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PURE INJECTIVE REPRESENTATIONS OF QUIVERS

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Abstract. Let R be a ring and $\mathcal{Q}$ be a quiver. In this paper we give another definition of purity in the category of quiver representations. Under such definition we prove that the class of all pure injective representations of $\mathcal{Q}$ by R-modules is preenveloping. In case $\mathcal{Q}$ is a left rooted semi-cobarren quiver and R is left Noetherian, we show that every cotorsion flat representation of $\mathcal{Q}$ is pure injective. If, furthermore, R is n-perfect and $\mathcal{F}$ is a flat representation $\mathcal{Q}$, then the pure injective dimension of $\mathcal{F}$ is at most n.

1. Introduction

Throughout the paper rings are associative with identity and modules are unital (unless otherwise specified). If $\mathcal{Q}$ is a *quiver* (a directed graph), then an arrow from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \longrightarrow v_2$. The set of vertices (resp. arrows) of a quiver $\mathcal{Q}$ is denoted by $\mathrm{V}_{\mathcal{Q}}$ (resp. $\mathrm{E}_{\mathcal{Q}}$). For a given arrow a of $\mathcal{Q}$, $i(a)$ denotes the initial vertex of a and $t(a)$ denotes the terminal vertex of a. A quiver may be thought as a category in which the objects are vertices and the morphisms are paths, a path is a sequence of arrows. A representation $\mathcal{X}$ by modules of a quiver $\mathcal{Q}$ is then a covariant functor $\mathcal{X} : \mathcal{Q} \longrightarrow$R-Mod. Thus a representation $\mathcal{X}$ is determined by giving a module $\mathcal{X}(v)$ to each vertex $v \in \mathrm{V}_{\mathcal{Q}}$ and a homomorphism $\mathcal{X}(a) : \mathcal{X}(v_1) \longrightarrow \mathcal{X}(v_2)$ to each arrow $a \in \mathrm{E}_{\mathcal{Q}}$. A morphism f between representations $\mathcal{X}$ and $\mathcal{Y}$ is a natural transformation. The category of representations of a quiver $\mathcal{Q}$ by left R-modules over a ring R is denoted by $\mathrm{Rep}(\mathcal{Q}, R)$. This is a Grothendieck category with projective generators and injective cogenerators (see [1] and [2]). A quiver $\mathcal{Q}$ is called left(right) rooted if there is no path of the form $\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet (\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots)$ in $\mathcal{Q}$. The study of special objects in the category of representations of quivers has a long history in the literature. Especially flat representations of a left rooted quiver, and existence of flat covers in the category $\mathrm{Rep}(\mathcal{Q}, R)$, when $\mathcal{Q}$ is left rooted, have been studied in [3], [8].

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Let $\text{Flat}(\mathcal{Q})$ be the class of all flat representations of a left rooted quiver $\mathcal{Q}$ and $\text{Flat}(\mathcal{Q})^{\perp}$, be the class of all object $\mathcal{C}$ of $\text{Rep}(\mathcal{Q}, R)$ such that $\text{Ext}^1_{\mathcal{Q}}(\mathcal{F}, \mathcal{C}) = 0$ for every $\mathcal{F} \in \text{Flat}(\mathcal{Q})$. A representation $\mathcal{C}$ of $\mathcal{Q}$ is called cotorsion if $\mathcal{C} \in \text{Flat}(\mathcal{Q})^{\perp}$. Therefore $\text{Flat}(\mathcal{Q})^{\perp}$ is the class of all cotorsion representations of $\mathcal{Q}$. Cotorsion representations of a left rooted quiver are characterized in [11].

Let us first recall some notations and results of [2, Corollary 6.7] that we need throughout. Let $\mathcal{Q}$ be a quiver. We can define the opposite quiver $\mathcal{Q}^{\text{op}} = (\text{V}_{\mathcal{Q}}, \text{E}^{\text{op}}_{\mathcal{Q}})$ to $\mathcal{Q}$ such that its set of vertices is $\text{V}_{\mathcal{Q}}$ and its set of arrows is $\text{E}^{\text{op}}_{\mathcal{Q}}$, in which $v \longrightarrow w \in \text{E}^{\text{op}}_{\mathcal{Q}}$ if and only if $w \longrightarrow v \in \text{E}_{\mathcal{Q}}$. Note that $\text{Rep}(\mathcal{Q}^{\text{op}}, R^{\text{op}})$ is the category of representations of $\mathcal{Q}^{\text{op}}$ by right R-modules.

Let $\mathcal{X}$ be a representation of $\mathcal{Q}$, the representation $\mathcal{X}^+ \in \text{Rep}(\mathcal{Q}^{\text{op}}, R^{\text{op}})$ is given by the following:

i) For any $v \in \text{V}_{\mathcal{Q}}$, $\mathcal{X}^+ = \text{Hom}_{\mathbb{Z}}(\mathcal{X}(v), \mathbb{Q}/\mathbb{Z})$,

ii) For any $a \in \text{E}_{\mathcal{Q}}$ such that $a : v \longrightarrow w$, $\mathcal{X}^+(a) : \mathcal{X}^+(w) \longrightarrow \mathcal{X}^+(v)$.

Let $\text{Inj}(\mathcal{Q}^{\text{op}})$ be the class of all injective representations of $\mathcal{Q}^{\text{op}}$. By the following proposition, there is a fully faithful functor $\text{Flat}(\mathcal{Q}) \xrightarrow{(-)^+} \text{Inj}(\mathcal{Q}^{\text{op}})$.

Proposition 1.1. *Let $\mathcal{Q}$ be a left rooted quiver. A representation $\mathcal{F}$ of $\mathcal{Q}$ is flat in* $\text{Rep}(\mathcal{Q}, R)$ *if and only $\mathcal{F}^+$ is an injective object of* $\text{Rep}(\mathcal{Q}^{\text{op}}, R^{\text{op}})$.

Proof. See [2, Corollary 6.7]. □

The category $\text{Rep}(\mathcal{Q}, R)$ is a locally finitely presented additive category, so there is a categorical notion of purity in terms of the finitely presented representations. By [9], every representation of $\mathcal{Q}$ has a pure injective envelope. Actually if $\mathcal{F}$ is a flat representation, it can be easily shown that the pure injective envelope (in the sense of [9]) is flat if, and only if, it coincides with its cotorsion envelope. Therefore the main result of this work, Theorem 3.6, can not be followed from [9]. So, for this end, we had to make a new definition of purity. We give many propositions to show that our notion of purity is well-behaved. For example in Theorem 2.13 we prove that the classical relation between flatness and purity is true in $\text{Rep}(\mathcal{Q}, R)$.

It is possible that the categorical notion of purity and our notion of purity are the same, but our notion of purity has some advantage. For instance, let $0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$ be an exact sequence of representations, then there exists the following commutative diagram with exact rows and pure exact columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X}^{++} & \longrightarrow & \mathcal{Y}^{++} & \longrightarrow & \mathcal{Z}^{++} & \longrightarrow & 0, \end{array}$$

in $\text{Rep}(\mathcal{Q}, R)$. But from the categorical notion of purity it is not clear if we can deduce this important diagram.

In Section 2 we give the definition of purity in $\mathrm{Rep}(\mathcal{Q}, R)$ and study its properties, and show that under such definition each representation $\mathcal{X}$ of $\mathcal{Q}$ is a pure subrepresentation of a pure injective representation. In Section 3, over a left Noetherian ring we give necessary and sufficient conditions for $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q}, R)$ to be flat and pure injective when $\mathcal{Q}$ is a left rooted *semi-co-barren* quiver. If R is n-perfect, then the finiteness of the length of pure injective resolution of $\mathcal{X}$ will be discussed.

Setup: Throughout this paper $\mathcal{Q}$ is a left rooted quiver.

2. Purity and pure injectivity

This section is devoted to the study of purity in $\mathrm{Rep}(\mathcal{Q}, R)$. As a rich reference to the concepts of purity and pure injectivity and its properties in the category of R-modules, see [10], [12] and [13].

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ be a class of objects of $\mathcal{A}$. For an object A of $\mathcal{A}$, an object $\mathrm{C} \in \mathcal{C}$ is called a $\mathcal{C}$-envelope of A if there is a morphism $\varphi : \mathrm{A} \longrightarrow \mathrm{C}$ such that the following hold.

(i) For any morphism $\varphi' : \mathrm{A} \longrightarrow \mathrm{C}'$ with $\mathrm{C}' \in \mathcal{C}$, there is a morphism $f : \mathrm{C} \longrightarrow \mathrm{C}'$ with $\varphi' = f\varphi$.

(ii) If an endomorphism $f : \mathrm{C} \longrightarrow \mathrm{C}$ is such that $\varphi = f\varphi$, then f must be an automorphism.

If (i) holds, $\varphi : \mathrm{A} \longrightarrow \mathrm{C}$ is called a $\mathcal{C}$-preenvelope. Sometimes we call C or the map φ a $\mathcal{C}$-envelope (preenvelope) of A. For more details on the concept of (pre)enveloping classes and their properties, see [6] and [12].

A morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of representations is called a monomorphism if f has a cancelation property from the left, that is if for each representation $\mathcal{Z}$ of $\mathcal{Q}$ and each morphisms $g, h : \mathcal{Z} \longrightarrow \mathcal{X}$ which $fh = fg$, then $g = h$. On the other hand a morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a monomorphism if and only if for every $v \in \mathrm{V}_{\mathcal{Q}}$, the morphism $f(v) : \mathcal{X}(v) \longrightarrow \mathcal{Y}(v)$ is a monomorphism of R-modules. Now we make our definition of purity in $\mathrm{Rep}(\mathcal{Q}, R)$.

Definition 2.2. A monomorphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathrm{Rep}(\mathcal{Q}, R)$ is a pure monomorphism if $f^+ : \mathcal{Y}^+ \longrightarrow \mathcal{X}^+$ is a split epimorphism in the category $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$.

Remark 2.3. If $\mathcal{X} \longrightarrow \mathcal{Y}$ is a pure monomorphism in $\mathrm{Rep}(\mathcal{Q}, R)$, then $\mathcal{X}(v) \longrightarrow \mathcal{Y}(v)$ is a pure monomorphism of R-modules for any vertex v of $\mathcal{Q}$. But the converse is not true. To see this let M be an arbitrary R-module. Consider the quiver $\mathcal{Q} : \; v_1 \xrightarrow{a} v_2$, and the representations $\mathcal{M}_1 : \; M \xrightarrow{\mathcal{M}_1(a)} M \oplus M$ (where $\mathcal{M}_1(a)$ is injection) and $\mathcal{M}_2 : \; M \oplus M \xrightarrow{\mathcal{M}_2(a)} M \oplus M$ (where $\mathcal{M}_2(a)$ is identity). We see that $\mathcal{M}_1(v_1) \longrightarrow \mathcal{M}_2(v_1)$ and $\mathcal{M}_1(v_2) \longrightarrow \mathcal{M}_2(v_2)$ are pure monomorphisms of R-modules but $\mathcal{M}_1 \xrightarrow{\theta} \mathcal{M}_2$, where $\theta(a) = (\mathcal{M}_1(a), \mathcal{M}_2(a))$, is not pure in $\mathrm{Rep}(\mathcal{Q}, R)$.

Example 2.4. Let $\mathcal{X} \longrightarrow \mathcal{Y}$ be a monomorphism in $\mathrm{Rep}(\mathcal{Q}, R)$ such that for each vertex v of $\mathcal{Q}$, $\mathcal{X}(v) \longrightarrow \mathcal{Y}(v)$ is a pure monomorphism of R-modules, and for each arrow $a : v \longrightarrow w$, $\mathcal{X}(a) : \mathcal{X}(v) \longrightarrow \mathcal{X}(w)$ is a split epimorphism. Then $\mathcal{X} \longrightarrow \mathcal{Y}$ is a pure monomorphism in $\mathrm{Rep}(\mathcal{Q}, R)$.

Proposition 2.5. (i) *Any split exact sequence*

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

in $\mathrm{Rep}(\mathcal{Q}, R)$ *is pure exact.*

(ii) *Any direct limit of pure exact sequences is pure.*

(iii) *Let* $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{Z}$ *be a sequence of subrepresentations of* $\mathcal{Z}$*. If* $\mathcal{X}$ *is a pure subrepresentation of* $\mathcal{Z}$*, then it is also pure as a subrepresentation of* $\mathcal{Y}$*.*

Proof. (i) Since the exact sequence

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

is split, $\mathcal{Y} = \mathcal{X} \oplus \mathcal{Z}$. Therefore $\mathcal{Y}^+ = \mathcal{X}^+ \oplus \mathcal{Z}^+$ and so

$$0 \longrightarrow \mathcal{Z}^+ \longrightarrow \mathcal{Y}^+ \longrightarrow \mathcal{X}^+ \longrightarrow 0$$

is split exact in $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$. Thus

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

is pure exact sequence in $\mathrm{Rep}(\mathcal{Q}, R)$.

(ii) Let

$$(0 \longrightarrow \mathcal{X}_i \longrightarrow \mathcal{Y}_i \longrightarrow \mathcal{Z}_i \longrightarrow 0)_{i \in I}$$

be a direct system of pure exact sequences in $\mathrm{Rep}(\mathcal{Q}, R)$. Then

$$(0 \longrightarrow \mathcal{Z}_i{}^+ \longrightarrow \mathcal{Y}_i{}^+ \longrightarrow \mathcal{X}_i{}^+ \longrightarrow 0)_{i \in I}$$

is an inverse system of split exact sequences in $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$. Therefore, for each $i \in$ I we have the following split short exact sequence

$$0 \longrightarrow \varprojlim \mathcal{Z}_i{}^+ \longrightarrow \varprojlim \mathcal{Y}_i{}^+ \longrightarrow \varprojlim \mathcal{X}_i{}^+ \longrightarrow 0,$$

and then we have the split short exact sequence

$$0 \longrightarrow (\varinjlim_{i \in I} \mathcal{Z}_i)^+ \longrightarrow (\varinjlim_{i \in I} \mathcal{Y}_i)^+ \longrightarrow (\varinjlim_{i \in I} \mathcal{X}_i)^+ \longrightarrow 0.$$

This implies that the short exact sequence

$$0 \longrightarrow \varinjlim_{i \in I} \mathcal{X}_i \longrightarrow \varinjlim_{i \in I} \mathcal{Y}_i \longrightarrow \varinjlim_{i \in I} \mathcal{Z}_i \longrightarrow 0,$$

is pure exact sequence in $\mathrm{Rep}(\mathcal{Q}, R)$.

(iii) Let

$$\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$$

be a sequence of subrepresentations in $\mathrm{Rep}(\mathcal{Q}, R)$. We have the following commutative diagram in $\mathrm{Rep}(\mathcal{Q}, R)$:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i_2} & \mathcal{Y} \\ \downarrow{\scriptstyle i_1} & \swarrow{\scriptstyle i_3} & \\ \mathcal{Z} & & \end{array}$$

Therefore we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}^+ & \xrightarrow{i_3^+} & \mathcal{Y}^+ \\ \downarrow{\scriptstyle i_1^+} & \swarrow{\scriptstyle i_2^+} & \\ \mathcal{X}^+ & & \end{array}$$

$\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$, and there exists $k_1 : \mathcal{X}^+ \longrightarrow \mathcal{Z}^+$ such that $i_1^+ k_1 = 1_{X^+}$. So

$$i_2^+(i_3^+ k_1) = (i_2^+ i_3^+) k_1 = i_1^+ k_1 = 1_{\mathcal{X}^+}.$$

Thus the composition

$$\mathcal{Y}^+ \xrightarrow{i_3^+} \mathcal{X}^+ \xrightarrow{i_2^+ k_1} \mathcal{Y}^+$$

is $1_{\mathcal{X}^+}$. Therefore

$$\mathcal{Y}^+ \xrightarrow{i_3^+} \mathcal{X}^+$$

admits a section, and hence $\mathcal{X} \hookrightarrow \mathcal{Y}$ is a pure monomorphism in $\mathrm{Rep}(\mathcal{Q}, R)$. □

Proposition 2.6. *Let*

$$\mathcal{E} : 0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

be an exact sequence in $\mathrm{Rep}(\mathcal{Q}, R)$.

(i) *Let* $\mathcal{E}$ *be a pure exact sequence in* $\mathrm{Rep}(\mathcal{Q}, R)$. *Then* $\mathcal{Y}$ *is flat in* $\mathrm{Rep}(\mathcal{Q}, R)$ *if and only if* $\mathcal{X}$ *and* $\mathcal{Z}$ *are flat.*

(ii) *Let* $\mathcal{Z}$ *be a flat object of* $\mathrm{Rep}(\mathcal{Q}, R)$. *Then* $\mathcal{X}$ *is a flat object of* $\mathrm{Rep}(\mathcal{Q}, R)$ *if and only if* $\mathcal{Y}$ *is flat.*

Proof. (i) Let $\mathcal{E}$ be a pure exact sequence of $\mathrm{Rep}(\mathcal{Q}, R)$. Then the exact sequence

$$0 \longrightarrow \mathcal{Z}^+ \longrightarrow \mathcal{Y}^+ \longrightarrow \mathcal{X}^+ \longrightarrow 0$$

is split. So $\mathcal{Y}^+$ is injective if and only if $\mathcal{X}^+$ and $\mathcal{Z}^+$ are injective. Therefore $\mathcal{Y}$ is a flat representation if and only if $\mathcal{X}$ and $\mathcal{Z}$ are flat.

(ii) The exact sequence

$$0 \longrightarrow \mathcal{Z}^+ \longrightarrow \mathcal{Y}^+ \longrightarrow \mathcal{X}^+ \longrightarrow 0$$

is split. Thus

$$\mathcal{Y}^+ = \mathcal{X}^+ \oplus \mathcal{Z}^+.$$

So $\mathcal{X}^+$ is injective if and only if $\mathcal{Y}^+$ is injective. Therefore $\mathcal{X}$ is flat if and only if $\mathcal{Y}$ is flat. □

Recall that a representation $\mathcal{Z}$ of $\mathcal{Q}$ is called *pure injective* if for any pure exact sequence

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y}$$

in $\mathrm{Rep}(\mathcal{Q}, R)$, the sequence

$$\mathrm{Hom}_{\mathcal{Q}}(\mathcal{Y}, \mathcal{Z}) \longrightarrow \mathrm{Hom}_{\mathcal{Q}}(\mathcal{X}, \mathcal{Z}) \longrightarrow 0$$

is exact in the category of abelian groups.

Proposition 2.7. *Let $\mathcal{Q}$ be any quiver and $\mathcal{X}$ be a representation of $\mathcal{Q}$. Then*

(i) *The canonical monomorphism $\mathcal{X} \longrightarrow \mathcal{X}^{++}$ is pure and $\mathcal{X}^{++}$ is a pure injective representation.*

(ii) *A representation $\mathcal{X}$ is pure injective if and only if it is a direct summand of $\mathcal{X}^{++}$. Moreover, $\mathcal{Y}^{+}$ is pure injective, for any representation $\mathcal{Y}$ of* $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$.

Proof. In the first place we show that $\mathcal{X} \longrightarrow \mathcal{X}^{++}$ is a pure injection. By definition, it suffices to give a section $\mathcal{X}^{+++} \longrightarrow \mathcal{X}^{+}$. But the canonical map $\mathcal{X}^{+} \longrightarrow \mathcal{X}^{+++}$ does the job. Suppose then that $0 \longrightarrow \mathcal{N} \xrightarrow{i} \mathcal{M}$ is a pure monomorphism and $f : \mathcal{N} \longrightarrow \mathcal{X}^{++}$ is a map. Let $s : \mathcal{N}^{+} \longrightarrow \mathcal{M}^{+}$ be such that $i^{+}s = 1_{\mathcal{N}^{+}}$ and set $g = sf^{+} : \mathcal{X}^{+++} \longrightarrow \mathcal{M}^{+}$. Clearly $g^{+}i^{++} = f^{++}$. Note that since $\mathcal{X} \xrightarrow{\varphi_{\mathcal{X}}} \mathcal{X}^{++}$ is a pure monomorphism, there exists a map $t : \mathcal{X}^{++++} \longrightarrow \mathcal{X}^{++}$ which is a retraction for $\varphi_{\mathcal{X}}^{++}$, i.e., $t\varphi_{\mathcal{X}}^{++} = 1_{\mathcal{X}^{++}}$. On the other hand, since the canonical map $\varphi_{\mathcal{N}}$ is natural, we infer that $\varphi_{\mathcal{M}} i = i^{++}\varphi_{\mathcal{N}}$ and $f^{++}\varphi_{\mathcal{N}} = \varphi_{\mathcal{X}}^{++} f$. Now define $h = tg^{+}\varphi_{\mathcal{M}}$ and observe that $hi = f$. Thus the map $\mathrm{Hom}(\mathcal{M}, \mathcal{X}^{++}) \longrightarrow \mathrm{Hom}(\mathcal{N}, \mathcal{X}^{++})$ is a surjection and therefore $\mathcal{X}^{++}$ is pure injective.

(ii) Suppose $\mathcal{X}$ is a pure injective representation. Then from the canonical monomorphism $\varphi_{\mathcal{X}}$, which is pure by (i), one obtains a map $f : \mathcal{X}^{++} \longrightarrow \mathcal{X}$ satisfying $1_{\mathcal{X}} = f\varphi_{\mathcal{X}}$. The converse is obvious from (i). Meanwhile, Since $\mathcal{X}^{+}$ is a summand of $\mathcal{X}^{+++} = (\mathcal{X}^{++})^{+}$, we deduce that $\mathcal{X}^{+}$ is pure injective. □

Corollary 2.8. *Every injective representation of $\mathcal{Q}$ is pure injective.*

Proof. Let $\mathcal{I}$ be an injective representation of $\mathcal{Q}$. Then the canonical monomorphism $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}^{++}$ is a split monomorphism and hence $\mathcal{I}$ is pure injective. □

Remark 2.9. If $\mathcal{X}$ is a pure injective representation, then it possesses a pure injective R-module in each vertex. But the converse need not be true.

Example 2.10. If $\mathcal{X}$ is a representation of $\mathcal{Q}$ such that for any vertex v of $\mathcal{Q}$, $\mathcal{X}(v)$ is pure injective and for any arrow a of $\mathcal{Q}$, $\mathcal{X}(a)$ is a split epimorphism, then $\mathcal{X}$ is a pure injective representation of $\mathcal{Q}$.

Corollary 2.11. *Every representation of $\mathcal{Q}$ has pure injective preenvelope.*

Proof. Let $\mathcal{X}$ be a representation of $\mathcal{Q}$ and $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of representations such that $\mathcal{Y}$ is pure injective. It is known that $\mathcal{X} \longrightarrow \mathcal{X}^{++}$ is a pure monomorphism. So there exists a morphism of representations $g : \mathcal{X}^{++} \longrightarrow \mathcal{Y}$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi_{\mathcal{X}}} & \mathcal{X}^{++} \\ \downarrow f & \swarrow g & \\ \mathcal{Y} & & \end{array}$$

is commutative. This completes the proof. □

Remark 2.12. By [11, Theorem 2.6], a representation $\mathcal{C}$ of $\mathcal{Q}$ is cotorsion if and only if it is cotorsion in each vertex. Therefore by Remark 2.9 every pure injective object of $\mathrm{Rep}(\mathcal{Q}, R)$ is a cotorsion representation of $\mathcal{Q}$.

Theorem 2.13. *An object $\mathcal{Z}$ of* $\mathrm{Rep}(\mathcal{Q}, R)$ *is flat if and only if any exact sequence*

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

is pure in $\mathrm{Rep}(\mathcal{Q}, R)$.

Proof. If $\mathcal{Z}$ is flat in $\mathrm{Rep}(\mathcal{Q}, R)$, then for any exact sequence

$$o \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0 \tag{2.13.1}$$

in $\mathrm{Rep}(\mathcal{Q}, R)$, the sequence $0 \longrightarrow \mathcal{Z}^+ \longrightarrow \mathcal{Y}^+ \longrightarrow \mathcal{X}^+ \longrightarrow 0$ is split exact in $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$, because $\mathcal{Z}^+$ is injective. Hence (2.13.1) is pure.

Let every exact sequence

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

be pure in $\mathrm{Rep}(\mathcal{Q}, R)$. It suffices to show that $\mathcal{Z}^+$ is injective in $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$. For this end, let

$$o \longrightarrow \mathcal{Z}^+ \xrightarrow{f} \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow 0 \tag{2.13.2}$$

be an exact sequence in $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$. Consider the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Y}^+ & \longrightarrow & \mathcal{P} & \xrightarrow{h} & \mathcal{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow i & & \\ 0 & \longrightarrow & \mathcal{Y}^+ & \longrightarrow & \mathcal{X}^+ & \xrightarrow{f^+} & \mathcal{Z}^{++} & \longrightarrow & 0 \end{array}$$

in $\mathrm{Rep}(\mathcal{Q}, R)$ with exact rows. By assumption the top row is pure and hence split, because $\mathcal{Y}^+$ is pure injective. So there is a morphism $h' : \mathcal{Z} \longrightarrow \mathcal{P}$ of representations such that $hh' = 1_{\mathcal{Z}}$. It follows that $g_1 = gh' : \mathcal{Z} \longrightarrow \mathcal{X}^+$ is a morphism of representations such that $f^+g_1 = f^+gh' = ihh' = i$. Now consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Z}^{+} & \xrightarrow{f} & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{Z}^{+++} & \xrightarrow{f^{++}} & \mathcal{X}^{++} & \longrightarrow & \mathcal{Y}^{++} & \longrightarrow & 0. \end{array}$$

It follows that $g_1^{+}kf = g_1^{+}f^{++}j = i^{+}j = 1_{\mathcal{Z}^{+}}$. Therefore (2.13.2) is split and hence $\mathcal{Z}^{+}$ is an injective object of $\mathrm{Rep}(\mathcal{Q}^{\mathrm{op}}, R^{\mathrm{op}})$. Then by Proposition 1.1, $\mathcal{Z}$ is flat in $\mathrm{Rep}(\mathcal{Q}, R)$. $\square$

3. Pure injective dimension of flat representations

In this section we define the notion of a *semi-co-barren* quiver and give a characterization of a pure injective flat object in the category of representations of a semi-co-barren quiver.

Definition 3.1. A quiver $\mathcal{Q}$ is called semi-co-barren if for every $v \in \mathrm{V}_{\mathcal{Q}}$, $\{a \in \mathrm{E}_{\mathcal{Q}} \mid t(a) = v\}$ is a finite set.

Example 3.2. Let $\mathcal{Q}$ be a quiver whose connected components are barren trees. Then $\mathcal{Q}^{\mathrm{op}}$ is a semi-co-barren quiver. Recall that a tree T with a root v is said to be barren if the number of vertices n_i of the ith state of T is finite for every natural number i and the sequence of positive natural numbers n_1, $n_2, \ldots$ stabilizes, for more details see [4] and [5].

Set up: In this section we let R be a left Noetherian ring and $\mathcal{Q}$ be a semi-co-barren quiver or a quiver of type A_∞ in the sense of [3].

Lemma 3.3. *Let $\mathcal{F}$ be a representation of $\mathcal{Q}$. Then $\mathcal{F}$ is flat if and only if $\mathcal{F}^{++}$ is a flat representation of $\mathcal{Q}$.*

Proof. Let $\mathcal{F}$ be a flat representation of $\mathcal{Q}$. Since $\mathcal{Q}$ is a semi-co-barren quiver and R is left Noetherian then for every $v \in \mathrm{V}_{\mathcal{Q}}$, $\bigoplus_{t(a)=v} \mathcal{F}^{++}(i(a)) \longrightarrow \mathcal{F}^{++}(v)$ is a split monomorphism of flat R-modules. Therefore by [8] and [3], $\mathcal{F}^{++}$ is a flat representation of $\mathcal{Q}$.

The converse is a direct consequence of Proposition 2.6(i). $\square$

Theorem 3.4. *Let $\mathcal{F}$ be a flat representation of $\mathcal{Q}$. Then the followings are equivalent*:

(i) *$\mathcal{F}$ is pure injective.*
(ii) *$\mathcal{F}$ is cotorsion.*
(iii) *$\mathcal{F}$ is isomorphic to a direct summand of $\mathcal{F}^{++}$.*

Proof. (i)⇒(ii) Let $\mathcal{F}$ be a pure injective representation of $\mathcal{Q}$. $\mathcal{F}$ is cotorsion in each vertex and hence by [11, Theorem 2.6] it is cotorsion object of $\mathrm{Rep}(\mathcal{Q}, R)$.

(ii)⇒(iii) Let $\mathcal{F}$ be a cotorsion representation of $\mathcal{Q}$. By Lemma 3.3, $\mathcal{F}^{++}$ and $\mathcal{F}^{++}/\mathcal{F}$ are flat representations of $\mathcal{Q}$. So

$$0 \longrightarrow \mathcal{F} \xrightarrow{\lambda_{\mathcal{F}}} \mathcal{F}^{++} \longrightarrow \mathrm{Coker}\lambda_{\mathcal{F}} \longrightarrow 0,$$

is split and hence $\mathcal{F}$ is pure injective.

(iii)$\Rightarrow$(i) This is trivial. □

Corollary 3.5. *Let $\mathcal{F}$ be a flat object of* $\mathrm{Rep}(\mathcal{Q}, R)$*. Then $\mathcal{F}$ is pure injective if and only if $\mathcal{F}(v)$ is pure injective R-module for all $v \in \mathrm{V}_{\mathcal{Q}}$.*

Proof. Assume that for any $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathcal{F}(v)$ is pure injective R-module. Thus for any $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathcal{F}(v)$ is cotorsion. So by [11, Theorem 2.6], $\mathcal{F}$ is cotorsion object in $\mathrm{Rep}(\mathcal{Q}, R)$. Then by Theorem 3.4 it is pure injective.

The converse is trivial. □

Let $\mathrm{Pinj}(\mathcal{Q})$ be the class of all pure injective objects in $\mathrm{Rep}(\mathcal{Q}, R)$. In Section 2, we proved that $\mathrm{Pinj}(\mathcal{Q})$ is preenveloping. So every object $\mathcal{X}$ in $\mathrm{Rep}(\mathcal{Q}, R)$ has a unique(up to homotopy equivalence) pure injective resolution. Then for a given representation $\mathcal{X}$ of $\mathcal{Q}$, the pure injective dimension of $\mathcal{X}$ can be defined as follows

$$\mathrm{pid}\mathcal{X} = \min\{n \mid \mathcal{X} \text{ has a pure injective resolution of lenght } n\}.$$

Recall that a ring R is called n-perfect if for each flat R-module F, $\mathrm{cd}F$(the cotorsion dimension of F) is at most n (for more details see [7]). In the following theorem we show that, if R is n-perfect, then

$$\begin{aligned} & \sup\{\mathrm{cd}\mathcal{F} \mid \text{for each flat representation } \mathcal{F}\} \\ = \; & \sup\{\mathrm{pid}\mathcal{F} \mid \text{for each flat representation } \mathcal{F}\}. \end{aligned}$$

Theorem 3.6. *Let R be an n-perfect ring and $\mathcal{F}$ be a flat representation of $\mathcal{Q}$. Then* $\mathrm{pid}\mathcal{F} \leq n$*.*

Proof. Let $\mathcal{F}$ be a flat representation of $\mathcal{Q}$ and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\sigma} \mathcal{C}^0 \xrightarrow{\delta^0} \mathcal{C}^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \mathcal{C}^n \xrightarrow{\delta^n} \cdots,$$

be a pure injective resolution of $\mathcal{X}$ such that

$$\mathcal{C}^i = (\mathcal{C}^{i-1}/\mathrm{Im}\delta^{i-2})^{++}$$

for each $i \geq 2$. By Proposition 2.6(i) and Lemma 3.3, for each $i \geq 1$, $\mathrm{Coker}\delta^{i-1}$ is a flat representation of $\mathcal{Q}$. Furthermore, for all $v \in \mathrm{V}_{\mathcal{Q}}$, the exact sequence

$$0 \longrightarrow \mathcal{F}(v) \xrightarrow{\sigma(v)} \mathcal{C}^0(v) \xrightarrow{\delta^0(v)} \mathcal{C}^1(v) \xrightarrow{\delta^1(v)} \cdots \xrightarrow{\delta^{n-1}(v)} \mathcal{C}^n(v) \xrightarrow{\delta^n(v)} \cdots,$$

is a pure injective resolution of $\mathcal{F}(v)$ by pure injective flat R-modules. We know that for any $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathrm{cd}\mathcal{F}(v) \leq n$. Then for any $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathrm{Coker}\delta^{n-1}(v)$ is a cotorsion flat R-module. Therefore by [11, Theorem 2.6] $\mathrm{Coker}\delta^{n-1}$ is cotorsion flat, and by Theorem 3.4, it is pure injective flat in $\mathrm{Rep}(\mathcal{Q}, R)$. Then $\mathrm{pid}\mathcal{F} \leq n$. □

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