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THE FIRST POSITIVE EIGENVALUE OF THE DIRAC OPERATOR ON 3-DIMENSIONAL SASAKIAN MANIFOLDS

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Abstract. Let (M^3, g) be a 3-dimensional closed Sasakian spin manifold. Let $S_{\min}$ denote the minimum of the scalar curvature of (M^3, g). Let $\lambda_1^+ > 0$ be the first positive eigenvalue of the Dirac operator of (M^3, g). We proved in [13] that if λ_1^+ belongs to the interval $\lambda_1^+ \in \left(\frac{1}{2}, \frac{5}{2}\right)$, then λ_1^+ satisfies $\lambda_1^+ \geq \frac{S_{\min}+6}{8}$. In this paper, we remove the restriction "if λ_1^+ belongs to the interval $\lambda_1^+ \in (\frac{1}{2}, \frac{5}{2})$" and prove

$$\lambda_1^+ \;\geq\; \begin{cases} \frac{S_{\min}+6}{8} & \text{for} \quad -\frac{3}{2} < S_{\min} \leq 30, \\ \frac{1+\sqrt{2\,S_{\min}+4}}{2} & \text{for} \quad S_{\min} \geq 30. \end{cases}$$

1. Introduction

Let (M^n, g) be a closed Riemannian spin manifold. The Levi-Civita connection ∇ and the Dirac operator D, acting on sections $\psi \in \Gamma(\Sigma(M))$ of the spinor bundle $\Sigma(M)$ over M^n, are respectively expressed as

$$\nabla_X \psi \;=\; X(\psi) + \frac{1}{4}\sum_{u=1}^{n} E_u \cdot \nabla_X E_u \cdot \psi$$

and

$$D\psi \;=\; \sum_{u=1}^{n} E_u \cdot \nabla_{E_u} \psi,$$

where $X(\psi)$ is the directional derivative of ψ along a vector field $X \in \Gamma(T(M))$, $(E_1, \ldots, E_n)$ is a local orthonormal frame on (M^n, g) and the dot "$\cdot$" indicates the Clifford multiplication [6]. Since (M^n, g) is a closed manifold, the spectrum $\mathrm{Spec}(D)$ of the Dirac operator D is discrete and real and will be written as

$$\cdots \leq \lambda_2^- \leq \lambda_1^- \leq 0 \leq \lambda_1^+ \leq \lambda_2^+ \leq \cdots,$$

where each eigenvalue except zero is repeated as many times as its multiplicity. The nonzero eigenvalue $\lambda_1^- \neq 0$ and $\lambda_1^+ \neq 0$ are called the *first negative eigenvalue* and the *first positive eigenvalue*, respectively. The eigenvalue

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$\lambda_1 \in \mathrm{Spec}(D)$ with $|\lambda_1| = \min\{|\lambda_1^-|, |\lambda_1^+|\}$ is called the *first eigenvalue*. A classical result about the first Dirac eigenvalue is the Friedrich inequality

$$|\lambda_1| \geq \sqrt{\frac{n\,S_{\min}}{4(n-1)}}, \tag{1.1}$$

where $S_{\min}$ denotes the minimum of the scalar curvature [5, 9]. (1.1) holds for all closed Riemannian spin manifolds (M^n, g) with positive scalar curvature $S > 0$ and the limiting case of this inequality is characterized by the existence of a *Killing spinor* ψ,

$$\nabla_X \psi = a\,X \cdot \psi, \qquad a \in \mathbb{R}.$$

If (M^n, g) is of odd dimension n, $n \equiv 3 \bmod 4$, then $\mathrm{Spec}(D)$ is generally asymmetric with respect to zero [2, 11]. In that case, a problem of interest is to find an optimal estimate for λ_1^- and that for λ_1^+, respectively [7, 13].

A Sasakian manifold is an odd-dimensional Riemannian manifold (M^{2m+1}, g), $m \geq 1$, equipped with a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η that satisfy

$$\begin{aligned}
&\eta(\xi) = 1, \qquad \phi^2(X) = -X + \eta(X)\xi,\\
&g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),\\
&(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(\Sigma(M))$. Over Sasakian spin manifolds, a special class of spinors deserves attention.

Definition 1.1. A spinor field ψ on Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called an *eta-Killing spinor* with Killing pair (a, b) if it satisfies

$$\nabla_X \psi = a\,X \cdot \psi + b\eta(X)\xi \cdot \psi$$

for some real numbers $a, b \in \mathbb{R}$ and for all vector fields X.

For the relations between the Killing pair (a, b) of an eta-Killing spinor and the geometry of the Sasakian manifold, we refer to [8, 12]. It turns out in Section 3 that the existence of an eta-Killing spinor characterizes the limiting case of inequalities (3.4)-(3.6).

As discussed in the introduction of [13], an observation of the Dirac spectrum of a round sphere S^{2m+1} with Berger metric gives rise to the following two questions:

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional closed Sasakian spin manifold.

(1) Does the first negative Dirac eigenvalue λ_1^- on $(M^3, \phi, \xi, \eta, g)$ satisfies

$$\lambda_1^- \leq \frac{1 - \sqrt{2\,S_{\min} + 4}}{2} \qquad \text{for} \quad S_{\min} > -\frac{3}{2}\ ? \tag{1.2}$$

(2) Does the first positive Dirac eigenvalue λ_1^+ on $(M^3, \phi, \xi, \eta, g)$ satisfies

$$\lambda_1^+ \geq \begin{cases} \frac{S_{\min}+6}{8} & \text{for} \quad -2 < S_{\min} \leq 30, \\ \frac{1+\sqrt{2\,S_{\min}+4}}{2} & \text{for} \quad S_{\min} \geq 30\ ? \end{cases} \tag{1.3}$$

Note that both (1.2) and (1.3) improve Friedrich's inequality (1.1). We proved in [13] that the answer to the first question is positive. But as to the second question, we gave only a partial answer (see Theorem 3.2 in [13]):

If the first positive Dirac eigenvalue λ_1^+ *belongs to the interval* $\lambda_1^+ \in (\frac{1}{2}, \frac{5}{2})$, *then* λ_1^+ *satisfies* $\lambda_1^+ \geq \frac{S_{\min}+6}{8}$.

The aim of this paper is to give a complete answer to the second question, removing the uncomfortable restriction "if λ_1^+ belongs to the interval $\lambda_1^+ \in (\frac{1}{2}, \frac{5}{2})$". Namely, we will prove Theorem 3.1. When comparing (3.10) with (1.3), there is a slight change in the scalar curvature restriction from $-2 < S_{\min} \leq 30$ to $-\frac{3}{2} < S_{\min} \leq 30$. It is pointed out in Remark 3.3 that the latter restriction is more reasonable.

2. A natural deformation of the Levi-Civita connection

To prove Proposition 3.1 in the next section, we will apply a deformation technique for spin connections. In the former part of this section we briefly review some general properties of regular Sasakian manifolds [3, 4, 14, 15], which help clarify the geometric implication of our deformation of the Levi-Civita connection (see (2.9)).

A Sasakian manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called *regular* if every point of M^{2m+1} has a neighbourhood through which any integral curve of the unit vector field ξ passes at most once. In that case, all the orbits of ξ have the same period and M^{2m+1} turns out to be the total space of a principal S^1-bundle $\pi : M^{2m+1} \longrightarrow N^{2m}$. Regarding the contact form η as a $U(1)$-connection form $\sqrt{-1}\eta$ with values in $\sqrt{-1}\,\mathbb{R}$, we can realize a closed 2-form representing the first Chern class of the principal S^1-bundle $\pi : M^{2m+1} \longrightarrow N^{2m}$ as the curvature form

$$c_1\left(M^{2m+1} \longrightarrow N^{2m}\right) = -\frac{1}{2\pi}\,[\,\pi_* d\eta\,] \;\in H^2(N^{2m}; \mathbb{Z}).$$

The Sasakian structure (ϕ, ξ, η, g) on the total space M^{2m+1} then induces a Kähler structure (J, g_N) on the base manifold N^{2m} via the relations

$$\pi_* \circ \phi = J \circ \pi_*, \tag{2.1}$$

$$g = \pi^* g_N + \eta \otimes \eta. \tag{2.2}$$

As a consequence of (2.1)-(2.2), the fundamental form $\Phi = \frac{1}{2}d\eta$ on the total space $(M^{2m+1}, \phi, \xi, \eta, g)$ coincides with the pull-back $\Phi = \pi^*\Omega$ of the fundamental form Ω associated to (N^{2m}, J, g_N). Let $(F_1, \ldots, F_{2m})$ be a local orthonormal fame on (N^{2m}, J, g_N) and consider its horizontal lift $(F_1^H, \ldots, F_{2m}^H, \xi)$. Proceeding as in Example 6.1 of [8], we find that the Levi-Civita connection ∇ of $(M^{2m+1}, \phi, \xi, \eta, g)$ is related to that ∇^N of (N^{2m}, J, g_N) by

$$\nabla_{F_u^H} F_v^H = \left(\nabla^N_{F_u} F_v\right)^H - \Omega_{uv}\,\xi, \tag{2.3}$$

where $\Omega_{uv} := \Omega(F_u, F_v) = g_N(F_u, J(F_v))$. Moreover, it holds that

$$\nabla_{F_v^H}\,\xi \;=\; \nabla_\xi F_v^H \;=\; -\,\phi(F_v^H). \tag{2.4}$$

The Ricci tensor Ric and scalar curvature S of $(M^{2m+1}, \phi, \xi, \eta, g)$ are related to those Ric_N, S_N of (N^{2m}, J, g_N) by

$$\mathrm{Ric}(W_1^H, W_2^H) = \mathrm{Ric}_N(W_1, W_2) - 2g_N(W_1, W_2), \quad W_1, W_2 \in \Gamma(T(N)), \tag{2.5}$$

and

$$S \;=\; S_N - 2m, \tag{2.6}$$

respectively. Let us assume that (N^{2m}, J, g_N) is a spin manifold and $(M^{2m+1}, \phi, \xi, \eta, g)$ is equipped with a spin structure obtained by pull-back from (N^{2m}, J, g_N). Let φ be a spinor field on N^{2m} and φ^H be its horizontal lift. Then the spinor derivative $\nabla\varphi^H$ on $(M^{2m+1}, \phi, \xi, \eta, g)$ is related to that $\nabla^N\varphi$ on (N^{2m}, J, g_N) by

$$\nabla_{W^H}\varphi^H = \big(\nabla^N_W\varphi\big)^H + \frac{1}{2}\phi(W^H)\cdot\xi\cdot\varphi^H, \quad W \in \Gamma(T(N)), \tag{2.7}$$

$$\nabla_\xi\varphi^H = \frac{1}{2}\Phi\cdot\varphi^H. \tag{2.8}$$

A spinor field ψ on M^{2m+1} is *projectable* onto N^{2m}, i.e., there exists some spinor field φ on N^{2m} with $\psi = \varphi^H$ if the directional derivative $\xi(\psi)$ vanishes identically. From (2.4), it follows that $\psi \in \Gamma(\Sigma(M))$ is projectable if and only if $\nabla_\xi\psi = \frac{1}{2}\Phi\cdot\psi$.

Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a (possibly irregular) Sasakian spin manifold. Let $\xi^\perp$ denote the orthogonal complement of the vector field ξ in the tangent bundle $T(M)$. We deform the Levi-Civita connection ∇ in the subbundle $\xi^\perp \subset T(M)$,

$$\overline{\nabla}_V\psi \;=\; \nabla_V\psi - \frac{1}{2}\phi(V)\cdot\xi\cdot\psi, \qquad V \in \Gamma(\xi^\perp), \quad \psi \in \Gamma(\Sigma(M)). \tag{2.9}$$

The deformed connection $\overline{\nabla}_V$ has a remarkable property in that it commutes with the fundamental form Φ

$$\overline{\nabla}_V \circ \Phi \;=\; \Phi \circ \overline{\nabla}_V.$$

For more interesting information about the connection $\overline{\nabla}_V$, we refer to [1]. Define first-order operators $\overline{C}$, $\overline{Q}$ acting on spinor fields $\psi \in \Gamma(\Sigma(M))$ by

$$\overline{C}\psi = \sum_{u=1}^{2m} E_u\cdot\overline{\nabla}_{E_u}\psi,$$

$$\overline{Q}\psi = \sum_{u=1}^{2m} \phi(E_u)\cdot\overline{\nabla}_{E_u}\psi,$$

where $(E_1, \ldots, E_{2m}, \xi)$ is a local orthonormal frame on $(M^{2m+1}, \phi, \xi, \eta, g)$. Both $\overline{C}$ and $\overline{Q}$ are self-adjoint with respect to the L^2-Hermitian product. But neither $\overline{C}$ nor $\overline{Q}$ is elliptic.

Suppose that $(M^{2m+1}, \phi, \xi, \eta, g)$ is regular, i.e., it is the total space of a circle bundle $M^{2m+1} \longrightarrow N^{2m}$ and the spin structure of M^{2m+1} is obtained by pull-back from N^{2m}. Let D_N be the Dirac operator of (N^{2m}, J, g_N) and let $\widetilde{D}_N$ be the J-twist of D_N defined by

$$\widetilde{D}_N \varphi \;=\; \sum_{u=1}^{2m} J(F_u) \cdot \nabla^N_{F_u} \varphi.$$

Then, from (2.7) we see that $\overline{\nabla}_{W^H}$, $\overline{C}$, $\overline{Q}$ coincide with the pull-back of ∇^N_W, $D_N, \widetilde{D}_N$, respectively. Inspired by this correspondence, we define a Sasakian analogue of the Kählerian twistor spinors [10].

Definition 2.1. A spinor field $\psi \in \Gamma(\Sigma(M))$ on Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called a *Sasakian twistor spinor* of type (a, b) if

$$\overline{\nabla}_V \psi \;=\; aV \cdot \overline{C}\psi + b\phi(V) \cdot \overline{Q}\psi$$

holds for some numbers $a, b \in \mathbb{R}$ and for all vector fields $V \in \Gamma(\xi^{\perp})$ orthogonal to ξ.

Example 2.1. Using Lemma 3.2 and Proposition 3.1 in [12], one verifies that, on a Sasakian spin manifold of dimension $2m+1 \geq 5$, any eta-Killing spinor with Killing pair (a, b), $a \neq 0, b \neq 0$, is a Sasakian twistor spinor of type $(0, 0)$.

We shall see in the next section that, on a 3-dimensional closed Sasakian spin manifold, the existence of a Sasakian twistor spinor characterizes the limiting case of inequalities (3.4)-(3.6).

We close the section with three lemmata that we will need in the next section. To state Lemma 2.3 we use the notation $(\, , \,) = \mathrm{Re}\langle \, , \, \rangle$ denoting the real part of the standard Hermitian product $\langle \, , \, \rangle$ on the spinor bundle $\Sigma(M)$ over M^{2m+1}.

Lemma 2.1. *On a Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$, the operator identity*

$$\overline{C}^{\,2} \;=\; D^2 + \nabla_\xi \nabla_\xi - \xi \circ \overline{Q} - 2\Phi \circ \nabla_\xi + \Phi \circ \Phi \tag{2.10}$$

holds.

Lemma 2.2. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold. Let $(E_1, \ldots, E_{2m}, \xi)$ be a local orthonormal frame on M^{2m+1} and let $\overline{\nabla}^*_{E_u}$ denote the adjoint of $\overline{\nabla}_{E_u}$ with respect to the L^2-Hermitain product. Then we have*

$$\sum_{u=1}^{2m} \overline{\nabla}^*_{E_u} \overline{\nabla}_{E_u} \;=\; \overline{C}^{\,2} - \frac{1}{4}S + 2\Phi \circ \nabla_\xi - \Phi \circ \Phi - \frac{m}{2}. \tag{2.11}$$

Lemma 2.3 ([13])**.** *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a closed Sasakian spin manifold and let μ denote the volume form. Then, for any eigenspinor ψ of the Dirac operator D with eigenvalue λ, we have*

$$0 \;=\; \int_{M^{2m+1}} \Big[2(\nabla_\xi \psi,\, \nabla_\xi \psi) - 3(\nabla_\xi \psi,\, \Phi \cdot \psi) + 2\lambda(\nabla_\xi \psi,\, \xi \cdot \psi)$$

$$-\lambda(\Phi\cdot\psi,\,\xi\cdot\psi)+(\Phi\cdot\psi,\,\Phi\cdot\psi)\Big]\mu. \tag{2.12}$$

3. Estimates of small Dirac eigenvalues on 3-dimensional Sasakian manifolds revisited

Let us realize the three-dimensional Clifford algebra using the matrices

$$E_1=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\qquad E_2=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\qquad E_3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Then the relations

$$E_1\cdot E_2=-E_3,\qquad E_2\cdot E_3=-E_1,\qquad E_3\cdot E_1=-E_2$$

are valid. It follows that, on a 3-dimensional closed Sasakian spin manifold (M^3,ϕ,ξ,η,g), the operator identities

$$\Phi \;=\; \xi,\qquad \xi\cdot\overline{Q}\;=\;D-\xi\cdot\nabla_\xi-1$$

hold and hence (2.10)-(2.12) simplify to

$$\overline{C}^{\,2}\;=\;D^2+\nabla_\xi\nabla_\xi-D-\xi\cdot\nabla_\xi \tag{3.1}$$

and

$$\sum_{u=1}^{2}\overline{\nabla}^{*}_{E_u}\overline{\nabla}_{E_u}\;=\;D^2-\frac{1}{4}S+\nabla_\xi\nabla_\xi-D+\xi\cdot\nabla_\xi+\frac{1}{2} \tag{3.2}$$

and

$$0\;=\;\int_{M^3}\Big[2(\nabla_\xi\psi,\,\nabla_\xi\psi)+(2\lambda-3)(\nabla_\xi\psi,\,\xi\cdot\psi)+(1-\lambda)(\psi,\,\psi)\Big]\mu, \tag{3.3}$$

respectively.

Proposition 3.1. *Let (M^3,ϕ,ξ,η,g) be a 3-dimensional closed Sasakian spin manifold and suppose that the scalar curvature satisfies $S_{\min}>-2$. Let λ be an eigenvalue of the Dirac operator D.*

(i) *If $\lambda<\frac{1}{2}$, then the inequality*

$$\lambda\;\le\;\frac{1-\sqrt{2\,S_{\min}+4}}{2} \tag{3.4}$$

holds. The limiting case of (3.4) *occurs if and only if the scalar curvature is constant and there exists an eta-Killing spinor with Killing pair $\left(\frac{-2+\sqrt{2S+4}}{4},\frac{4-\sqrt{2S+4}}{4}\right)$.*

(ii) *If $\frac{1}{2}<\lambda\le\frac{9}{2}$, then the inequality*

$$\lambda\;\ge\;\frac{S_{\min}+6}{8} \tag{3.5}$$

holds. The limiting case of (3.5) *occurs if and only if the scalar curvature is constant and there exists an eta-Killing spinor with Killing pair $\left(-\frac{1}{2},-\frac{S}{8}+\frac{3}{4}\right)$.*

(iii) *If $\lambda\ge\frac{9}{2}$, then the inequality*

$$\lambda\;\ge\;\frac{1+\sqrt{2\,S_{\min}+4}}{2} \tag{3.6}$$

holds. The limiting case of (3.6) *occurs if and only if the scalar curvature is constant and there exists an eta-Killing spinor with Killing pair* $\left(\frac{-2-\sqrt{2S+4}}{4}, \frac{4+\sqrt{2S+4}}{4}\right)$.

Proof. Let $(E_1, E_2 = \phi(E_1), \xi)$ be an adapted local orthonormal frame on $(M^3, \phi, \xi, \eta, g)$. Let ψ be an eigenspinor of D with eigenvalue λ. Introducing free parameters $\kappa, \tau \in \mathbb{R}$ to control the unnecessary terms, we compute

$$\begin{aligned} H &:= \sum_{u=1}^{2} \int_{M^3} \left(\overline{\nabla}_{E_u}\psi + \frac{1}{2}E_u \cdot \overline{C}\psi,\ \overline{\nabla}_{E_u}\psi + \frac{1}{2}E_u \cdot \overline{C}\psi\right)\mu \\ &\quad + \kappa^2 \int_{M^3} (\nabla_\xi \psi - \tau\xi\cdot\psi,\ \nabla_\xi\psi - \tau\xi\cdot\psi)\,\mu \\ &= \sum_{u=1}^{2} \int_{M^3} (\overline{\nabla}_{E_u}\psi, \overline{\nabla}_{E_u}\psi)\mu - \frac{1}{2}\int_{M^3} (\overline{C}^2\psi,\ \psi)\mu \\ &\quad + \int_{M^3} \left[\kappa^2(\nabla_\xi\psi,\ \nabla_\xi\psi) - 2\kappa^2\tau(\nabla_\xi\psi,\ \xi\cdot\psi) + \kappa^2\tau^2(\psi,\ \psi)\right]\mu. \end{aligned}$$

We apply (3.1)-(3.2) to obtain

$$\begin{aligned} H &= \int_{M^3} \left(\frac{\lambda^2}{2} - \frac{\lambda}{2} + \frac{1}{2} - \frac{S}{4} + \kappa^2\tau^2\right)(\psi,\ \psi)\mu \\ &\quad + \int_{M^3} \left[\left(\kappa^2 - \frac{1}{2}\right)(\nabla_\xi\psi,\ \nabla_\xi\psi) - \left(2\kappa^2\tau + \frac{3}{2}\right)(\nabla_\xi\psi,\ \xi\cdot\psi)\right]\mu. \end{aligned}$$

Due to (3.3) we have

$$\begin{aligned} H = &\int_{M^3} \left[\frac{\lambda^2}{2} - \frac{\lambda}{2} + \frac{1}{2} - \frac{S}{4} + \kappa^2\tau^2 + \left(\kappa^2 - \frac{1}{2}\right)\left(\frac{\lambda}{2} - \frac{1}{2}\right)\right](\psi,\ \psi)\mu \\ &- \int_{M^3} \left[\left(\kappa^2 - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right) + 2\kappa^2\tau + \frac{3}{2}\right](\nabla_\xi\psi,\ \xi\cdot\psi)\mu. \end{aligned} \tag{3.7}$$

Let us now choose

$$\kappa^2 = \frac{2\lambda - 9}{2(2\lambda - 3 + 4\tau)} \ \geq\ 0.$$

Then the latter integral of (3.7) vanishes and we obtain

$$H \ =\ \frac{1}{4}\int_{M^3} \left[2\lambda^2 - 2\lambda + 2 - S + 4\kappa^2\tau^2 + (2\kappa^2 - 1)(\lambda - 1)\right](\psi,\ \psi)\mu \ \geq\ 0. \tag{3.8}$$

The proof idea for Theorem 3.1 and that for Theorem 3.2 in [13] suggest us to consider the case $\tau = \frac{1}{2}$ and $\tau = 1 - \lambda$, respectively. In case of $\tau = \frac{1}{2}$, we have

$$\kappa^2 = \frac{2\lambda - 9}{2(2\lambda - 1)}$$

and obtain

$$H = \frac{1}{4}\int_{M^3}\left[2\lambda^2 - 2\lambda - S - \frac{3}{2}\right](\psi,\,\psi)\mu,$$

which proves part (i) as well as part (iii) of the proposition. The limiting case of part (i) occurs if and only if there exists a solution to the system of equations

$$\overline{\nabla}_V\psi + \frac{1}{2}V\cdot\overline{C}\psi = 0, \qquad \nabla_\xi\psi = \frac{1}{2}\xi\cdot\psi, \qquad V\in\xi^\perp \tag{3.9}$$

$$\Longleftrightarrow\quad \nabla_V\psi = -\left(\frac{\lambda}{2}+\frac{1}{4}\right)V\cdot\psi, \qquad \nabla_\xi\psi = \frac{1}{2}\xi\cdot\psi, \qquad V\in\xi^\perp,$$

which is equivalent to the condition that the scalar curvature is constant and there exists an eta-Killing spinor with Killing pair $\left(\frac{-2+\sqrt{2S+4}}{4},\ \frac{4-\sqrt{2S+4}}{4}\right)$. In the same way, one verifies the condition for the limiting case of part (iii). Let us now consider the other case $\tau = 1-\lambda$. In that case, we have

$$\kappa^2 = -\frac{2\lambda-9}{2(2\lambda-1)}$$

and obtain

$$H = \frac{1}{4}\int_{M^3}(8\lambda - S - 6)\,(\psi,\,\psi)\mu,$$

which proves part (ii) of the proposition. The condition for the limiting case of part (ii) is easy to check. □

Remark 3.1. Any eta-Killing spinor with Killing pair $\left(-\frac{1}{2}, -\frac{S}{8}+\frac{3}{4}\right)$ is a Sasakian twistor spinor of type $(0,0)$. Any eta-Killing spinor with Killing pair $\left(\frac{-2\pm\sqrt{2S+4}}{4},\ \frac{4\mp\sqrt{2S+4}}{4}\right)$ is a Sasakian twistor spinor of type $(-\frac{1}{2},0)$.

Remark 3.2. Suppose that (M^3,ϕ,ξ,η,g) is a circle bundle $\pi : M^3 \longrightarrow N^2$ and admits an eta-Killing spinor ψ_1 with Killing pair $\left(\frac{-2+\sqrt{2S+4}}{4},\ \frac{4-\sqrt{2S+4}}{4}\right)$ as well as an eta-Killing spinor ψ_2 with Killing pair $\left(\frac{-2-\sqrt{2S+4}}{4},\ \frac{4+\sqrt{2S+4}}{4}\right)$. Then, due to (3.9), both ψ_1 and ψ_2 are projectable onto N^2 and there exist Killing spinors φ_1, φ_2 on N^2 with $\psi_k = \pi^*\varphi_k$, $k=1,2$. The base 2-manifold N^2 is in fact isometric to a sphere with constant scalar curvature $S+2$ (see (2.6)) and φ_1, φ_2 satisfy

$$\begin{aligned}\nabla^N_W\varphi_1 &= \frac{1}{4}\sqrt{2(S+2)}\,W\cdot\varphi_1\\ \nabla^N_W\varphi_2 &= -\frac{1}{4}\sqrt{2(S+2)}\,W\cdot\varphi_2, \qquad W\in\Gamma(T(N)).\end{aligned}$$

Proposition 3.2. *Let (M^3,ϕ,ξ,η,g) be a 3-dimensional closed Sasakian spin manifold. Then there exists an eigenspinor ψ of the Dirac operator D with eigenvalue $\lambda = \frac{1}{2}$ only if the minimum of the scalar curvature satisfies $S_{\min}\leq -2$.*

Proof. We see from (3.1) that

$$\int_{M^3} \left[(\overline{C}\varphi, \overline{C}\varphi) + \left(\nabla_\xi \varphi - \frac{1}{2}\xi \cdot \varphi,\, \nabla_\xi \varphi - \frac{1}{2}\xi \cdot \varphi \right) \right] \mu$$
$$= \int_{M^3} \left(D\varphi - \frac{1}{2}\varphi,\, D\varphi - \frac{1}{2}\varphi \right) \mu$$

holds for any spinor field φ on M^3. Thus, if there exists an eigenspinor ψ of D with eigenvalue $\lambda = \frac{1}{2}$, then we have

$$\overline{C}\psi = 0, \qquad \nabla_\xi \psi = \frac{1}{2}\xi \cdot \psi$$

and so

$$\sum_{i=1}^{2} \int_{M^3} (\overline{\nabla}_{E_i}\psi,\, \overline{\nabla}_{E_i}\psi)\, \mu$$
$$= \int_{M^3} \left[(\overline{C}\psi, \overline{C}\psi) - \frac{S}{4}(\psi,\psi) - 2(\nabla_\xi \psi, \xi \cdot \psi) + \frac{1}{2}(\psi,\psi) \right] \mu$$
$$= \int_{M^3} \left(-\frac{1}{4}S - \frac{1}{2} \right) (\psi,\psi)\mu,$$

which yields $S_{\min} \leq -2$. □

Theorem 3.1. *Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional closed Sasakian spin manifold. Then the first positive eigenvalue λ_1^+ of the Dirac operator D satisfies*

$$\lambda_1^+ \;\geq\; \begin{cases} \frac{S_{\min}+6}{8} & \textit{for } -\frac{3}{2} < S_{\min} \leq 30, \\ \frac{1+\sqrt{2\,S_{\min}+4}}{2} & \textit{for } S_{\min} \geq 30. \end{cases} \tag{3.10}$$

The limiting case of (3.10) *occurs if and only if the scalar curvature is constant and there exists an eta-Killing spinor with Killing pair*

$$\begin{cases} \left(-\frac{1}{2},\, -\frac{S}{8} + \frac{3}{4}\right) & \textit{for } -\frac{3}{2} < S_{\min} \leq 30, \\ \left(\frac{-2-\sqrt{2S+4}}{4},\, \frac{4+\sqrt{2S+4}}{4}\right) & \textit{for } S_{\min} \geq 30. \end{cases}$$

Proof. Because of the restriction $S_{\min} > -\frac{3}{2}$, it follows from Proposition 3.2 and part (i) of Proposition 3.1 that $\lambda_1^+ \leq \frac{1}{2}$ is not allowed. Consequently, part (ii) and (iii) of Proposition 3.1 together give

$$\lambda_1^+ \;\geq\; \min \left\{ \frac{S_{\min}+6}{8},\; \frac{1+\sqrt{2\,S_{\min}+4}}{2} \right\},$$

which we can equivalently rewrite as (3.10). The condition for the limiting case of (3.10) is easy to check. □

Remark 3.3. Let $(M^3, \phi, \xi, \eta, g)$ be a simply-connected Sasakian spin manifold of dimension 3 and suppose that the scalar curvature S is constant. We proved in [8] that if $S \geq -2$, then there exists an eta-Killing spinor ψ with Killing pair $\left(\frac{-2+\sqrt{2S+4}}{4}, \frac{4-\sqrt{2S+4}}{4}\right)$. In particular, if we choose $S = -\frac{3}{2}$, then ψ is a harmonic spinor. This means that our restriction $S_{\min} > -\frac{3}{2}$ in Theorem 3.1 is reasonable.

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