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CHARACTERIZATIONS OF GEOMETRICAL PROPERTIES OF BANACH SPACES USING ψ-DIRECT SUMS

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Abstract. Let X be a Banach space and ψ a continuous convex function on Δ_{K+1} satisfying certain conditions. Let $(X \bigoplus X \bigoplus \cdots \bigoplus X)_\psi$ be the ψ-direct sum of X. In this paper, we characterize the K strict convexity, K uniform convexity and uniform non-l_1^N-ness of Banach spaces using ψ-direct sums.

1. Introduction

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be absolute if

$$\|(x_1, x_2, \dots, x_n)\| = \|(|x_1|, |x_2|, \dots, |x_n|)\| \quad \text{for any } (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$$

and normalized if

$$\|(1, 0, \dots, 0)\| = \|(0, 1, 0, \dots, 0)\| = \cdots = \|(0, \dots, 0, 1)\|.$$

The l_p-norms are such examples:

$$\|(x_1, x_2, \dots x_n)\|_p = \begin{cases} (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max\{|x_1|, |x_2|, \dots, |x_n|\}, & p = \infty. \end{cases}$$

Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$. When $n = 2$ Bonsall and Duncan [2] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{C}^2$ is in one-to-one correspondence with the set Ψ_2 of all continuous convex functions on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1, 0 \le t \le 1$. The correspondence is given by

$$\psi(t) = \|(1 - t, t)\|, \quad 0 \le t \le 1. \tag{1}$$

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Indeed, for any $\psi \in \Psi_2$, define

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi(\frac{|w|}{|z|+|w|}), & (z,w) \neq (0,0) \\ 0, & (z,w) = (0,0). \end{cases}$$

By calculation we have $\|\cdot\|_\psi \in AN_2$ and $\|\cdot\|_\psi$ satisfies (1). From this result, there are plenty of concrete absolute normalized norms of $\mathbb{C}^2$ which are not l_p-type.

In [13] K.-S. Saito, M. Kato and Y. Takahashi generalized the result to $\mathbb{C}^n$. Before stating it, we give some notations. For each $n \in N$ with $n \geq 2$, we put

$$\Delta_n = \left\{ (t_1, t_2, t_3, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geq 0, \sum_{j=1}^{n-1} t_j \leq 1 \right\}$$

and define the set Ψ_n of all continuous convex functions on Δ_n satisfying the following conditions:

(A_0) $$\psi(0,0,\ldots,0) = \psi(1,0,\ldots,0) = \cdots = \psi(0,\ldots,0,1),$$

(A_1)

$$\psi(t_1,t_2,\ldots,t_{n-1}) \geq (t_1+t_2+\cdots+t_{n-1})\psi\left(\frac{t_1}{\sum\limits_{i=1}^{n-1} t_i}, \ldots, \frac{t_{n-1}}{\sum\limits_{i=1}^{n-1} t_i}\right), \text{ if } \sum_{i=1}^{n-1} t_i \neq 0,$$

(A_2) $$\psi(t_1,t_2,\ldots,t_{n-1}) \geq (1-t_1)\psi\left(0, \frac{t_2}{1-t_1}, \ldots, \frac{t_{n-1}}{1-t_1}\right), \text{ if } t_1 \neq 1,$$

(A_3) $$\psi(t_1,t_2,\ldots,t_{n-1}) \geq (1-t_2)\psi\left(\frac{t_1}{1-t_2}, 0, \ldots, \frac{t_{n-1}}{1-t_2}\right), \text{ if } t_2 \neq 1,$$

$$\vdots$$

(A_n)

$$\psi(t_1,t_2,\ldots,t_{n-1}) \geq (1-t_{n-1})\psi\left(\frac{t_1}{1-t_{n-1}}, \ldots, \frac{t_{n-2}}{1-t_{n-1}}, 0\right), \text{ if } t_{n-1} \neq 1.$$

K.-S. Saito, M. Kato and Y. Takahashi in [13] showed that, for each $n \in \mathbb{N}$ with $n \geq 2$, AN_n and Ψ_n are in one-to-one correspondence under the following equation:

(2) $$\psi(t_1,\ldots,t_{n-1}) = \left\|(1-\sum_{j=1}^{n-1} t_j, t_1, \ldots, t_{n-1})\right\|, (t_1,\ldots,t_{n-1}) \in \Delta_n.$$

Indeed, for any $\psi \in \Psi_n$, the norm $\|\cdot\|_\psi$ on $\mathbb{C}^n$ is defined as

$$\|(x_0, x_1, \ldots, x_{n-1})\|_\psi = \begin{cases} \left(\sum_{i=0}^{n-1} |x_i|\right) \psi\left(\frac{|x_1|}{\sum_{i=0}^{n-1}|x_i|}, \ldots, \frac{|x_{n-1}|}{\sum_{i=0}^{n-1}|x_i|}\right), \\ \qquad (x_0, x_1, \ldots, x_{n-1}) \neq (0, \ldots, 0) \\ 0, \quad (x_0, x_1, \ldots, x_{n-1}) = (0, \ldots, 0). \end{cases}$$

Moreover, M. Kato, K.-S. Saito and Tamura in [6] introduced the ψ-direct sums $(X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n)_\psi$ as follows. Let $X_1, X_2, \ldots, X_n$ be Banach spaces and let $\psi \in \Psi_n$. Then the product space $X_1 \times X_2 \times \cdots \times X_n$ with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_\psi = \|(\|x_1\|, \|x_2\|, \ldots, \|x_n\|)\|_\psi, \quad x_i \in X_i,\ 1 \leq i \leq n,$$

is a Banach space which is denoted by $(X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n)_\psi$. They showed that $(X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n)_\psi$ is strictly convex (uniformly convex) if and only if $X_1, X_2, \ldots, X_n$ is strictly convex (uniformly convex) and $\psi \in \Psi_n$ is strictly convex. In [7] the authors presented that $X \bigoplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$. Since the introduction of ψ-direct sums of Banach spaces, it has attracted plenty of attention and been treated by several authors (cf. [3, 4, 5, 12, 16]).

In particular, K.-I. Mitani and K.-S. Saito in [11] characterized the strict convexity, uniform convexity and uniform non-squareness of Banach spaces using ψ-direct sums $X \bigoplus_\psi X$. They showed that, if t_0 is a unique minimal point, a Banach space X is strictly convex if and only if, for each $x, y \in X$ with $x \neq y$, then

$$\|(1-t_0)x + t_0 y\| < \frac{1}{\psi(t_0)} \|((1-t_0)x, t_0 y)\|_\psi, \quad \psi \in \Psi_2.$$

As for the cases of uniform convexity and uniform non-squareness, they gained some similar results.

Our main purpose of this paper is to give the characterization of K strict convexity, K uniform convexity and uniform non-l_1^N-ness using ψ-direct sums $(X \bigoplus X \bigoplus \cdots \bigoplus X)_\psi$, we first characterize the K strict convexity using ψ-direct sums. We show that, if ψ has a minimal point $s_0 = (t_1, t_2, \ldots, t_K)$, and $0 < t_i < 1, i = 1, 2, \ldots, K$ and $0 < \sum_{i=1}^K t_i < 1$, then a Banach space X is K strictly convex if and only if for any $x_0, x_1, \ldots, x_K \in X$, with $x_0, x_1, \ldots, x_K$ linearly independent, we have

$$\|t_0 x_0 + t_1 x_1 + \cdots + t_K x_K\| < \frac{1}{\psi(s_0)} \|(t_0 x_0, t_1 x_1, \ldots, t_K x_K)\|_\psi,$$

where $\sum_{i=0}^K t_i = 1$. As a result, we can give different characterization by choosing different ψ. In contrast with the result of K.-I. Mitani and K.-S. Saito [11], the uniqueness of t_0 is not required, but the linear independence of x and y is necessary. Moreover when $K = 1$, we get the characterization of strict convexity. In Section 3, we also characterize the K uniform convexity and make Theorem 8 in [11] as our Corollary 3.5. In Section 4, the characterization of uniform non-l_1^N-ness is gained by adding the uniqueness of minimal point.

2. K strict convexity

A Banach space X is said to be K strictly convex (cf. [14]) if and only if for any $K+1$ elements $x_0, x_1, \ldots, x_K$ in X, whenever $\|\sum_{i=0}^K x_i\| = \sum_{i=0}^K \|x_i\|$, then $x_0, x_1, \ldots, x_K$ are linearly dependent.

The closed unit ball of a Banach space X is $\{x \in X : \|x\| \leq 1\}$ and is denoted by B_X, the unit sphere of X is $\{x \in X : \|x\| = 1\}$ and is denoted by S_X. It is obvious that when $K = 1$, X is strictly convex.

Proposition 2.1 (cf. [8])**.** *Let X be a Banach space. For all non-zero elements $x_1, x_2, \ldots, x_n \in X$, the following inequality holds*:

$$\|\sum_{j=1}^n x_j\| + \left(n - \|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\|\right) \min_{1\leq j\leq n} \|x_j\| \leq \sum_{j=1}^n \|x_j\|$$
$$\leq \|\sum_{j=1}^n x_j\| + \left(n - \|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\|\right) \max_{1\leq j\leq n} \|x_j\|.$$

Lemma 2.2. *Let X be a Banach space. Then the following assertions are equivalent.*

(1) *X is K strictly convex.*

(2) *For any $x_0, x_1, \ldots, x_K \in S_X$, whenever $\|\sum_{i=0}^K x_i\| = K+1$, then $x_0, x_1, \ldots, x_K$ are linearly dependent.*

(3) *If $x_0, x_1, \ldots, x_K \in S_X$ and $x_0, x_1, \ldots, x_K$ are linearly independent, then for any $\{t_i\}_{i=0}^K$ satisfying $0 < t_i < 1, \sum_{i=0}^K t_i = 1$, there holds $\|\sum_{i=0}^K t_i x_i\| < 1$.*

($3'$) *If $x_0, x_1, \ldots, x_K \in S_X$ and $x_0, x_1, \ldots, x_K$ are linearly independent, then there exists $\{t_i\}_{i=0}^K$ with $0 < t_i < 1, \sum_{i=0}^K t_i = 1$, such that $\|\sum_{i=0}^K t_i x_i\| < 1$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let any $x_0, x_1, \ldots, x_K \in X\backslash\{0\}$, and $\|\sum_{i=0}^K x_i\| = \sum_{i=0}^K \|x_i\|$. By Proposition 2.1 we have $\|\sum_{i=0}^K \frac{x_i}{\|x_i\|}\| = K+1$. Hence $\frac{x_0}{\|x_0\|}, \frac{x_1}{\|x_1\|}, \ldots, \frac{x_K}{\|x_K\|}$ are linearly dependent, so do $x_0, x_1, \ldots, x_K$.

(2) $\Rightarrow$ (3) Assume that the conclusion falls to hold. Then there exists $\{t_i\}_{i=0}^K$ satisfying $0 < t_i < 1, \sum_{i=0}^K t_i = 1$, but $\|\sum_{i=0}^K t_i x_i\| = 1$. Using Proposition 2.1 we have

$$\|\sum_{i=0}^K t_i x_i\| + \left(K+1 - \|\sum_{i=0}^K \frac{x_i}{\|x_i\|}\|\right) \min_{0\leq i\leq K} \|t_i x_i\| \leq \sum_{i=0}^K t_i \|x_i\|$$
$$\leq \|\sum_{i=0}^K t_i x_i\| + \left(K+1 - \|\sum_{i=0}^K \frac{x_i}{\|x_i\|}\|\right) \max_{0\leq i\leq K} \|t_i x_i\|.$$

Hence $\| \sum_{i=0}^{K} \frac{x_i}{\|x_i\|} \| = K+1$. So $x_0, x_1, \dots, x_K$ are linearly dependent. Contradiction.

(3) $\Rightarrow$ (2) Clearly.

(2) $\Rightarrow$ (3$'$) We just need to let $t_i = \frac{1}{K+1}$, $i = 0, 1, \dots, K$.

(3$'$) $\Rightarrow$ (2) If there are $x_0, x_1, \dots, x_K \in S_X$ and satisfying $\| \sum_{i=0}^{K} x_i \| = K+1$, but $x_0, x_1, \dots, x_K$ are linearly independent. Then there exists $\{t_i\}_{i=0}^{K}$, with $0 < t_i < 1, \sum_{i=0}^{K} t_i = 1$, and $\| \sum_{i=0}^{K} t_i x_i \| < 1$. Considering Proposition 2.1 we have

$$1 = \sum_{i=0}^{K} t_i \|x_i\| \leq \| \sum_{i=0}^{K} t_i x_i \| + (K+1 - \| \sum_{i=0}^{K} x_i \|) \max_{0 \leq i \leq K} \|t_i x_i\|,$$

that is $\| \sum_{i=0}^{K} t_i x_i \| \geq 1$, contradiction. □

Theorem 2.3. *Let $\psi \in \Psi_{K+1}$. Assume that ψ has a minimal point $s_0 = (t_1, t_2, \dots, t_K)$, and $0 < t_i < 1$, $i = 1, 2, \dots, K$ and $0 < \sum_{i=1}^{K} t_i < 1$. Then a Banach space X is K strictly convex if and only if for any $x_0, x_1, \dots, x_K \in X$, with $x_0, x_1, \dots, x_K$ linearly independent, we have*

$$\|t_0 x_0 + t_1 x_1 + \cdots + t_K x_K\| < \frac{1}{\psi(s_0)} \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_\psi,$$

where $\sum_{i=0}^{K} t_i = 1$.

Proof. Assume that X is K strictly convex. Since $\psi(s) \geq \psi(s_0)$ for all $s \in \Delta_{K+1}$, and $t_0 x_0, t_1 x_1, \dots, t_K x_K$ are linearly independent, then we have

$$\begin{aligned} & \|t_0 x_0 + t_1 x_1 + \cdots + t_K x_K\| \\ < & \|t_0 x_0\| + \|t_1 x_1\| + \cdots + \|t_K x_K\| \\ = & \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_1 \\ \leq & \max_{s \in \Delta_{K+1}} \frac{\psi_1(s)}{\psi(s)} \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_\psi \\ = & \frac{1}{\min\limits_{s \in \Delta_{K+1}} \psi(s)} \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_\psi \\ = & \frac{1}{\psi(s_0)} \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_\psi. \end{aligned}$$

Conversely for any $x_i \in S_X, i = 0, 1, \dots, K$ with $x_0, x_1, \dots, x_K$ linearly independent. We have

$$\begin{aligned} & \|t_0 x_0 + t_1 x_1 + \cdots + t_K x_K\| \\ < & \frac{1}{\psi(s_0)} \|(t_0 x_0, t_1 x_1, \dots, t_K x_K)\|_\psi \\ = & \frac{1}{\psi(s_0)} \|(t_0, t_1, \dots, t_K)\|_\psi \end{aligned}$$

$$= \frac{1}{\psi(s_0)}\|(1-\sum_{i=1}^{K} t_i, t_1, \ldots, t_K)\|_\psi = 1.$$

□

Corollary 2.4. *Let $\psi \in \Psi_2$. Assume that ψ has a minimal point t_0. Then a Banach space X is strictly convex if and only if, for each $x, y \in X$ with x, y linearly independent we have*

$$\|(1-t_0)x + t_0 y\| < \frac{1}{\psi(t_0)}\|((1-t_0)x, t_0 y)\|_\psi.$$

Corollary 2.5. *If $\psi = \psi_p \in \Psi_{K+1}$, when $1 < p < \infty$, $\psi_p(t_1, t_2, \ldots, t_K) = ((1-\sum_{i=1}^{K} t_i)^p + t_1^p + \cdots + t_K^p)^{\frac{1}{p}}$. Note that for any $s \neq (\frac{1}{K+1}, \ldots, \frac{1}{K+1}), s \in \Delta_{K+1}$, $\psi_p(s) > \psi_p(\frac{1}{K+1}, \ldots, \frac{1}{K+1}) = (K+1)^{\frac{1}{p}-1}$. Then a Banach space X is K strictly convex if and only if for any $x_0, x_1, \ldots, x_K \in X$ with $x_0, x_1, \ldots, x_K$ linearly independent, we have*

$$\|\frac{x_0 + x_1 + \cdots + x_K}{K+1}\|^p < \frac{\|x_0\|^p + \cdots + \|x_K\|^p}{K+1}.$$

Theorem 2.3 does not require that ψ is strictly convex. This should be contrasted with the result of [6], i.e., $(X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n)_\psi$ is strictly convex if and only if $X_1, X_2, \ldots, X_n$ are strictly convex respectively and ψ is a strictly convex function on Δ_n. Thus, let $\|\cdot\| = \max\{\|\cdot\|_2, \lambda\|\cdot\|_1\}$ $(\frac{1}{\sqrt{K+1}} < \lambda < 1)$. Let ψ be the corresponding convex function of $\|\cdot\|$. Then for any $s = (s_1, \ldots, s_K) \in \Delta_{K+1}$,

$$\begin{aligned}\psi(s) &= \|(1-\sum_{i=1}^{K} s_i, s_1, \ldots, s_K)\| \\ &= \max\left\{\|(1-\sum_{i=1}^{K} s_i, s_1, \ldots, s_K)\|_2, \lambda\|(1-\sum_{i=1}^{K} s_i, s_1, \ldots, s_K)\|_1\right\} \\ &= \max\{\psi_2(s), \lambda\}.\end{aligned}$$

Since $\min_{s\in\Delta_{K+1}} \psi_2(s) = \frac{1}{\sqrt{K+1}}$. Then

$$\psi(s) = \begin{cases} \lambda, & \frac{1}{\sqrt{K+1}} \leq \psi_2(s) \leq \lambda \\ \psi_2(s), & \lambda < \psi_2(s) \leq 1. \end{cases}$$

For $\psi_2(s)$ is continuous on Δ_{K+1}, we have $\min_{s\in\Delta_{K+1}} \psi(s) = \lambda$ and ψ is not strictly convex on Δ_{K+1}. Applying Theorem 2.3, we can give the following characterization using ψ above.

Corollary 2.6. *Let $\frac{1}{\sqrt{K+1}} < \lambda \leq 1$. Then a Banach space X is K strictly convex if and only if for any $x_0, x_1, \ldots, x_K \in X$, with $x_0, x_1, \ldots, x_K$ linearly independent, we have*

$$\|\frac{x_0 + x_1 + \cdots + x_K}{K+1}\|$$

$$< \frac{1}{\lambda} \max \left\{ \frac{(\|x_0\|^2 + \cdots + \|x_K\|^2)^{\frac{1}{2}}}{K+1}, \lambda \frac{\|x_0\| + \cdots + \|x_K\|}{K+1} \right\}$$
$$= \max \left\{ \frac{(\|x_0\|^2 + \cdots + \|x_K\|^2)^{\frac{1}{2}}}{\lambda(K+1)}, \frac{\|x_0\| + \cdots + \|x_K\|}{K+1} \right\}.$$

3. *K* uniform convexity

We say that a Banach space X is K uniformly convex (or K uniformly rotund see [15]) if for any $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) > 0$, such that whenever $x_0, x_1, \ldots, x_K \in S_X$ and $\|x_0 + x_1 + \cdots + x_K\| > (K+1) - \delta$, we have

$$A(x_0, x_1, \ldots, x_K)$$
$$\equiv \sup \left\{ \left| \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ f_1(x_0) & f_1(x_1) & \cdots & f_1(x_K) \\ \cdots & \cdots & \cdots & \cdots \\ f_K(x_0) & f_K(x_1) & \cdots & f_K(x_K) \end{array} \right|, \{f_i\}_{i=1}^K \subset B_{X^*} \right\} < \varepsilon.$$

In the case of $K = 1$, X is uniformly convex.

Proposition 3.1 (cf. [17])**.** *Let X be a Banach space. Then X is K uniformly convex if and only if for any $K+1$ sequences $\{x_0^n\}, \{x_1^n\}, \ldots, \{x_K^n\}$ in X, if $\|x_i^n\| \to a, n \to \infty, i = 0, 1, 2, \ldots, K$ and $\|x_0^n + x_1^n + \cdots + x_K^n\| \to (K+1)a$, then*

$$\lim_{n\to\infty} A(x_0^n, x_1^n, \ldots, x_K^n) = 0.$$

Proposition 3.2 (cf. [9])**.** *Let $\{x_1^k\}_k, \{x_2^k\}_k, \ldots, \{x_n^k\}_k$ be n sequences in a Banach space X for which the sequences of their norms are convergent. Then the following are equivalent.*

(1) $\lim_{k\to\infty} \| \sum_{j=1}^n x_j^k \| = \lim_{k\to\infty} \sum_{j=1}^n \|x_j^k\|$.

(2) $\lim_{k\to\infty} \|\alpha x_1^k + \sum_{j=2}^n x_j^k\| = \lim_{k\to\infty} (\alpha\|x_1^k\| + \sum_{j=2}^n \|x_j^k\|)$ *for all* $\alpha > 0$.

(3) $\lim_{k\to\infty} \|\alpha x_1^k + \sum_{j=2}^n x_j^k\| = \lim_{k\to\infty} (\alpha\|x_1^k\| + \sum_{j=2}^n \|x_j^k\|)$ *for some* $\alpha > 0$.

Proposition 3.3 (cf. [13])**.** *Let $\psi \in \Psi_n$ and let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$. Then*

(1) *If $|x| \leq |y|$, then $\|x\|_\psi \leq \|y\|_\psi$.*

(2) *If ψ is strictly convex and $|x| < |y|$, then $\|x\|_\psi < \|y\|_\psi$.*

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, denote $|x|$ by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$. We say that $|x| \leq |y|$ if $|x_j| \leq |y_j|$ for $1 \leq j \leq n$. Further, we say that $|x| < |y|$ if $|x| \leq |y|$ and $|x_j| < |y_j|$ for some j.

Theorem 3.4. *Let $\psi \in \Psi_{K+1}$. Assume that ψ has a unique minimal point $s_0 = (t_1, t_2, \ldots, t_K)$, with $0 < t_i < 1$, $\sum_{i=1}^K t_i < 1$. Then a Banach space X is K uniformly convex if and only if for any $\varepsilon > 0$, there exists some $\delta > 0$, such* *that for any* $x_0, x_1, \dots, x_K \in B_X$*, satisfying*

$$\|t_0x_0 + t_1x_1 + \cdots + t_Kx_K\| > (1-\delta)\frac{1}{\psi(s_0)}\|(t_0x_0, t_1x_1, \dots, t_Kx_K)\|_\psi,$$

where $\sum_{i=0}^K t_i = 1$*, then we have* $A(x_0, x_1, \dots, x_K) < \varepsilon$.

Proof. Let X be a K uniformly convex Banach space. Assume that there exists $\varepsilon_0 > 0$, for any $n \in \mathbb{N}$, there are sequences $\{x_0^n\}, \{x_1^n\}, \dots, \{x_K^n\}$ in B_X satisfying

$$(3)\quad \|t_0x_0^n + t_1x_1^n + \cdots + t_Kx_K^n\| > (1-\frac{1}{n})\frac{1}{\psi(s_0)}\|(t_0x_0^n, t_1x_1^n, \dots, t_Kx_K^n)\|_\psi.$$

But $A(x_0^n, x_1^n, \dots, x_K^n) \geq \varepsilon_0$.

Since $\{\|x_i^n\|\}_{n=1}^\infty$, $i = 0, 1, \dots, K$ and $\{\|\sum_{i=0}^K t_ix_i^n\|\}_{n=1}^\infty$ are bounded sequences, without loss of generality we can let $\|x_i^n\| \to a_i$ $(n \to \infty), i = 0, 1, \dots, K$ and $\|t_0x_0^n + t_1x_1^n + \cdots + t_Kx_K^n\| \to b$ $(n \to \infty)$. Moreover, we can choose $\{\|x_i^n\|\}_{n=1}^\infty$ such that $\max\{\|x_i^n\|, 0 \leq i \leq K\} = 1$. Thus $\max\{a_i, 0 \leq i \leq K\} = 1$. From this, $\sum_{i=0}^K t_ia_i > 0$. It is clear that $0 \leq a_i \leq 1$, $0 \leq b \leq 1$. Considering the equality (2), we have

$$\begin{aligned}
&(1-\frac{1}{n})\frac{1}{\psi(s_0)}\|(t_0x_0^n, t_1x_1^n, \dots, t_Kx_K^n)\|_\psi \\
&= (1-\frac{1}{n})\frac{1}{\psi(s_0)}\|(t_0\|x_0^n\|, t_1\|x_1^n\|, \dots, t_K\|x_K^n\|)\|_\psi \\
&< \|t_0x_0^n + t_1x_1^n + \cdots + t_Kx_K^n\| \\
&\leq t_0\|x_0^n\| + t_1\|x_1^n\| + \cdots + t_K\|x_K^n\|.
\end{aligned}$$

Let $n \to \infty$. Then $\frac{1}{\psi(s_0)}\|(t_0a_0, t_1a_1, \dots, t_Ka_K)\|_\psi \leq t_0a_0 + t_1a_1 + \cdots + t_Ka_K$ holds. Hence

$$\psi\left(\frac{t_1a_1}{\sum_{i=0}^K t_ia_i}, \dots, \frac{t_Ka_K}{\sum_{i=0}^K t_ia_i}\right) \leq \psi(s_0) = \psi(t_1, t_2, \dots, t_K).$$

From the uniqueness of s_0, we get $a_0 = a_1 = \cdots = a_K$. Let us denote them as a. Moreover,

$$\lim_{n\to\infty}\|\sum_{i=0}^K t_ix_i^n\| = \lim_{n\to\infty}\sum_{i=0}^K \|t_ix_i^n\|.$$

Using Proposition 3.2 we get

$$\lim_{n\to\infty}\|\frac{1}{t_0}t_0x_0^n + \sum_{i=1}^K t_ix_i^n\| = \lim_{n\to\infty}(\|x_0^n\| + \sum_{i=1}^K \|t_ix_i^n\|).$$

Repeat the similar process above for $K+1$ times, we have

$$\lim_{n\to\infty}\|\sum_{i=0}^K x_i^n\| = \lim_{n\to\infty}\sum_{i=0}^K \|x_i^n\| = (K+1)a.$$

Hence there is $\lim_{n\to\infty} A(x_0^n, x_1^n, \dots, x_K^n) = 0$. By Proposition 3.1, it is a contradiction.

Conversely, for any $\varepsilon > 0$ there exists some $\delta > 0$, such that for any $x_0, x_1, \dots, x_K$ in S_X with $A(x_0, x_1, \dots, x_K) \geq \varepsilon$, we have

$$\begin{aligned} &\|t_0x_0 + t_1x_1 + \cdots + t_Kx_K\| \\ &\leq (1-\delta)\frac{1}{\psi(s_0)}\|(t_0x_0, t_1x_1, \dots, t_Kx_K)\|_\psi \\ &\leq (1-\delta)\frac{1}{\psi(s_0)}\|(t_0, t_1, \dots, t_K)\|_\psi = 1-\delta. \end{aligned}$$

By Proposition 2.1 we have

$$\begin{aligned} 1 = \sum_{i=0}^K t_i\|x_i\| &\leq (K+1-\|\sum_{i=0}^K x_i\|)\max_{0\leq i\leq K} t_i\|x_i\| + \|\sum_{i=0}^K t_ix_i\| \\ &\leq (K+1-\|\sum_{i=0}^K x_i\|) + 1 - \delta. \end{aligned}$$

Hence $\|\sum_{i=0}^K x_i\| \leq (K+1) - \delta$. □

Corollary 3.5 (cf. [11])**.** *Let $\psi \in \Psi_2$. Assume that ψ has a unique minimal point t_0. Then a Banach space X is uniformly convex if and only if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\|x - y\| \geq \varepsilon, x, y \in B_X$ implies*

$$\|(1-t_0)x + t_0y\| \leq (1-\delta)\frac{1}{\psi(t_0)}\|((1-t_0)x, t_0y)\|_\psi.$$

Corollary 3.6. *Let $\psi(s) = \psi_p(s) = [(1-\sum_{i=1}^K s_i)^p + s_1^p + \cdots + s_K^p]^{\frac{1}{p}}$, $1 < p < \infty$. Then $\psi_p(s)$ has a unique minimal point $s_0 = (\frac{1}{K+1}, \frac{1}{K+1}, \dots, \frac{1}{K+1})$. A Banach space X is K uniformly convex if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x_0, x_1, \dots, x_K$ in B_X satisfying*

$$\|\frac{x_0 + x_1 + \cdots + x_K}{K+1}\|^p > (1-\delta)\frac{\|x_0\|^p + \cdots + \|x_K\|^p}{K+1}$$

implies $A(x_0, x_1, \dots, x_K) < \varepsilon$.

4. Uniform non-l_1^N-ness

A Banach space X is said to be uniformly non-l_1^N (cf. [1, 10]) provided there exists $\delta (0 < \delta < 1)$ such that for any $x_0, x_1, \dots, x_{N-1}$ in S_X, there exists an N-tuple of signs $\theta = (\theta_j)$ for which

$$\|\sum_{j=0}^{N-1} \theta_j x_j\| \leq N(1-\delta).$$

In the case of $N = 2$, X is called uniform non-squareness. As is well known, we may take $x_0, x_1, \dots, x_{N-1}$ from B_X in the definition (see [8]).

Lemma 4.1. *A Banach space X is uniformly non-l_1^N if and only if there exist some $s=(s_0,s_1,\dots,s_{N-1})$, with $\sum_{i=0}^{N-1} s_i = 1, 0<s_i<1, i=0,1,\dots,N-1$, and some $\delta (0<\delta<1)$, such that for any $x_0,x_1,\dots,x_{N-1}$ in B_X, there exists an N-tuple of signs $\theta=(\theta_j)$, for which $\|\sum_{j=0}^{N-1}\theta_j s_j x_j\| \le 1-\delta$.*

Proof. Assume that X is uniformly non-l_1^N. Let $s_i=\frac{1}{N}, i=0,1,\dots,N-1$. For any $x_0,x_1,\dots,x_{N-1}$ in S_X, there exists an N-tuple of signs $\theta=(\theta_j)$ and $s=(s_0,\dots,s_{N-1})$ with $\sum_{i=0}^{N-1} s_i=1$, $\|\sum_{j=0}^{N-1}\theta_j s_j x_j\|\le 1-\delta$. Use Proposition 2.1 we have

$$\begin{aligned}1 &= \sum_{j=0}^{N-1}\|\theta_j s_j x_j\| \\ &\le \|\sum_{j=0}^{N-1}\theta_j s_j x_j\| + (N-\|\sum_{j=0}^{N-1}\theta_j x_j\|)\max_{0\le i\le N-1}\|\theta_j s_j x_j\| \\ &\le 1-\delta+N-\|\sum_{j=0}^{N-1}\theta_j x_j\|.\end{aligned}$$

Let $\delta' = \frac{\delta}{N}$. Then for any $x_0,x_1,\dots,x_{N-1}$ in S_X, there exists an N-tuple of signs $\theta=(\theta_j)$, for which $\|\sum_{j=0}^{N-1}\theta_j x_j\|\le N(1-\delta')$. □

Lemma 4.2. *Let X be a Banach space. Then X is uniformly non-l_1^N if and only if for any N sequences $\{x_0^n\},\dots,\{x_{N-1}^n\}$ in X and $\|x_j^n\|\to a (a>0), n\to\infty, j=0,1,\dots,N-1$, $\|\sum_{j=0}^{N-1}\theta_j x_j^n\|\to A_\theta$ for any $\theta=(\theta_j)$, then there exists an N-tuple of signs $\theta=(\theta_j)$ for which*

$$\lim_{n\to\infty}\left\|\sum_{j=0}^{N-1}\theta_j x_j^n\right\| < Na.$$

Proof. It is equivalent to prove that: X is not uniformly non-l_1^N if and only if there exist N sequences $\{x_0^n\},\dots,\{x_{N-1}^n\}$ in X and $\|x_j^n\|\to a(a>0), n\to\infty, j=0,1,\dots,N-1$, for any N-tuple of signs $\theta=(\theta_j)$ there holds

$$\lim_{n\to\infty}\left\|\sum_{j=0}^{N-1}\theta_j x_j^n\right\| = Na.$$

Without loss of generality, let $a=1$. On one hand, since $\|x_j^n\|\to 1, j=0,1,\dots,N-1$, we can assume that $\|x_j^n\|>0$, then $\left\{\frac{x_j^n}{\|x_j^n\|}\right\}\subseteq S_X$. In addition, we have

$$\left|\|\sum_{j=0}^{N-1}\theta_j\frac{x_j^n}{\|x_j^n\|}\| - \|\sum_{j=0}^{N-1}\theta_j x_j^n\|\right|$$

$$\le \left\| \sum_{j=0}^{N-1} \theta_j \left(\frac{x_j^n}{\|x_j^n\|} - x_j^n \right) \right\| \le \sum_{j=0}^{N-1} \left\| \frac{x_j^n}{\|x_j^n\|} - x_j^n \right\|$$
$$= \sum_{j=0}^{N-1} \left| \frac{1}{\|x_j^n\|} - 1 \right| \cdot \|x_j^n\| \to 0 \ (n \to \infty).$$

Hence

$$\lim_{n\to\infty} \left\| \sum_{j=0}^{N-1} \theta_j \frac{x_j^n}{\|x_j^n\|} \right\| = \lim_{n\to\infty} \left\| \sum_{j=0}^{N-1} \theta_j x_j^n \right\| = N.$$

By definition X is not uniformly non-l_1^N.

The converse is obvious from the definition of uniform non-l_1^N-ness. □

Theorem 4.3. *Let $\psi \in \Psi_N$. Assume that ψ has a unique minimal point $s = (s_1, s_2, \ldots, s_{N-1})$ with $\sum_{i=1}^{N-1} s_i < 1, 0 < s_i < 1, i = 1, 2, \ldots, N-1$. Then a Banach space X is uniformly non-l_1^N if and only if there exists $\delta (0 < \delta < 1)$ such that for any $x_0, x_1, \ldots, x_{N-1}$ in B_X, there exists an N-tuple of signs $\theta = (\theta_j)$, for which*

$$\| \sum_{j=0}^{N-1} s_j \theta_j x_j \| \le (1-\delta) \frac{1}{\psi(s)} \|(s_0 x_0, s_1 x_1, \ldots, s_{N-1} x_{N-1})\|_\psi ,$$

where $s_0 = 1 - \sum_{i=1}^{N-1} s_j$.

Proof. Let X be a uniformly non-l_1^N Banach space. Assume that the conclusion fails to hold. Then for $\delta_n = \frac{1}{n}, n \in \mathbb{N}$, there exist sequences $\{x_j^n\}$ in B_X, $j = 0, 1, \ldots, N-1$, for any N-tuple of signs $\theta = (\theta_j)$, we have

$$\left\| \sum_{j=0}^{N-1} s_j \theta_j x_j^n \right\|$$
$$> (1 - \frac{1}{n}) \frac{1}{\psi(s)} \left\| (s_0 x_0^n, s_1 x_1^n, \ldots, s_{N-1} x_{N-1}^n) \right\|_\psi$$
$$= (1 - \frac{1}{n}) \frac{1}{\psi(s)} \left\| (s_0 \|x_0^n\|, s_1 \|x_1^n\|, \ldots, s_{N-1} \|x_{N-1}^n\|) \right\|_\psi . \tag{4}$$

Because $\{\|x_j^n\|\}_{n=1}^\infty$, $j = 0, 1, \ldots, N-1$ are bounded sequences, we just let $\|x_j^n\| \to a_j (n \to \infty)$, $j = 0, 1, \ldots, N-1$. Without loss of generality, we can choose $\{\|x_j^n\|\}_{n=1}^\infty$ such that $\max\{\|x_j^n\|, 0 \le j \le N-1\} = 1$. As a_j is the limit of $\{\|x_j^n\|\}_{n=1}^\infty$, we get $\max\{a_j, 0 \le j \le N-1\} = 1$. Thus $\sum_{j=0}^{N-1} s_j a_j > 0$. In (4) let $n \to \infty$, then there is

$$\frac{1}{\psi(s)} \|(s_0 a_0, s_1 a_1, \ldots, s_{N-1} a_{N-1})\|_\psi \le \sum_{j=0}^{N-1} s_j a_j.$$

From this we get

$$\psi\left(\frac{s_1a_1}{\sum_{j=0}^{N-1}s_ja_j},\ldots,\frac{s_{N-1}a_{N-1}}{\sum_{j=0}^{N-1}s_ja_j}\right)\le\psi(s_1,\ldots,s_{N-1}).$$

By the uniqueness of $s=(s_1,s_2,\ldots,s_{N-1})$, we get $a_0=a_1=\cdots=a_{N-1}$, denote them as a. In addition, from (4) we get $\lim_{n\to\infty}\left\|\sum_{j=0}^{N-1}s_j\theta_jx_j^n\right\|=1=\lim_{n\to\infty}\sum_{j=0}^{N-1}\left\|s_j\theta_jx_j^n\right\|$. Using Proposition 3.2 there holds

$$\lim_{n\to\infty}\left\|\sum_{j=0}^{N-1}\theta_jx_j^n\right\|=\lim_{n\to\infty}\sum_{j=0}^{N-1}\|\theta_jx_j^n\|=Na.$$

It's a contradiction by Lemma 4.2.

On the other hand, for any $x_0,x_1,\ldots,x_{N-1}$ in B_X

$$\begin{aligned}\left\|\sum_{j=0}^{N-1}s_j\theta_jx_j\right\|&\le(1-\delta)\frac{1}{\psi(s)}\|(s_0x_0,s_1x_1,\ldots,s_{N-1}x_{N-1})\|_\psi\\&\le(1-\delta)\frac{1}{\psi(s)}\|(s_0,s_1,\ldots,s_{N-1})\|_\psi\\&=1-\delta.\end{aligned}$$

We claim that X is uniformly non-l_1^N by Lemma 4.1. □

Corollary 4.4. *Let $\psi\in\Psi_2$. Assume that ψ has the unique minimum at $t=t_0(0<t_0<1)$. Then a Banach space X is uniformly non-square if and only if there exists some $\delta(0<\delta<1)$ such that for any $x,y\in B_X$ implies*

$$\min\{\|(1-t_0)x+t_0y\|,\|(1-t_0)x-t_0y\|\}\le(1-\delta)\frac{1}{\psi(t_0)}\|((1-t_0)x,t_0y)\|_\psi.$$

Corollary 4.5. *A Banach space X is uniformly non-l_1^N if and only if there exists some $\delta(0<\delta<1)$ such that for any $x_0,x_1,\ldots,x_{N-1}$ in B_X, there exists an N-tuple of signs $\theta=(\theta_j)$ for which*

$$\left\|\frac{\sum_{j=0}^{N-1}\theta_jx_j}{N}\right\|^p\le(1-\delta)\frac{\|x_0\|^p+\cdots+\|x_{N-1}\|^p}{N},$$

where $1<p<\infty$.

Proof. We only need to let $\psi(t)=\psi_p(t)=\left[(1-\sum_{i=1}^{N-1}t_i)^p+t_1^p+\cdots+t_{N-1}^p\right]^{\frac{1}{p}}$ in Theorem 4.3. □

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