory* (or *cotorsion pair*) provided that $\mathcal{A}^{\perp_1} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$. An R-module M is called *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat R-module F. Let $\mathscr{F}$ be the class of flat R-modules and $\mathscr{C}$ be the class of cotorsion R-modules, it is known that $(\mathscr{F}, \mathscr{C})$ is a cotorsion theory.

For any class $\mathcal{F}$ of R-modules. The following theorem, due to Eklof and

Trlifaj, says that every R-module has a special $\mathcal{F}^{\perp_1}$-preenvelope if there is a set $\mathcal{S}$ of R-modules such that $\mathcal{S}^{\perp_1} = \mathcal{F}^{\perp_1}$. Before stating the result, we need more notions:

A sequence of modules $\mathcal{A} = (A_\alpha |\ \alpha \leq \mu)$ is called a *continuous chain of modules* provided that $A_0 = 0, A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

Let M be a module and $\mathcal{C}$ a class of modules. Then M is called *$\mathcal{C}$-filtered* provided that there are an ordinal κ and a continuous chain, $(M_\alpha |\ \alpha \leq \kappa)$, consisting of submodules of M such that $M = M_\kappa$, and such that each of the modules $M_{\alpha+1}/M_\alpha$ $(\alpha < \kappa)$ is isomorphic to an element of $\mathcal{C}$. The chain $(M_\alpha |\ \alpha \leq \kappa)$ is called a *$\mathcal{C}$-filtration* of M.

Theorem 2.1 ([10], Theorem 3.2.1, p. 117)**.** *Let $\mathcal{S}$ be a set of R-modules and M an R-module. Then there is a short exact sequence* $0 \to M \hookrightarrow P \to N \to 0$, *where $P \in \mathcal{S}^{\perp_1}$ and N is $\mathcal{S}$-filtered. In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^{\perp_1}$-preenvelope of M.*

The following theorem from [10] gives a criterion to judge when an R-module M has a $\mathcal{C}^{\perp_1}$-envelope.

Theorem 2.2 ([10], Theorem 2.3.2, p. 107)**.** *Let R be a ring and M be an R-module. Let $\mathcal{C}$ be a class of R-modules closed under extensions and direct limits. Assume that M has a special $\mathcal{C}^{\perp_1}$-preenvelope ν with* $\mathrm{Coker}\ \nu \in \mathcal{C}$. *Then M has a $\mathcal{C}^{\perp_1}$-envelope.*

A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called *pure* if the induced sequence $0 \to \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(F, C) \to 0$ of abelian groups is exact for every finitely presented R-module F. A submodule A of an R-module B is called a *pure submodule* of B if the canonical exact sequence $0 \to A \to B \to B/A \to 0$ is pure. An R-module M is called *pure injective* if the sequence $0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact for every pure exact sequence $0 \to A \to B \to C \to 0$ of R-modules.

Let M be an R-module. M is said to be Σ-*pure injective* if for every index set I the direct sum $M^{(I)}$ is pure injective. M is said to be Σ-*self orthogonal* if $\mathrm{Ext}^1_R(M, M^{(I)}) = 0$ for every index set I.

The following property of Σ-pure injective modules will be used in this paper.

Proposition 2.3 ([9], Corollary 1.42, p. 30)**.** *Every pure submodule of a Σ-pure injective module B is a direct summand of B.*

For unexplained terminology and notation, we refer the reader to [1, 3, 8, 10, 13].

3. Properties of Matlis injective modules

We start with the following definition.

Definition 3.1. Let R be a ring and M an R-module. M is said to be *Matlis injective* if $\mathrm{Ext}_R^1(E(R), M) = 0$. An R-module N is said to be *Matlis projective* if $\mathrm{Ext}_R^1(E(R), C) = 0$ implies $\mathrm{Ext}_R^1(N, C) = 0$ for any R-module C. R is said to be a *right Matlis ring* if $E(R)$ is flat and $pd(E(R)) \leq 1$.

In what follows, we denote by $\mathcal{MI}$ ($\mathcal{MP}$) the class of Matlis injective (projective) R-modules. For $\mathcal{C} = \mathcal{MI}$, $\mathcal{C}$-(pre)envelopes will simply be called Matlis injective (pre)envelopes.

Proposition 3.2. *Let R be a ring. Then $\mathcal{MI}$ is closed under extensions, direct products and direct summands*; *$\mathcal{MI}$=Mod-R if and only if $E(R)$ is projective.*

Proof. It is easy to see that the assertion holds by definition. □

Corollary 3.3. *Let R be an integral domain. Then every R-module is Matlis injective if and only if R is a field.*

Proof. " $\Longleftarrow$ " is trivial.
" $\Longrightarrow$ ". By Proposition 3.2, $E(R)$ is projective, then there exists a non-zero homomorphism $f \in \mathrm{Hom}_R(E(R), R)$. So $f(E(R))$ is a non-zero divisible submodule of R. Let r be any non-zero element from R. We choose a non-zero element $x \in f(E(R))$. Since rx is non-zero and $f(E(R))$ is divisible, there is an element $y \in f(E(R))$ with $(rx)y = x$, and so $(ry - 1)x = 0$. But R is an integral domain, then $ry - 1 = 0$, i.e., $ry = 1$. Hence R is a field. □

Remark 3.4. Recall that a commutative domain R is called *almost perfect* provided that R/I is a perfect ring for each ideal $0 \neq I \neq R$. We will show that $\mathcal{MI}$ is not closed under direct sums if R is an almost perfect domain but not a field. If R is an almost perfect domain, then $\mathcal{MI}$ coincides with the class of cotorsion R-modules by [10, Theorem 4.4.16, p. 172]. But the class of cotorsion R-modules is closed under direct sums if and only if R is a perfect ring by [11, Theorem 19]. Note that $E(R)$ is flat when R is a commutative domain, and so R is a perfect ring if and only if R is a field by Corollary 3.3. Hence $\mathcal{MI}$ is not closed under direct sums when R is an almost perfect domain but not a field. Then we will show that there exist a ring R and an R-module M such that M doesn't have a Matlis injective precover. For example, let R be an almost perfect domain but not a field, then there exists a family $\{M_i\}_{i\in I}$ of Matlis injective R-modules such that $\bigoplus_{i\in I} M_i$ is not Matlis injective. But since $\mathcal{MI}$ is closed under direct summands by Proposition 3.2, it is easy to check that $\bigoplus_{i\in I} M_i$ doesn't have a Matlis injective precover.

Lemma 3.5. *Let R be a ring. Then every cotorsion R-module is Matlis injective if and only if $E(R)$ is flat.*

Proof. " $\Longleftarrow$ " is clear.
" $\Longrightarrow$ ". Let C be any cotorsion R-module. By hypothesis, we have

$$\mathrm{Ext}_R^1(E(R), C) = 0.$$

Hence $E(R)$ is flat by the fact that $(\mathscr{F}, \mathscr{C})$ is a cotorsion theory. □

Proposition 3.6. *Let R be a ring. Then $\mathcal{MI} = \mathscr{C}$ if and only if $E(R)$ is flat and every Matlis injective R-module is cotorsion.*

Proof. " $\Longleftarrow$ " holds by assumption and Lemma 3.5.
" $\Longrightarrow$ ". By assumption, we have M is cotorsion if and only if it is Matlis injective. Then the assertion holds by Lemma 3.5. □

Proposition 3.7. *Let R be a ring. Then the following are equivalent.*

(1) *Every quotient module of any Matlis injective R-module is Matlis injective.*
(2) *Every quotient module of any injective R-module is Matlis injective.*
(3) *The projective dimension of $E(R)$ is at most 1.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3). Let K be any R-module. It is enough to show that $\mathrm{Ext}^2_R(E(R), K) = 0$. Let us consider the exact sequence $0 \to K \to E(K) \to E(K)/K \to 0$. We then have the exact sequence $\mathrm{Ext}^1_R(E(R), E(K)/K) \to \mathrm{Ext}^2_R(E(R), K) \to \mathrm{Ext}^2_R(E(R), E(K)) = 0$. Note that $\mathrm{Ext}^1_R(E(R), E(K)/K) = 0$ by (2), we get $\mathrm{Ext}^2_R(E(R), K) = 0$.

(3) $\Longrightarrow$ (1). Let M be a Matlis injective R-module and N a submodule of M. Let us consider the exact sequence $0 \to N \to M \to M/N \to 0$. Applying the functor $\mathrm{Hom}_R(E(R), -)$ to the above exact sequence, we get the exact sequence $0 = \mathrm{Ext}^1_R(E(R), M) \to \mathrm{Ext}^1_R(E(R), M/N) \to \mathrm{Ext}^2_R(E(R), N)$. Note that $\mathrm{Ext}^2_R(E(R), N) = 0$ by (3), so $\mathrm{Ext}^1_R(E(R), M/N) = 0$ and (1) follows. □

Remark 3.8. If $E(R)$ is flat, then the condition that every quotient module of any cotorsion R-module is Matlis injective is also equivalent to the conditions of Proposition 3.7.

Lemma 3.9. *Let R be a ring. Then $(\mathcal{MP}, \mathcal{MI})$ is a cotorsion theory.*

Proof. Straightforward. □

Theorem 3.10. *Let R be a ring. Then the following are equivalent.*

(1) *R is a right Matlis ring.*
(2) *Every quotient module of any Matlis injective R-module is Matlis injective and every cotorsion R-module is Matlis injective.*
(3) *Every quotient module of any injective R-module is Matlis injective and every cotorsion R-module is Matlis injective.*
(4) *Every Matlis projective R-module is flat and its projective dimension is at most 1.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) hold by Lemma 3.5 and Proposition 3.7.

(1) $\Longrightarrow$ (4). By Lemma 3.9 and [10, Corollary 3.2.4, p. 119], every Matlis projective R-module is a direct summand of some $\{E(R), R\}$-filtered R-module. Note that every $\{E(R), R\}$-filtered R-module is flat and its projective dimension is at most $pd(E(R))$ by (1) and [10, Lemma 3.1.2, p. 113]. So every Matlis projective R-module is flat and its projective dimension is at most 1.

(4) $\Longrightarrow$ (1). Obviously, $E(R)$ is Matlis projective by definition. So (1) holds by assumption. □

Recall that a submodule N of a module M of projective dimension k is said to be a *tight submodule* if the projective dimension of M/N is at most k. We now have the following simple fact:

Proposition 3.11. *Let R be a ring. If $pd(E(R)) \leq 1$, then tight submodules of Matlis projective R-modules are also Matlis projective.*

Proof. Let us consider the exact sequence $0 \to N \to M \to M/N \to 0$, where M is Matlis projective and N is a tight submodule of M. Then $pd(M) \leq 1$ by hypothesis and the proof of Theorem 3.10. For any Matlis injective R-module C, we have the induced exact sequence

$$\mathrm{Ext}^1_R(M, C) \to \mathrm{Ext}^1_R(N, C) \to \mathrm{Ext}^2_R(M/N, C).$$

The two ends vanish, since M is Matlis projective and $pd(M/N) \leq pd(M) \leq 1$. So the middle term is 0, and hence the assertion holds. □

Proposition 3.12. *Let R be a ring. Then the following are equivalent.*

(1) *$C \in \mathcal{MI}$ whenever $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules such that $A, B \in \mathcal{MI}$.*
(2) *$E(M)/M$ is Matlis injective when M is Matlis injective.*
(3) *For any R-module M, $\mathrm{Ext}^1_R(E(R), M) = 0$ implies $\mathrm{Ext}^2_R(E(R), M) = 0$.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3). Let M be an R-module such that $\mathrm{Ext}^1_R(E(R), M) = 0$, i.e., M is Matlis injective. Then $E(M)/M$ is Matlis injective by (2). Applying the functor $\mathrm{Hom}_R(E(R), -)$ to the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$, we have the exact sequence $0 = \mathrm{Ext}^1_R(E(R), E(M)/M) \to \mathrm{Ext}^2_R(E(R), M) \to \mathrm{Ext}^2_R(E(R), E(M)) = 0$. So $\mathrm{Ext}^2_R(E(R), M) = 0$.

(3) $\Longrightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules such that $A, B \in \mathcal{MI}$. Applying the functor $\mathrm{Hom}_R(E(R), -)$ to the above sequence, we have the exact sequence $0 = \mathrm{Ext}^1_R(E(R), B) \to \mathrm{Ext}^1_R(E(R), C) \to \mathrm{Ext}^2_R(E(R), A) = 0$ by (3). So $\mathrm{Ext}^1_R(E(R), C) = 0$, i.e., C is Matlis injective. Hence (1) holds. □

Proposition 3.13. *Let R be a commutative Artinian ring. Then $\mathcal{MI}$ is closed under direct sums, pure submodules and direct limits. Moreover, $\mathcal{MI}$ is a definable class, i.e., it is closed under pure submodules, direct products and direct limits.*

Proof. By hypothesis, $E(R)$ is finitely presented by [12, Theorem 3.64, p. 90]. Then $\mathcal{MI}$ is closed under direct sums by the isomorphism

$$\bigoplus \mathrm{Ext}^1_R(F, M_\alpha) \cong \mathrm{Ext}^1_R(F, \bigoplus M_\alpha)$$

for any finitely presented R-module F and any family $\{M_\alpha\}$ of R-modules. Suppose that A is a pure submodule of a Matlis injective R-module B. Then we have the exact sequences $0 \longrightarrow \mathrm{Hom}_R(E(R), A) \longrightarrow \mathrm{Hom}_R(E(R), B) \longrightarrow \mathrm{Hom}_R(E(R), B/A) \longrightarrow 0$ and $\mathrm{Hom}_R(E(R), B) \longrightarrow \mathrm{Hom}_R(E(R), B/A) \longrightarrow \mathrm{Ext}^1_R(E(R), A) \longrightarrow \mathrm{Ext}^1_R(E(R), B) = 0$. Hence $\mathrm{Ext}^1_R(E(R), A) = 0$, i.e., A is Matlis injective. So $\mathcal{MI}$ is closed under pure submodules. That $\mathcal{MI}$ is closed under direct limits follows from the isomorphism $\mathrm{Ext}^1_R(F, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}^1_R(F, M_i)$ for any finitely presented R-module F and any family $\{M_i\}$ of R-modules since R is a commutative Artinian ring. So $\mathcal{MI}$ is definable by Proposition 3.2. □

Proposition 3.14. *Let R be a commutative Artinian ring and $S \subset R$ be a multiplicative set. If M is a Matlis injective R-module, then $S^{-1}M$ is a Matlis injective $S^{-1}R$-module.*

Proof. By assumption, $E(R)$ is finitely generated by [12, Theorem 3.64, p. 90] and R is a Noetherian ring. So,

$$\mathrm{Ext}^1{}_{S^{-1}R}(S^{-1}E_R(R), S^{-1}M) \cong S^{-1}\mathrm{Ext}^1{}_R(E_R(R), M)$$

by [8, Theorem 3.2.5, p. 76]. But $S^{-1}E_R(R) \cong E_{S^{-1}R}(S^{-1}R)$ by [8, Theorem 3.3.3, p. 84]. Thus $S^{-1}M$ is a Matlis injective $S^{-1}R$-module when M is a Matlis injective R-module. □

Proposition 3.15. *Let R be a commutative Noetherian ring and $S \subset R$ be a multiplicative set. If M is a Matlis projective R-module, then $S^{-1}M$ is a Matlis projective $S^{-1}R$-module.*

Proof. Note that $S^{-1}E_R(R) \cong E_{S^{-1}R}(S^{-1}R)$ by [8, Theorem 3.3.3, p. 84] and by hypothesis. Then, every Matlis projective $S^{-1}R$-module is a direct summand of some $\{S^{-1}E_R(R), S^{-1}R\}$-filtered $S^{-1}R$-module by [10, Corollary 3.2.4, p. 119]. Since M is a Matlis projective R-module, M is a direct summand of some $\{E(R), R\}$-filtered R-module by [10, Corollary 3.2.4, p. 119]. Let N be an $\{E(R), R\}$-filtered R-module and the chain $(N_\alpha |\ \alpha \le \kappa)$ be a $\{E(R), R\}$-filtration of N. Then $S^{-1}N$ is an $\{S^{-1}E_R(R), S^{-1}R\}$-filtered $S^{-1}R$-module and the chain $(S^{-1}N_\alpha |\ \alpha \le \kappa)$ is a $\{S^{-1}E_R(R), S^{-1}R\}$-filtration of $S^{-1}N$ by [8, Theorem 1.5.7, p. 33, and Proposition 2.2.4, p. 44] and by definition. So $S^{-1}M$ is a Matlis projective $S^{-1}R$-module and the assertion holds. □

4. The existence of Matlis injective (pre)envelopes

According to Theorem 2.1, we immediately have the following proposition.

Proposition 4.1. *Let R be a ring. Then every R-module has a special Matlis injective preenvelope.*

The following lemmas are needed to prove the main result of this paper.

Lemma 4.2. *Let R be a ring and M an R-module. If M is Σ-pure injective and Σ-self orthogonal, then Add(M) is closed under extensions and direct limits.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules such that both A and C are in $\mathrm{Add}(M)$. Without loss of generality, we may assume that both A and C are direct summands of $M^{(I)}$ for an index set I. Since M is Σ-self orthogonal, we have $\mathrm{Ext}_R^1(C, A) = 0$. Then the exact sequence $0 \to A \to B \to C \to 0$ splits, and so $B \cong A \bigoplus C$. Obviously, $A \bigoplus C \in \mathrm{Add}(M)$. Therefore, $B \in \mathrm{Add}(M)$. So $\mathrm{Add}(M)$ is closed under extensions. We claim that any R-module N from $\mathrm{Add}(M)$ is Σ-pure injective. It is clear that N is pure injective since M is Σ-pure injective. In addition, $\mathrm{Add}(M)$ is closed under direct sums. Thus N is Σ-pure injective. Let $((M_i)_{i\in I}, (f_{ji}))$ be a direct system of R-modules from $\mathrm{Add}(M)$ where I is a directed set. Then there exists a short exact sequence $0 \to K \hookrightarrow \bigoplus_{i\in I} M_i \to \varinjlim M_i \to 0$ with K a pure submodule of $\bigoplus_{i\in I} M_i$. But $\bigoplus_{i\in I} M_i$ is Σ-pure injective, then the exact sequence $0 \to K \hookrightarrow \bigoplus_{i\in I} M_i \to \varinjlim M_i \to 0$ splits by Proposition 2.3. So $\varinjlim M_i$ is isomorphic to a direct summand of $\bigoplus_{i\in I} M_i$, i.e., $\varinjlim M_i \in \mathrm{Add}(M)$. Hence $\mathrm{Add}(M)$ is closed under direct limits. □

Lemma 4.3. *Let R be a ring and M an R-module. Assume that M is Σ-pure injective and Σ-self orthogonal. Then every R-module N has an $M^{\perp_1}$-envelope.*

Proof. Obviously, $M^{\perp_1} = (\mathrm{Add}(M))^{\perp_1}$. Thus it is equivalent to show that every R-module N has an $(\mathrm{Add}(M))^{\perp_1}$-envelope. By Theorem 2.1, N has a special $(\mathrm{Add}(M))^{\perp_1}$-preenvelope f with Coker f is $\{M\}$-filtered. Note that every $\{M\}$-filtered R-module is in $\mathrm{Add}(M)$ by Lemma 4.2 and transfinite induction. So N has an $(\mathrm{Add}(M))^{\perp_1}$-envelope by Lemma 4.2 and Theorem 2.2. □

We are now in a position to prove the following

Theorem 4.4. *Let R be a right Noetherian ring and $\mathcal{F}$ a class of injective R-modules. Then every R-module M has an $\mathcal{F}^{\perp_1}$-envelope; in particular, every R-module M has a Matlis injective envelope.*

Proof. If R is right Noetherian, then every injective R-module is the direct sum of indecomposable injective R-modules. Each such module is the injective envelope of a cyclic R-module. Hence, we can find a representative set of such modules. So there is a family $\{E_i\}_{i\in I}$ of indecomposable injective R-modules such that every injective R-module is the direct sum of copies of E_i.

Let $\mathcal{S} = \{E_i \mid E_i \text{ is isomorphic to a direct summand of an element of } \mathcal{F}\}$. It is easy to see that $(\bigoplus_{E_i\in\mathcal{S}} E_i)^{\perp_1} = \mathcal{F}^{\perp_1}$. Note that $\bigoplus_{E_i\in\mathcal{S}} E_i$ is Σ-pure injective and Σ-self orthogonal by the fact that the class of right injective R-modules is closed under direct sums when R is right Noetherian. So the assertion holds by Lemma 4.3. □

We end this paper with the following remark.

Remark 4.5. If R is a commutative Artinian ring, then every R-module has a Matlis injective cover by Proposition 3.13 and [2, Corollary 2.6 and Proposition 4.3(3)].

Acknowledgements. The author would like to thank the referees for the very helpful comments and suggestions.

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