

WEAK AND STRONG CONVERGENCE THEOREMS FOR A SYSTEM OF MIXED EQUILIBRIUM PROBLEMS AND A NONEXPANSIVE MAPPING IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce an iterative sequence for finding solution of a system of mixed equilibrium problems and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Then, the weak and strong convergence theorems are proved under some parameters controlling conditions. Moreover, we apply our result to fixed point problems, system of equilibrium problems, general system of variational inequalities, mixed equilibrium problem, equilibrium problem and variational inequality.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . Recall that a mapping $f : C \rightarrow C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A mapping $T : C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $F(T)$. Fixed point iterations process for nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations. In 1953, W. R. Mann [14] introduced Mann iterative process defined by

$$(1.1) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n,$$

where $\alpha_n \in [0, 1]$ and satisfies the assumptions $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and proved the convergence of $\{x_n\}$ to a point y implies that $Ty = y$.

Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function and $F : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction, that is, $F(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x^* \in C$ such that

$$(1.2) \quad F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C.$$

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Denote the set of solution of (1.2) by $MEP(F, \varphi)$.

In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x^* \in C$ such that

$$(1.3) \quad F(x^*, y) \geq 0 \text{ for all } y \in C.$$

The set of solutions of (1.3) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3). The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases (see, e.g., [2, 3, 5, 6, 7, 8, 9, 13, 15, 20, 21, 22, 27, 31, 32]).

Let $F_1, F_2 : C \times C \rightarrow \mathbb{R}$ be two monotone bifunctions and $\lambda, \mu > 0$ are two constants. In 2009, Moudafi [16] considered the following problem for finding $(x^*, y^*) \in C \times C$ such that finding $(x^*, y^*) \in C \times C$ such that

$$(1.4) \quad \begin{cases} F_1(x^*, z) + \frac{1}{\lambda} \langle y^* - x^*, x^* - z \rangle \geq 0, & \forall z \in C, \\ F_2(y^*, z) + \frac{1}{\lambda} \langle x^* - y^*, y^* - z \rangle \geq 0, & \forall z \in C. \end{cases}$$

Using the following alternating equilibrium algorithm: $(x_0, y_0) \in H \times H$ given, $\{x_k\}$ and $\{y_k\}$ be sequences in H defined by

$$(1.5) \quad \begin{cases} \lambda F_1(x_{k+1}, z) + \langle y_k - x_{k+1}, x_{k+1} - z \rangle \geq 0, & \forall z \in C, \\ \lambda F_2(y_{k+1}, z) + \langle x_{k+1} - y_{k+1}, y_{k+1} - z \rangle \geq 0, & \forall z \in C. \end{cases}$$

Moudafi proved that the sequence (x_k, y_k) weakly converges to a solution of the problem (1.4). Further, if we add up the requirement that $x^* = y^*$, then problem (1.4) reduce to the equilibrium problem (1.3).

In this paper, we consider the following problem for finding $(x^*, y^*) \in C \times C$ such that

$$(1.6) \quad \begin{cases} F_1(x^*, z) + \varphi(z) - \varphi(x^*) + \frac{1}{\lambda} \langle y^* - x^*, x^* - z \rangle \geq 0, & \forall z \in C, \\ F_2(y^*, z) + \varphi(z) - \varphi(y^*) + \frac{1}{\mu} \langle x^* - y^*, y^* - z \rangle \geq 0, & \forall z \in C, \end{cases}$$

which is called a system of mixed equilibrium problems. In particular, if $\lambda = \mu$, then problem (1.6) reduce to finding $(x^*, y^*) \in C \times C$ such that

$$(1.7) \quad \begin{cases} F_1(x^*, z) + \varphi(z) - \varphi(x^*) + \frac{1}{\lambda} \langle y^* - x^*, x^* - z \rangle \geq 0, & \forall z \in C, \\ F_2(y^*, z) + \varphi(z) - \varphi(y^*) + \frac{1}{\lambda} \langle x^* - y^*, y^* - z \rangle \geq 0, & \forall z \in C. \end{cases}$$

If $\varphi = 0$ and $\lambda = \mu$, then the problem (1.6) reduce to problems (1.7).

The system of nonlinear variational inequalities close to these introduce by Verma [28] are also a special case: by taking $\varphi = 0$, $F_1(x, y) = \langle A(x), y - x \rangle$ and $F_2(x, y) = \langle B(x), y - x \rangle$, where $A, B : C \rightarrow H$ are two nonlinear mappings. In this case, we can reformulate problem (1.7) to finding $(x^*, y^*) \in C \times C$ such that

$$(1.8) \quad \begin{cases} \langle \lambda A(x^*) + x^* - y^*, z - x^* \rangle \geq 0, & \forall z \in C, \\ \langle \mu B(y^*) + y^* - x^*, z - y^* \rangle \geq 0, & \forall z \in C, \end{cases}$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. Moreover, if we add up the requirement that $x^* = y^*$, then problem (1.8) reduce to the classical variational inequality $VI(A, C)$.

In this paper, we organize as follows. In Section 2, we present some basic concepts and useful lemmas for proving the convergence results of this paper. In Section 3, we introduce two iterative sequences (3.1) and (3.5). Moreover, we prove weak and strong convergence theorems for finding solution of a system of mixed equilibrium problems and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Finally, we apply our result to fixed point problems, system of equilibrium problems, general system of variational inequalities, mixed equilibrium problem, equilibrium problem and variational inequality.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H . For every point $x \in H$, there exists a unique nearest point in C , denote by P_Cx , such that

$$\|x - P_Cx\| \leq \|x - y\| \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C . It is well-known that P_C is a nonexpansive mapping of H onto C and satisfies

$$(2.1) \quad \langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2$$

for all $x, y \in H$. Moreover, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$(2.2) \quad \langle x - P_Cx, y - P_Cy \rangle \leq 0,$$

$$(2.3) \quad \|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cy\|^2$$

for all $x \in H, y \in C$. Further, for all $x \in H$ and $y \in C, y = P_Cx$ if and only if $\langle x - y, y - z \rangle \geq 0, \forall z \in C$.

A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}_{n=1}^\infty$ in X which converges weakly to point $x \in X$, we have

$$(2.4) \quad \liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \forall y \in X, y \neq x$$

and

$$(2.5) \quad \limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \forall y \in X, y \neq x.$$

Next, we collect some lemmas which will be use in the next section.

Lemma 2.1 ([18]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2.$$

Lemma 2.2 ([12]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - T)x_n\}$ converges strongly to y , then $(I - T)x = y$.*

Lemma 2.3 ([26]). *Let C be a closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H . Suppose that for all $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|$$

for every $n = 0, 1, 2, \dots$. Then, $\{P_C x_n\}$ converges strongly to some $z \in C$.

Lemma 2.4 ([29]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

For solving the mixed equilibrium problems for an equilibrium bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\lim_{t \rightarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;
- (B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0;$$

- (B2) C is a bounded set.

The following lemmas appears implicitly in [3] and [11].

Lemma 2.5 ([3]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into \mathbb{R} satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.6 ([11]). *Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

- (1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

By a similar argument as in the proof of Lemma 2.3 in [19], we have the following result.

Lemma 2.7 ([19]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $F : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction satisfying (A1)-(A4) and let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. For $r > 0$ and $x \in H$, define a mapping $S_r(x) : H \rightarrow C$ as follows.*

(2.6)

$$S_r(x) = \{y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle y - x, z - y \rangle \geq 0, \forall z \in C\} \quad \forall x \in H.$$

Then, the following results hold:

(i) For each $x \in H$, $S_r(x) \neq \emptyset$;

(ii) S_r is single-valued;

(iii) S_r is firmly nonexpansive, i.e., for any $x, y \in H$

$$(2.7) \quad \|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle;$$

(iv) $F(S_r) = MEF(F, \varphi)$;

(v) $MEF(F, \varphi)$ is closed and convex.

Lemma 2.8. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two mappings from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4) and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.7 associated to F_1 and F_2 , respectively. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.6) if and only if x^* is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$G(x) = S_{1,\lambda}(S_{2,\mu}x), \quad \forall x \in C,$$

where $y^* = S_{2,\mu}x^*$.

Proof. By a similar argument as in the proof of Proposition 2.1 in [16], we have obtain the result. \square

We note from Lemma 2.7 that the mapping G is nonexpansive. Moreover, if C is a closed bounded convex subset of H , then the solution of problem (1.6) always exists. Throughout this paper, we denote the set of solutions of (1.6) and (1.7) by Ω and Ω_1 , respectively.

3. Main result

In this section, we prove weak and strong convergence theorems for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the system of mixed equilibrium problems.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda, \mu > 0$ and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.7 associated to F_1 and F_2 , respectively. Let T be a nonexpansive mapping of C into itself such that $F(T) \cap \Omega \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.1) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(T) \cap \Omega} x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.6), where $\bar{y} = S_{2,\mu} \bar{x}$.

Proof. Let $x^* \in F(T) \cap \Omega$. Then $x^* = T x^*$ and $x^* = S_{1,\lambda}(S_{2,\mu} x^*)$. Put $y^* = S_{2,\mu} x^*$, $y_n = S_{1,\lambda} z_n$ and $z_n = S_{2,\mu} x_n$. Since

$$\begin{aligned} \|y_n - x^*\| &= \|S_{1,\lambda} z_n - S_{1,\lambda} y^*\| \\ &\leq \|z_n - y^*\| \\ &= \|S_{2,\mu} x_n - S_{2,\mu} x^*\| \\ &\leq \|x_n - x^*\|, \end{aligned}$$

it follows by Lemma 2.1 that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &= \|\alpha_n x_n + (1 - \alpha_n) T y_n\|^2 \\ &= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(T y_n - x^*)\|^2 \\ &= \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|T y_n - x^*\|^2 - \alpha_n(1 - \alpha_n) \|T y_n - x_n\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 - \alpha_n(1 - \alpha_n) \|T y_n - x_n\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - \alpha_n(1 - \alpha_n) \|T y_n - x_n\|^2 \\ &= \|x_n - x^*\|^2 - \alpha_n(1 - \alpha_n) \|T y_n - x_n\|^2 \\ (3.2) \quad &\leq \|x_n - x^*\|^2. \end{aligned}$$

Hence $\{\|x_{n+1} - x^*\|\}$ is a decreasing sequence and therefore $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. This implies that $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{T y_n\}$ are bounded. From (3.2), we note that

$$\alpha_n(1 - \alpha_n) \|T y_n - x_n\| \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Since $0 < a \leq \alpha_n \leq b < 1$ and $\lim_{n \rightarrow \infty} \|x_n - x^*\|^2 = \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\|^2$, we have

$$a(1 - b) \|T y_n - x_n\| \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \rightarrow 0.$$

This implies that $\lim_{n \rightarrow \infty} \|Ty_n - x_n\| = 0$. Since $S_{1,\lambda}$ and $S_{2,\mu}$ are firmly nonexpansive, it follows that

$$\begin{aligned} \|z_n - y^*\|^2 &= \|S_{2,\mu}x_n - S_{2,\mu}x^*\|^2 \\ &\leq \langle S_{2,\mu}x_n - S_{2,\mu}x^*, x_n - x^* \rangle \\ &= \frac{1}{2}(\|z_n - y^*\|^2 + \|x_n - x^*\|^2 - \|z_n - y^* - x_n + x^*\|^2), \end{aligned}$$

and so $\|z_n - y^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n + x^* - y^*\|^2$. By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Ty_n\|^2 \\ &= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(Ty_n - x^*)\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|Ty_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|z_n - y^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) [\|x_n - x^*\|^2 - \|z_n - x_n + x^* - y^*\|^2]. \end{aligned}$$

This implies that

$$(1 - \alpha_n) \|z_n - x_n + x^* - y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Since $0 < a \leq \alpha_n \leq b < 1$ and $\lim_{n \rightarrow \infty} \|x_n - x^*\|^2 = \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\|^2$, we obtain $\lim_{n \rightarrow \infty} \|z_n - x_n + x^* - y^*\| = 0$. Similarly, we note that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|S_{1,\lambda}z_n - S_{1,\lambda}y^*\|^2 \\ &\leq \langle S_{1,\lambda}z_n - S_{1,\lambda}y^*, z_n - y^* \rangle \\ &= \frac{1}{2}(\|y_n - x^*\|^2 + \|z_n - y^*\|^2 - \|y_n - x^* - z_n + y^*\|^2) \\ &\leq \frac{1}{2}(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2), \end{aligned}$$

and so $\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2$. Thus, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) [\|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2]. \end{aligned}$$

Hence

$$(1 - \alpha_n) \|y_n - z_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Since $0 < a \leq \alpha_n \leq b < 1$ and $\lim_{n \rightarrow \infty} \|x_n - x^*\|^2 = \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\|^2$, it follows that $\lim_{n \rightarrow \infty} \|y_n - z_n - x^* + y^*\| = 0$. Hence

$$\|Ty_n - y_n\| \leq \|Ty_n - x_n\| + \|x_n - z_n - x^* + y^*\| + \|z_n - y_n + x^* - y^*\| \rightarrow 0,$$

and therefore

$$\|y_n - x_n\| \leq \|y_n - Ty_n\| + \|Ty_n - x_n\| \rightarrow 0.$$

Since

$$\|Tx_n - x_n\| \leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \leq \|x_n - y_n\| + \|Ty_n - x_n\|,$$

we get $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to \bar{x} . By Lemma 2.2, we have $\bar{x} \in F(T)$. Let G be a mapping which defined as in Lemma 2.8. Thus, we have

$$\|y_n - G(y_n)\| = \|S_{1,\lambda}S_{2,\mu}x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \leq \|x_n - y_n\|,$$

and so

$$\|x_n - G(x_n)\| \leq \|x_n - y_n\| + \|y_n - G(y_n)\| + \|G(y_n) - G(x_n)\| \leq 3\|x_n - y_n\| \rightarrow 0.$$

By Lemma 2.2 and Lemma 2.8, we have $\bar{x} \in \Omega$ and hence $\bar{x} \in F(T) \cap \Omega$. Let $\{x_{n_j}\}$ be another sequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to \hat{x} . We next show that $\bar{x} = \hat{x}$. Suppose that $\bar{x} \neq \hat{x}$. By the Opial's condition, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - \bar{x}\| &= \liminf_{i \rightarrow \infty} \|x_{n_i} - \bar{x}\| < \liminf_{i \rightarrow \infty} \|x_{n_i} - \hat{x}\| = \lim_{n \rightarrow \infty} \|x_n - \hat{x}\| \\ &= \liminf_{j \rightarrow \infty} \|x_{n_j} - \hat{x}\| < \liminf_{j \rightarrow \infty} \|x_{n_j} - \bar{x}\| = \lim_{n \rightarrow \infty} \|x_n - \bar{x}\|. \end{aligned}$$

This is a contradiction. Thus, we have $\bar{x} = \hat{x}$. This implies that $\{x_n\}$ converges weakly to $\bar{x} \in F(T) \cap \Omega$. Put $u_n = P_{F(T) \cap \Omega}x_n$. Finally, we show that $\bar{x} = \lim_{n \rightarrow \infty} u_n$. From (2.1) and $\bar{x} \in F(T) \cap \Omega$, we have

$$\langle \bar{x} - u_n, u_n - x_n \rangle \geq 0.$$

By Lemma 2.3, $\{u_n\}$ converges strongly to $\hat{x} \in F(T) \cap \Omega$ and hence

$$\langle \bar{x} - \hat{x}, \hat{x} - \bar{x} \rangle \geq 0.$$

This conclude that $\bar{x} = \hat{x}$. □

Setting $\mu = \lambda$ in Theorem 3.1, we have following result.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda > 0$ and let $S_{1,\lambda}$ and $S_{2,\lambda}$ be defined as in Lemma 2.7 associated to F_1 and F_2 respectively. Let T be a nonexpansive mapping of C into itself such that $F(T) \cap \Omega \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.3) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\lambda} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(T) \cap \Omega}x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.7), where $\bar{y} = S_{2,\lambda}\bar{x}$.

Setting $\varphi = 0$ in Theorem 3.1, we have following result.

Corollary 3.3. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda, \mu > 0$ and let $T_{1,\lambda}$ and $T_{2,\mu}$ be defined as in Lemma 2.6 associated to F_1 and F_2 , respectively. Let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega_1 \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.4) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(S) \cap \Omega_1} x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.7), where $\bar{y} = T_{2,\mu} \bar{x}$.

Theorem 3.4. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda, \mu > 0$ and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.7 associated to F_1 and F_2 respectively. Let T be a nonexpansive mapping of C into itself such that $F(T) \cap \Omega \neq \emptyset$. Suppose $x_0, u \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.5) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. If $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(T) \cap \Omega} u$ and (\bar{x}, \bar{y}) is a solution of problem (1.6), where $\bar{y} = S_{2,\mu} \bar{x}$.

Proof. Let $x^* \in F(T) \cap \Omega$. Then $x^* = T x^*$ and $x^* = S_{1,\lambda}(S_{2,\mu} x^*)$. Put $y^* = S_{2,\mu} x^*, y_n = S_{1,\lambda} z_n$ and $z_n = S_{2,\mu} x_n$. Thus, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + (1 - \alpha_n) T y_n - x^*\| \\ &\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|y_n - x^*\| \\ &= \alpha_n \|u - x^*\| + (1 - \alpha_n) \|S_{1,\lambda} z_n - S_{1,\lambda} y^*\| \\ &\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|z_n - y^*\| \\ &= \alpha_n \|u - x^*\| + (1 - \alpha_n) \|S_{2,\mu} x_n - S_{2,\mu} x^*\| \\ &\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}. \end{aligned}$$

By introduction, we obtain

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_0 - x^*\|\}$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded and hence the sequences $\{y_n\}$, $\{z_n\}$ and $\{Ty_n\}$ are also bounded. Moreover, we observe that

$$(3.6) \quad \begin{aligned} \|y_{n+1} - y_n\| &= \|S_{1,\lambda}z_{n+1} - S_{1,\lambda}z_n\| \leq \|z_{n+1} - z_n\| \\ &= \|S_{2,\mu}x_{n+1} - S_{2,\mu}x_n\| \leq \|x_{n+1} - x_n\| \end{aligned}$$

and

$$\begin{aligned} &\|x_{n+2} - x_{n+1}\| \\ &= \|\alpha_{n+1}u + (1 - \alpha_{n+1})Ty_{n+1} - \alpha_n u - (1 - \alpha_n)Ty_n\| \\ &= \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})(Ty_{n+1} - Ty_n) + (\alpha_n - \alpha_{n+1})Ty_n\| \\ &\leq |\alpha_{n+1} - \alpha_n|(\|u\| + \|Ty_n\|) + (1 - \alpha_{n+1})\|y_{n+1} - y_n\| \\ &\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\|u\| + \|Ty_n\|). \end{aligned}$$

By Lemma 2.4, we obtain $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Consequently, $\|y_{n+1} - y_n\| \rightarrow 0$ and $\|z_{n+1} - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since

$$x_{n+1} - x_n = \alpha_n(u - x_n) + (1 - \alpha_n)(Ty_n - x_n),$$

it follows from $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ that

$$\lim_{n \rightarrow \infty} \|Ty_n - x_n\| = 0.$$

As in the proof of Theorem 3.1, we obtain

$$\|z_n - y^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n + x^* - y^*\|^2$$

and

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2.$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|z_n - y^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) [\|x_n - x^*\|^2 - \|z_n - x_n + x^* - y^*\|^2] \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) [\|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2]. \end{aligned}$$

This implies that

$$\begin{aligned} &(1 - \alpha_n) \|z_n - x_n + x^* - y^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| - \|x_{n+1} - x^*\|) \end{aligned}$$

and

$$\begin{aligned} &(1 - \alpha_n) \|y_n - z_n - x^* + y^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \end{aligned}$$

$$\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| (\|x_n - x^*\| - \|x_{n+1} - x^*\|).$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, it follows that

$$\lim_{n \rightarrow \infty} \|z_n - x_n + x^* - y^*\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|y_n - z_n - x^* + y^*\| = 0.$$

Hence

$$\|Ty_n - y_n\| \leq \|Ty_n - x_n\| + \|x_n - z_n - x^* + y^*\| + \|z_n - y_n + x^* - y^*\| \rightarrow 0$$

and therefore

$$\|y_n - x_n\| \leq \|y_n - Ty_n\| + \|Ty_n - x_n\| \rightarrow 0.$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{x}, x_n - \bar{x} \rangle \leq 0,$$

where $\bar{x} = P_{F(T) \cap \Omega} u$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{x}, Ty_n - \bar{x} \rangle = \lim_{i \rightarrow \infty} \langle u - \bar{x}, Ty_{n_i} - \bar{x} \rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z . Without loss of generality, we can assume that $y_{n_i} \rightharpoonup z$. From $\|Ty_n - y_n\| \rightarrow 0$, we obtain $Ty_{n_i} \rightharpoonup z$. By Lemma 2.2, we have $z \in F(T)$. Let G be a mapping which defined in Lemma 2.8. Thus, we have

$$\|y_n - G(y_n)\| = \|S_{1,\lambda} S_{2,\mu} x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \leq \|x_n - y_n\| \rightarrow 0.$$

According to Lemma 2.2 and Lemma 2.8, we have $z \in \Omega$ and hence $z \in F(T) \cap \Omega$. Now from $\bar{x} = P_{F(T) \cap \Omega} u$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - \bar{x}, x_n - \bar{x} \rangle &= \limsup_{n \rightarrow \infty} \langle u - \bar{x}, Ty_n - \bar{x} \rangle \\ &= \lim_{i \rightarrow \infty} \langle u - \bar{x}, Ty_{n_i} - \bar{x} \rangle \\ (3.7) \qquad \qquad \qquad &= \langle u - \bar{x}, z - \bar{x} \rangle \leq 0. \end{aligned}$$

Finally, we note that

$$\begin{aligned} &\|x_{n+1} - \bar{x}\|^2 \\ &= \langle \alpha_n u + (1 - \alpha_n)Ty_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &= \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - \alpha_n) \langle Ty_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - \alpha_n) [\|Ty_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2] \\ &\leq \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2}(1 - \alpha_n) \|y_n - \bar{x}\|^2 + \frac{1}{2}(1 - \alpha_n) \|x_{n+1} - \bar{x}\|^2 \\ &\leq \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - \alpha_n) \|x_n - \bar{x}\|^2 + \frac{1}{2} \|x_{n+1} - \bar{x}\|^2 \\ (3.8) \quad &\leq (1 - \alpha_n) \|x_n - \bar{x}\|^2 + 2\alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle. \end{aligned}$$

It follows from (3.7), (3.8) and Lemma 2.4, that $x_n \rightarrow \bar{x}$. This completes the proof. \square

Setting $\mu = \lambda$ in Theorem 3.4, we have following result.

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H . Let F_1, F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda > 0$ and let $S_{1,\lambda}, S_{2,\lambda}$ be defined as in Lemma 2.7 associated to F_1, F_2 respectively. Let T be a nonexpansive mapping of C into itself such that $F(T) \cap \Omega_1 \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.9) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\lambda} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(T) \cap \Omega_1} x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.7), where $\bar{y} = S_{2,\lambda} \bar{x}$.

Setting $\varphi = 0$ in Theorem 3.4, we have following result.

Corollary 3.6. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $\lambda, \mu > 0$ and let $T_{1,\lambda}$ and $T_{2,\mu}$ be defined as in Lemma 2.6 associated to F_1 and F_2 , respectively. Let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega_1 \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by*

$$(3.10) \quad \begin{cases} z_n \in C; F_2(z_n, z) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle \geq 0, & \forall z \in C, \\ y_n \in C; F_1(y_n, z) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle \geq 0, & \forall z \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) S y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(S) \cap \Omega_1} x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.7), where $\bar{y} = T_{2,\mu} \bar{x}$.

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