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BIFURCATION ANALYSIS OF A DELAYED PREDATOR-PREY MODEL OF PREY MIGRATION AND PREDATOR SWITCHING

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Abstract. In this paper, a class of delayed predator-prey models of prey migration and predator switching is considered. By analyzing the associated characteristic transcendental equation, its linear stability is investigated and Hopf bifurcation is demonstrated. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Some numerical simulations for justifying the theoretical analysis are also provided. Finally, biological explanations and main conclusions are given.

1. Introduction

In recent years, population dynamics (including stable, unstable, persistent and oscillatory behavior) has become very popular since Vito Volterra and James Lotka proposed the seminal models of predator-prey models in the mid-1920s [5, 7-15]. Great attention has been paid to the dynamics properties of the predator-prey models which have significant biological background. Many excellent and interesting results have been obtained [6-26]. It is well known that in a prey-predator environment, there is an inherent tendency among the predator species to feed itself in a habit for some duration and then change its preference to some other habit (this preferential phenomenon of change of habit by the predator is called switching). Tansky [14] investigated the mathematical

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model of one-predator-prey system which has switching property of predation in the following form:

$$\text{(1)}\qquad \begin{cases} \dot{x} = \left[E_1 - \frac{az}{1+(y/x)^n}\right] x, \\ \dot{y} = \left[E_2 - \frac{bz}{1+(x/y)^n}\right] y, \\ \dot{z} = -E_3 + \frac{axz}{1+(y/x)^n} + \frac{byz}{1+(x/y)^n}, \end{cases}$$

$(n = 1, 2, 3, \ldots)$, where x, y represent the prey densities and z the predator density. The functions $[a/(1+(y/x)^n]$ and $[b/(1+(x/y)^n]$ have the characteristic property of switching mechanism. Biologically, these functions signify the fact that the predator rate, i.e., the frequency with which an individual of the prey species is attacked by a predator, decreases when the population of species becomes rare compared to the population of the other species. For $n = 1$, these functions represent a simple multiplicative effect [14], whereas for $n > 1$, these functions exhibit an effect that is stronger than the multiplicative one [17].

It is known to all that a commonly observed phenomenon is the migration of populations of differential species. Considering the seasonal migration of prey population and switching mechanism of the predator, Bhattacharyya and Mukhopadhyay [16] studied the following model under the conditions: $n = 1$ and $n = 2$.

$$\text{(2)}\qquad \begin{cases} \dot{x}_1(t) = x_1(t)\left[g_1(1 - \frac{x_1}{k_1}) - \frac{\beta_1 y}{1+(x_1/x_2)^n}\right], \\ \dot{x}_2(t) = x_2(t)\left[g_2(1 - \frac{x_2}{k_2}) - \frac{\beta_2 y}{1+(x_2/x_1)^n}\right], \\ \dot{y}(t) = -\mu y + \frac{\delta_1 x_1 y}{1+(x_1/x_2)^n} + \frac{\delta_2 x_2 y}{1+(x_2/x_1)^n}, \end{cases}$$

where x_1 and x_2 denote prey density in two habits, and y the predator density. The prey population is assumed to grow logistically with a specific growth rate g_i and environmental carrying k_i, β_1 and β_2 represent the predation rate in the two habitats, δ_1, δ_2 are the corresponding conversion rates. The predation functions $\beta_1 x_1 y/(1 + (x_1/x_2)^n)$ and $\beta_2 x_2 y/(1 + (x_2/x_1)^n)$ model the switching behavior of the predator in the realm of prey group defence, i.e., there will be less predation in the habitat having larger prey density.

Inspired by the work of [14, 16] and considering the factor that the reproduction of predator after predating the prey will not be instantaneous, but mediated by some discrete time-delay required for gestation of predator [13], we revise model (2) into the following delayed predator-prey model of prey migration and predator switching:

$$\text{(3)}\qquad \begin{cases} \dot{x}_1(t) = x_1(t)\left[g_1(1 - \frac{x_1}{k_1}) - \frac{\beta_1 y(t-\tau) x_2(t-\tau)}{x_1+x_2}\right], \\ \dot{x}_2(t) = x_2(t)\left[g_2(1 - \frac{x_2}{k_2}) - \frac{\beta_2 y(t-\tau) x_1(t-\tau)}{x_1+x_2}\right], \\ \dot{y}(t) = -\mu y + \frac{\delta_1 x_1 x_2(t-\tau) y}{x_1+x_2} + \frac{\delta_2 x_1(t-\tau) x_2(t-\tau) y}{x_1+x_2}, \end{cases}$$

where $g_i,\ k_i,\ \beta_i,\ \delta_i,\ \mu$ are all positive constants, $n = 1, 2, 3, \ldots$. The more detail biological meaning of the coefficients of the system (3), one can see [14] or [16].

In this paper, we study the stability, the local Hopf bifurcation for system (3). To the best of our knowledge, it is the first time to deal with the research of Hopf bifurcation for model (3).

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Biological explanations and some main conclusions are drawn in Section 5.

2. Stability of the positive equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations. It is easy to see that Eq.(3) has an interior equilibrium $E_0(x_1^*, x_2^*, y^*)$, where

$$\begin{aligned}
x_1^* &= \frac{\mu(1+x_r)}{\delta}, \\
x_2^* &= \frac{\mu(1+x_r)}{x_r\delta}, \\
y^* &= \frac{g_1}{\beta_1}(1+x_r)\left[1-\frac{\mu}{\delta k_1}(1+x_r)\right] \\
&= \frac{g_2}{\beta_2}(1+x_r)\left[1-\frac{\mu(1+x_r)}{\delta k_2 x_r}\right],
\end{aligned}$$

where $\delta = \delta_1 + \delta_2$ and $x_r = x_1^*/x_2^*$ is the real positive root of the following cubic equation:

$$(g_1\beta_2 k_2\mu)x_r^3 + [g_1\beta_2 k_2(\mu - \delta k_1)]x_r^2 + [g_2\beta_1 k_1(k_2\delta - \mu)]x_r - g_2\mu\beta_1 k_1 = 0.$$

We make the following assumptions:

(H1) $\qquad \mu(1+x_r) < \min\{\delta k_1, \delta k_2 x_r\},$

(H2) $\qquad \mu(1+x_r) > \delta_1.$

Obviously, the interior equilibrium $E_0(x_1^*, x_2^*, y^*)$ is positive equilibrium if the condition (H1) holds.

Let $\bar{x}_1(t) = x_1(t) - x_1^*$, $\bar{x}_2(t) = x_2(t) - x_2^*$, $\bar{y}(t) = y(t) - y^*$ and still denote $\bar{x}_i(t)(i = 1, 2)$, $\bar{y}(t)$ by $x_i(t)(i = 1, 2)$, $y(t)$, respectively. Then (3) becomes

$$(4)\quad \begin{cases} \dot{x}_1(t) = m_1x_1(t) + m_2x_2(t) + m_3x_2(t-\tau) + m_4y(t-\tau) \\ \qquad + n_1x_1^2(t) + n_2x_1(t)x_2(t) + n_3x_2^2(t) + n_4x_1(t)x_2(t-\tau) \\ \qquad + n_5x_1(t)y(t-\tau) + n_6x_2(t-\tau)y(t-\tau) + n_7x_2(t)x_2(t-\tau) \\ \qquad + n_8x_2(t)y(t-\tau) + l_1x_1(t)x_2(t-\tau)y(t-\tau) \\ \qquad + l_1x_2(t)x_2(t-\tau)y(t-\tau) + l_2x_1^2(t)x_2(t-\tau) \\ \qquad + l_3x_1(t)x_2(t)x_2(t-\tau) + l_4x_2^2(t)x_2(t-\tau) + l_5x_1^2(t)y(t-\tau) \\ \qquad + l_6x_1(t)x_2(t)y(t-\tau) + l_5x_2^2(t)y(t-\tau) + l_7x_1^3(t) \\ \qquad + l_8x_1^2(t)x_2(t) + l_8x_1(t)x_2^2(t) + l_7x_2^3(t), \\ \dot{x}_2(t) = p_1x_2(t) + p_2x_1(t) + p_3x_1(t-\tau) + p_4y(t-\tau) \\ \qquad + q_1x_2^2(t) + q_2x_2(t)x_1(t) + q_3x_1^2(t) + q_4x_2(t)x_1(t-\tau) \\ \qquad + q_5x_2(t)y(t-\tau) + q_6x_1(t-\tau)y(t-\tau) + q_7x_1(t)x_1(t-\tau) \\ \qquad + q_8x_1(t)y(t-\tau) + s_1x_2(t)x_1(t-\tau)y(t-\tau) \\ \qquad + s_1x_1(t)x_1(t-\tau)y(t-\tau) + s_2x_2^2(t)x_1(t-\tau) \\ \qquad + s_3x_2(t)x_1(t)x_1(t-\tau) + s_4x_1^2(t)x_2(t-\tau) + s_5x_2^2(t)y(t-\tau) \\ \qquad + s_6x_2(t)x_1(t)y(t-\tau) + s_5x_1^2(t)y(t-\tau) + s_7x_2^3(t) \\ \qquad + s_8x_2^2(t)x_1(t) + s_8x_2(t)x_1^2(t) + s_7x_1^3(t), \\ \dot{y}(t) = u_1y(t) + u_2x_1(t)y(t) + u_3x_2(t-\tau)y(t) + u_4x_2(t)y(t) \\ \qquad + v_1x_1^2(t)y(t) + v_2x_1(t)x_2(t)y(t) + v_3x_2^2(t)y(t) \\ \qquad + v_4x_1(t)x_2(t-\tau)y(t) + v_4x_2(t)x_2(t-\tau)y(t), \end{cases}$$

where

$$m_1 = g_1 - \frac{x_1^*}{k_1} - \frac{\beta_1x_2^*y^*}{x_1^*+x_2^*} - \frac{\beta_1x_1^*x_2^*y^*}{(x_1^*+x_2^*)^2},\ m_2 = -\frac{\beta_1x_1^*x_2^*y^*}{(x_1^*+x_2^*)^2},$$

$$m_3 = -\frac{\beta_1x_1^*y^*}{x_1^*+x_2^*},\ m_4 = -\frac{\beta_1x_1^*x_2^*}{x_1^*+x_2^*},\ n_1 = -\left[\frac{\beta_1x_2^*y^*}{(x_1^*+x_2^*)^2} + \frac{\beta_1x_1^*x_2^*y^*}{(x_1^*+x_2^*)^3}\right],$$

$$n_2 = -\left[\frac{\beta_1x_2^*y^*}{(x_1^*+x_2^*)^2} + \frac{2\beta_1x_1^*x_2^*y^*}{(x_1^*+x_2^*)^3}\right],\ n_3 = -\frac{2\beta_1x_1^*x_2^*y^*}{(x_1^*+x_2^*)^3},$$

$$n_4 = -\left[\frac{\beta_1y^*}{x_1^*+x_2^*} + \frac{2\beta_1x_1^*y^*}{(x_1^*+x_2^*)^2}\right],\ n_5 = -\frac{2\beta_1x_2^*}{x_1^*+x_2^*},$$

$$n_6 = -\frac{\beta_1x_2^*}{(x_1^*+x_2^*)^2},\ n_7 = -\frac{\beta_1x_1^*y^*}{(x_1^*+x_2^*)^2},\ n_8 = -\left[\frac{\beta_1x_1^*}{x_1^*+x_2^*} + \frac{\beta_1x_1^*x_2^*}{(x_1^*+x_2^*)^2}\right],$$

$$l_1 = -\frac{\beta_1x_1^*}{(x_1^*+x_2^*)^2},\ l_2 = -\frac{\beta_1x_1^*y^*}{(x_1^*+x_2^*)^2},\ l_3 = -\frac{2\beta_1x_1^*y^*}{(x_1^*+x_2^*)^3},\ l_4 = -\frac{\beta_1x_1^*y^*}{(x_1^*+x_2^*)^3},$$

$$l_5=-\frac{\beta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^3},\ l_6=-\frac{2\beta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^3},\ l_7=-\frac{\beta_1 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^4},\ l_8=\frac{3\beta_1 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^4},$$
$$p_1=g_2-\frac{x_2^*}{k_2}-\frac{\beta_2 x_1^* y^*}{x_1^*+x_2^*}-\frac{\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^2},\ p_2=-\frac{\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^2},\ p_3=-\frac{\beta_2 x_2^* y^*}{x_1^*+x_2^*},$$
$$p_4=-\frac{\beta_2 x_1^* x_2^*}{x_1^*+x_2^*},\ q_1=-\left[\frac{\beta_2 x_1^* y^*}{(x_1^*+x_2^*)^2}+\frac{\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^3}\right],$$
$$q_2=-\left[\frac{\beta_2 x_1^* y^*}{(x_1^*+x_2^*)^2}+\frac{2\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^3}\right],\ q_3=-\frac{2\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^3},$$
$$q_4=-\left[\frac{\beta_2 y^*}{x_1^*+x_2^*}+\frac{2\beta_2 x_2^* y^*}{(x_1^*+x_2^*)^2}\right],\ q_5=-\frac{2\beta_2 x_1^*}{x_1^*+x_2^*},\ q_6=-\frac{\beta_2 x_1^*}{(x_1^*+x_2^*)^2},$$
$$q_7=-\frac{\beta_2 x_2^* y^*}{(x_1^*+x_2^*)^2},\ q_8=-\left[\frac{\beta_2 x_2^*}{x_1^*+x_2^*}+\frac{\beta_2 x_1^* x_2^*}{(x_1^*+x_2^*)^2}\right],\ s_1=-\frac{\beta_2 x_2^*}{(x_1^*+x_2^*)^2},$$
$$s_2=-\frac{\beta_2 x_2^* y^*}{(x_1^*+x_2^*)^2},\ s_3=-\frac{2\beta_2 x_2^* y^*}{(x_1^*+x_2^*)^3},\ s_4=-\frac{\beta_2 x_2^* y^*}{(x_1^*+x_2^*)^3},$$
$$s_5=-\frac{\beta_2 x_2^* x_1^*}{(x_1^*+x_2^*)^3},\ s_6=-\frac{2\beta_2 x_1^* x_2^*}{(x_1^*+x_2^*)^3},\ s_7=-\frac{\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^4},$$
$$l_8=\frac{3\beta_2 x_1^* x_2^* y^*}{(x_1^*+x_2^*)^4},\ u_1=\frac{\delta_1 x_1^* x_2^*}{x_1^*+x_2^*}-\mu,\ u_2=\frac{\delta_1 x_1^*}{x_1^*+x_2^*}+\frac{\delta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^2},$$
$$u_3=\frac{\delta_1 x_1^*}{x_1^*+x_2^*},\ u_4=\frac{\delta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^2},\ v_1=\frac{\delta_1 x_2^*}{(x_1^*+x_2^*)^2}+\frac{\delta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^3},$$
$$v_2=\frac{\delta_1 x_2^*}{x_1^*+x_2^*}+\frac{2\delta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^3},\ v_3=\frac{\delta_1 x_1^* x_2^*}{(x_1^*+x_2^*)^3},\ v_4=\frac{\delta_1}{x_1^*+x_2^*}.$$

The linearization of Eq.(4) at $(0,0,0)$ is

$$\text{(5)}\qquad \begin{cases} \dot{x}_1(t)=m_1x_1(t)+m_2x_2(t)+m_3x_2(t-\tau)+m_4y(t-\tau),\\ \dot{x}_2(t)=p_2x_1(t)+p_1x_2(t)+p_3x_1(t-\tau)+p_4y(t-\tau),\\ \dot{y}(t)=u_1y(t), \end{cases}$$

whose characteristic equation is

$$\text{(6)}\quad (\lambda-u_1)\left[\lambda^2-(m_1+p_1)\lambda+m_1p_1-(p_2+p_3)m_2-(p_2+p_3)m_3e^{-\lambda\tau}\right]=0.$$

Obviously, Eq.(6) has the root $\lambda=u_1$. Under the assumption (H2), we know that $\lambda=u_1<0$.

In the following, we only need to investigate the distribution of roots of the following equation:

$$\text{(7)}\qquad \lambda^2-(m_1+p_1)\lambda+m_1p_1-(p_2+p_3)m_2-(p_2+p_3)m_3e^{-\lambda\tau}=0.$$

In order to investigate the distribution of roots of the transcendental equation (7), the following lemma is useful.

Lemma 2.1 ([2])**.** *For the transcendental equation*

$$P(\lambda,e^{-\lambda\tau_1},\cdots,e^{-\lambda\tau_m})=\lambda^n+p_1^{(0)}\lambda^{n-1}+\cdots+p_{n-1}^{(0)}\lambda+p_n^{(0)}$$

$$+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots$$
$$+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0,$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

For $\tau = 0$, (7) becomes

$$\lambda^2 - (m_1 + p_1)\lambda + m_1p_1 - (p_2 + p_3)(m_2 + m_3) = 0. \tag{8}$$

A set of necessary and sufficient conditions for all roots of (8) to have a negative real part is given by the well-known Routh-Hurwitz criteria in the following form:

$$m_1 + p_1 < 0,\ m_1p_1 - (p_2 + p_3)(m_2 + m_3) < 0. \tag{H3}$$

For $\omega > 0, i\omega$ is a root of (7) if and only if

$$-\omega^2 - i(m_1 + p_1)\omega + m_1p_1 - (p_2 + p_3)m_2 - (p_2 + p_3)m_3(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} (p_2 + p_3)m_3 \cos\omega\tau = m_1p_1 - (p_2 + p_3)m_2 - \omega^2, \\ (p_2 + p_3)m_3 \sin\omega\tau = (m_1 + p_1)\omega, \end{cases} \tag{9}$$

which leads to

$$\omega^4 + [m_1^2 + p_1^2 + 2(p_2 + p_3)m_2]\omega^2 + [m_1p_1 - (p_2 + p_3)m_2]^2 - [(p_2 + p_3)m_3]^2 = 0. \tag{10}$$

Let us denote

$$\Delta = [m_1^2 + p_1^2 + 2(p_2 + p_3)m_2]^2 - 4\{[m_1p_1 - (p_2 + p_3)m_2]^2 - [(p_2 + p_3)m_3]^2\}.$$

Then the roots of biquadratic equation (10) are given by

$$\omega_\pm^2 = \frac{1}{2}\Big\{ - [m_1^2 + p_1^2 + 2(p_2 + p_3)m_2] \pm \sqrt{\Delta}\Big\}.$$

In the sequel, we consider the five cases:

(K1) $\Delta < 0$ implies that Eq.(10) has no purely imaginary roots of the form $\pm i\omega$;

(K2) $\Delta > 0, [m_1p_1 - (p_2+p_3)m_2]^2 > [(p_2+p_3)m_3]^2, m_1^2+p_1^2+2(p_2+p_3)m_2 > 0$ imply that Eq.(10) has no purely imaginary roots of the form $\pm i\omega$;

(K3) $\Delta > 0, [m_1p_1 - (p_2+p_3)m_2]^2 < [(p_2+p_3)m_3]^2, m_1^2+p_1^2+2(p_2+p_3)m_2 > 0$ imply that Eq.(10) has one purely imaginary roots of the form $\pm i\omega_+$;

(K4) $\Delta > 0, [m_1p_1 - (p_2+p_3)m_2]^2 < [(p_2+p_3)m_3]^2, m_1^2+p_1^2+2(p_2+p_3)m_2 < 0$ imply that Eq.(10) has a pair of purely imaginary roots of the form $\pm i\omega_+$;

(K5) $\Delta > 0, [m_1p_1 - (p_2+p_3)m_2]^2 > [(p_2+p_3)m_3]^2, m_1^2+p_1^2+2(p_2+p_3)m_2 < 0$ imply that Eq.(10) has two purely imaginary roots of the form $\pm i\omega_\pm$.

For cases (K1) and (K2), the characteristic Eq.(10) has no purely imaginary roots. This shows that the positive interior equilibrium point E_0 is absolutely stable (locally asymptotically stable for all $\tau \geq 0$) under the assumptions (H1)-(H3) with the condition (K1) or (K2).

We now consider cases (K3), (K4) and (K5). In cases (K3) and (K4), Eq.(10) has a pair of purely imaginary roots $\pm i\omega_+$ and ω_+ are given by

$$\omega_+ = \sqrt{\frac{1}{2}\Big\{ - [m_1^2 + p_1^2 + 2(p_2+p_3)m_2] + \sqrt{\Delta}\Big\}}.$$

From (9), we have

$$\sin\omega_+\tau = \frac{(m_1+p_1)\omega_+}{(p_2+p_3)m_3} < 0, \ \cos\omega_+\tau = \frac{m_1p_1 - (p_2+p_3)m_2 - \omega_+^2}{(p_2+p_3)m_3} < 0,$$

and thus

$$\tau_k^+ = \frac{1}{\omega_+}\left[\arcsin\frac{(m_1+p_1)\omega_+}{(p_2+p_3)m_3} + 2k\pi\right] \ (k = 0,1,2,\ldots).$$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (7) near $\tau = \tau_k^+$, and $\alpha(\tau_k^+) = 0$, and $\omega(\tau_k^+) = \omega_+$. Due to functional differential equation theory, for every $\tau_k^+, k = 0,1,2,3,\ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^+| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand of (7) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{2e^{\lambda\tau}}{(p_2+p_3)m_3} + \frac{(m_1+p_1)e^{\lambda\tau}}{(p_2+p_3)m_3} - \frac{\tau}{\lambda}. \tag{11}$$

Then we obtain

$$\begin{aligned}\left[\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_k^+} &= \mathrm{Re}\Big\{-\frac{2e^{\lambda\tau}}{(p_2+p_3)m_3}\Big\}_{\tau=\tau_k^+} + \mathrm{Re}\Big\{\frac{(m_1+p_1)e^{\lambda\tau}}{(p_2+p_3)m_3}\Big\}_{\tau=\tau_k^+} \\ &= \frac{(m_1+p_1)\sin\omega_+\tau_k^+ - 2\omega_+\cos\omega_+\tau_k^+}{\omega_+(p_2+p_3)m_3} > 0.\end{aligned}$$

In case (K5), Eq.(10) has two pair of purely imaginary roots $\pm i\omega_\pm$ and $\omega_\pm$ is given by

$$\omega_\pm = \sqrt{\frac{1}{2}\Big\{ - [m_1^2 + p_1^2 + 2(p_2+p_3)m_2] \pm \sqrt{\Delta}\Big\}}.$$

From (9), we have

$$\sin\omega_\pm\tau = \frac{(m_1+p_1)\omega_\pm}{(p_2+p_3)m_3} < 0, \ \cos\omega_\pm\tau = \frac{m_1p_1 - (p_2+p_3)m_2 - \omega_\pm^2}{(p_2+p_3)m_3} < 0,$$

and thus

$$\tau_k^\pm = \frac{1}{\omega_\pm}\left[\arcsin\frac{(m_1+p_1)\omega_\pm}{(p_2+p_3)m_3} + 2k\pi\right] \ (k = 0,1,2,\ldots). \tag{12}$$

Similarly, let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (7) near $\tau = \tau_k^\pm$, and $\alpha(\tau_k^\pm) = 0$, and $\omega(\tau_k^\pm) = \omega_\pm$. Due to functional differential equation theory, for every

$\tau_k^{\pm}, k = 0, 1, 2, 3, \ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{\pm}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand of (7) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{2e^{\lambda\tau}}{(p_2+p_3)m_3} + \frac{(m_1+p_1)e^{\lambda\tau}}{(p_2+p_3)m_3} - \frac{\tau}{\lambda}.$$

Then we obtain

$$\begin{aligned}
\left[\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_k^+} &= \mathrm{Re}\left\{-\frac{2e^{\lambda\tau_k^+}}{(p_2+p_3)m_3}\right\}_{\tau=\tau_k^+} + \mathrm{Re}\left\{\frac{(m_1+p_1)e^{\lambda\tau_k^+}}{(p_2+p_3)m_3}\right\}_{\tau=\tau_k^+} \\
&= \frac{(m_1+p_1)\sin\omega_+\tau_k^+ - 2\omega_+\cos\omega_+\tau_k^+}{\omega_+(p_2+p_3)m_3} \\
&= \frac{\sqrt{\Delta}}{(p_2+p_3)^2 m_3^2} > 0, \\
\left[\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_k^-} &= \mathrm{Re}\left\{-\frac{2e^{\lambda\tau_k^-}}{(p_2+p_3)m_3}\right\}_{\tau=\tau_k^-} + \mathrm{Re}\left\{\frac{(m_1+p_1)e^{\lambda\tau_k^-}}{(p_2+p_3)m_3}\right\}_{\tau=\tau_k^-} \\
&= \frac{(m_1+p_1)\sin\omega_-\tau_k^- - 2\omega_+\cos\omega_-\tau_k^-}{\omega_-(p_2+p_3)m_3} \\
&= \frac{-\sqrt{\Delta}}{(p_2+p_3)^2 m_3^2} < 0.
\end{aligned}$$

The above analysis leads to the following results on the stability and Hopf bifurcation.

Theorem 2.2. *For system* (3)*,*

(i) *under the conditions* (H1)-(H3)*, if* (K1) *or* (K2) *holds, then the positive interior equilibrium point* E_0 *is locally asymptotically stable for all* $\tau \geq 0$;

(ii) *under the conditions* (H1)-(H3)*, if* (K3) *or* (K4) *holds, the positive interior equilibrium point* E_0 *is locally asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau \geq \tau_0$*. System* (3) *undergoes a Hopf bifurcation at the positive interior equilibrium point* E_0 *when* $\tau = \tau_k^+, k = 0.1, 2, \ldots$;

(iii) *under the conditions* (H1)-(H3)*, if* (K5) *holds, then there exists a positive integer* n *such that the positive interior equilibrium point* E_0 *switches* n *times from stability to instability to stability and so on and the positive interior equilibrium point* E_0 *is locally asymptotically stable whenever* $\tau \in [0, \tau_0^+) \bigcup (\tau_0^-, \tau_1^+) \bigcup \cdots \bigcup (\tau_{n-1}^-, \tau_n^+)$ *and is unstable whenever* $\tau \in (\tau_0^+ \bigcup \tau_0^-) \bigcup (\tau_1^+ \bigcup \tau_1^-) \bigcup \cdots \bigcup (\tau_{n-1}^+, \tau_{n-1}^-)$ *and* $\tau > \tau_n^+$*. System* (3) *undergoes a Hopf bifurcation at the positive interior equilibrium point* E_0 *when* $\tau = \tau_k^{\pm}, k = 0, 1, 2, \ldots$.

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{\pm}, k = 0, 1, 2, \ldots$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ at these critical value of τ, by using techniques from normal form and center manifold theory [1]. Throughout this section, we always assume that system (3) undergoes Hopf bifurcation at the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ for $\tau = \tau_k^{\pm}, k = 0, 1, 2, \ldots$, and then $\pm i\omega_0$ are corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$.

For convenience, let $\bar{x}_i(t) = x_i(\tau t)$ $(i = 1, 2, 3)$ and $\tau = \tau_k^{\pm} + \mu$, where $\tau_k^{\pm}$ is defined by (12) and $\mu \in R$, drop the bar for the simplification of notations, then the system (4) can be written as an FDE in $C = C([-1, 0]), R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{13}$$

where $u(t) = (x_1(t), x_2(t), y(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (x_1(t+\theta), x_2(t+\theta), y(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$\begin{aligned} L_\mu \phi = {} & (\tau_k^{\pm} + \mu) \begin{pmatrix} m_1 & m_2 & 0 \\ p_2 & p_1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ & + (\tau_k^{\pm} + \mu) \begin{pmatrix} 0 & m_3 & m_4 \\ p_3 & 0 & p_4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \end{aligned} \tag{14}$$

and

$$F(\mu, \phi) = (\tau_k^{\pm} + \mu) \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{15}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$ and

$$\begin{aligned} F_1 = {} & n_1\phi_1^2(0) + n_2\phi_1(0)\phi_2(0) + n_3\phi_2^2(0) + n_4\phi_1(0)\phi_2(-1) \\ & + n_5\phi_1(0)\phi_3(-1) + n_6\phi_2(-1)\phi_3(-1) + n_7\phi_2(0)\phi_2(-1) \\ & + n_8\phi_2(0)\phi_3(-1) + l_1\phi_1(0)\phi_2(-1)\phi_3(-1) + l_1\phi_2(0)\phi_2(-1)\phi_3(-1) \\ & + l_2\phi_1^2(0)\phi_2(-1) + l_3\phi_1(0)\phi_2(0)\phi_2(-1) + l_4\phi_2^2(0)\phi_2(-1) \\ & + l_5\phi_1^2(0)\phi_3(-1) + l_6\phi_1(0)\phi_2(0)\phi_3(-1) + l_5\phi_2^2(0)\phi_3(-1) \\ & + l_7\phi_1^3(0) + l_8\phi_1^2(0)\phi_2(0) + l_8\phi_1(0)\phi_2^2(0) + l_7\phi_2^3(0), \\ F_2 = {} & + q_1\phi_2^2(0) + q_2\phi_2(0)\phi_1(0) + q_3\phi_1^2(0) + q_4\phi_2(0)\phi_1(-1) \\ & + q_5\phi_2(0)\phi_3(-1) + q_6\phi_1(-1)\phi_3(-1) + q_7\phi_1(0)\phi_1(-1) \\ & + q_8\phi_1(0)\phi_3(-1) + s_1\phi_2(0)\phi_1(-1)\phi_3(-1) + s_1\phi_1(0)\phi_1(-1)\phi_3(-1) \\ & + s_2\phi_2^2(0)\phi_1(-1) + s_3\phi_2(0)\phi_1(0)\phi_1(-1) + s_4\phi_1^2(0)\phi_2(-1) \end{aligned}$$

$$
\begin{aligned}
&+ s_5\phi_2^2(0)\phi(-1) + s_6\phi_2(0)\phi_1(0)\phi_3(-0) + s_5\phi_1^2(0)\phi_3(-1)\\
&+ s_7\phi_2^3(0) + s_8\phi_2^2(0)\phi_1(0) + s_8\phi_2(0)\phi_1^2(0) + s_7\phi_1^3(0),\\
F_3 = &+ u_2\phi_1(0)\phi_3(0) + u_3\phi_2(-1)\phi_3(0) + u_4\phi_2(0)\phi_3(0) + v_1\phi_1^2(0)\phi_3(0)\\
&+ v_2\phi_1(0)\phi_2(0)\phi_3(0) + v_3\phi_2^2(0)\phi_3(0) + v_4\phi_1(0)\phi_2(-1)\phi_3(0)\\
&+ v_4\phi_2(0)\phi_2(-1)\phi_3(0).
\end{aligned}
$$

From the discussion in Section 2, we know that if $\mu = 0$, then the system (13) undergoes a Hopf bifurcation at the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ and the associated characteristic equation of system (13) has a pair of simple imaginary roots $\pm\omega_0\tau_k^{\pm}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta,\mu)$, $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{16}$$

In fact, we can choose

$$
\begin{aligned}
\eta(\theta,\mu) = {} & (\tau_k^{\pm} + \mu)\begin{pmatrix} m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\\ 0 & 0 & l_1 \end{pmatrix}\delta(\theta)\\
& - (\tau_k^{\pm} + \mu)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ l_2 & l_3 & l_4 \end{pmatrix}\delta(\theta+1),
\end{aligned}
\tag{17}
$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0,\\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases} \tag{18}$$

and

$$R\phi = \begin{cases} 0, & -1 \le \theta < 0,\\ F(\mu,\phi), & \theta = 0. \end{cases} \tag{19}$$

Then (13) is equivalent to the abstract differential equation

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t, \tag{20}$$

where $u_t(\theta) = u(t+\theta), \theta \in [-1, 0]$.
For $\psi \in C([0,1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1],\\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1,0], R^3)$ and $\psi \in C([0,1], (R^3)^*)$, define the bilinear form

$$\langle\psi,\phi\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0\tau_k^\pm$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k^\pm$ and $-i\omega_0\tau_k^\pm$, respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_k^\pm\theta}, q^*(s) = M(1, \alpha^*, \beta^*)e^{i\omega_0\tau_k^\pm s}, M = \frac{1}{B},$$

where

$$\begin{aligned}
\alpha &= \frac{(m_1 - i\omega_0)p_4 - (p_2 + p_3 e^{-i\omega_0\tau_k^\pm})m_4}{(m_2 + m_3 e^{-i\omega_0\tau_k^\pm})p_4 + (i\omega_0 - p_1)m_4},\\
\beta &= \frac{(m_2 + m_3 e^{-i\omega_0\tau_k^\pm})(p_2 + p_3 e^{-i\omega_0\tau_k^\pm}) + (i\omega_0 - p_1)(m_1 - i\omega_0)}{(m_2 + m_3 e^{-i\omega_0\tau_k^\pm})p_4 e^{-i\omega_0\tau_k^\pm} + (i\omega_0 - p_1)m_4 e^{-i\omega_0\tau_k^\pm}},\\
\alpha^* &= -\frac{m_1 + i\omega_0}{p_2 + p_3 e^{-i\omega_0\tau_k^\pm}},\\
\beta^* &= -\frac{(m_4 + p_4\alpha^*)e^{-i\omega_0\tau_k^\pm}}{i\omega_0 + \mu_1},\\
B &= 1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \tau_k^\pm[p_3\alpha^* + m_3\bar{\alpha} + (m_4 + p_4\alpha^*)\bar{\beta}].
\end{aligned}$$

Furthermore, $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

Next, we use the same notations as those in Hassard [1] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq.(13) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t\rangle,\ W(t, \theta) = u_t(\theta) - 2\text{Re}\{z(t)q(\theta)\} \tag{21}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{22}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{23}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (13),

$$\dot{z}(t) = i\omega_0\tau_k^\pm z + \bar{q}^*(\theta)f(0, W(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\} \stackrel{\text{def}}{=} i\omega_0\tau_k^\pm z + \bar{q}^*(0)f_0.$$

That is

$$\dot{z}(t) = i\omega_0\tau_k^\pm z + g(z, \bar{z}),$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots.$$

Hence, we have

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = f(0, y_t) = K_{20}z^2 + K_{11}z\bar{z} + K_{02}\bar{z}^2 + K_{21}z^2\bar{z} + \text{h.o.t.},$$

where

$$
\begin{aligned}
K_{20} = {}& \bar{D}\tau_k^{\pm}\Big[n_1 + n_2\alpha + n_3\alpha^2 + n_4\alpha e^{-i\omega_0\tau_k^{\pm}} + n_5\beta e^{-i\omega_0\tau_k^{\pm}} + n_6\alpha\beta e^{-2i\omega_0\tau_k^{\pm}} \\
& + n_7\alpha^2 e^{-i\omega_0\tau_k^{\pm}} + n_8\alpha\beta e^{-i\omega_0\tau_k^{\pm}} + \bar{\alpha^*}\Big(q_1\alpha^2 + q_2\alpha + q_3 + q_4\alpha e^{-i\omega_0\tau_k^{\pm}} \\
& + q_5\alpha\beta e^{-i\omega_0\tau_k^{\pm}} + q_6\beta e^{-2i\omega_0\tau_k^{\pm}} + q_7 e^{-i\omega_0\tau_k^{\pm}} + q_8\beta e^{-i\omega_0\tau_k^{\pm}}\Big) + \bar{\beta}^*\Big(u_2\beta \\
& + u_3\alpha\beta e^{-i\omega_0\tau_k^{\pm}} + u_4\alpha\beta\Big)\Big], \\
K_{11} = {}& \bar{D}\tau_k^{\pm}\Big[2n_1 + 2n_2\mathrm{Re}\{\alpha\} + 1n_3|\alpha|^2 + 2n_4\mathrm{Re}\{\bar{\alpha}e^{i\omega_0\tau_k^{\pm}}\} \\
& + 2n_5\mathrm{Re}\{\bar{\beta}e^{i\omega_0\tau_k^{\pm}}\} + 2n_6\mathrm{Re}\{\bar{\beta}\beta\} \\
& + 2n_7|\alpha|^2 e^{i\omega_0\tau_k^{\pm}} + 2n_8\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\} \\
& + \bar{\alpha}^*\Big(2|\alpha|^2 q_1 + 2q_2\mathrm{Re}\{\alpha\} + 2q_3 + 2q_4\mathrm{Re}\{\bar{\alpha}e^{-i\omega_0\tau_k^{\pm}}\} + q_5\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\} \\
& + 2q_6\mathrm{Re}\{\beta\} + q_7\Big(e^{i\omega_0\tau_k^{\pm}} + e^{-i\omega_0\tau_k^{\pm}} + q_8\mathrm{Re}\{\bar{\beta}e^{i\omega_0\tau_k^{\pm}}\}\Big)\Big) \\
& + \bar{\beta}^*\Big(2u_2\mathrm{Re}\{\beta\} + 2u_3\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\} + 2u_4\mathrm{Re}\{\bar{\beta}\beta\}\Big)\Big], \\
K_{02} = {}& \bar{D}\tau_k^{\pm}\Big[n_1 + n_2\bar{\alpha} + n_3\bar{\alpha}^2 + n_4\bar{\alpha}e^{i\omega_0\tau_k^{\pm}} + n_5\bar{\beta}e^{i\omega_0\tau_k^{\pm}} + n_6\bar{\alpha}\bar{\beta}e^{2i\omega_0\tau_k^{\pm}} \\
& + n_7\bar{\alpha}^2 e^{2i\omega_0\tau_k^{\pm}} + n_8\bar{\alpha}\bar{\beta}e^{i\omega_0\tau_k^{\pm}} + \bar{\alpha}^*\Big(q_1\bar{\alpha}^2 + q_2\bar{\alpha} + q_3 + q_4\bar{\alpha}e^{i\omega_0\tau_k^{\pm}} \\
& + q_5\bar{\alpha}\bar{\beta}e^{i\omega_0\tau_k^{\pm}} + q_6\bar{\beta}e^{2i\omega_0\tau_k^{\pm}} + q_7 e^{i\omega_0\tau_k^{\pm}} + q_8\bar{\beta}e^{i\omega_0\tau_k^{\pm}}\Big) \\
& + \bar{\beta}^*\Big(u_2\bar{\beta} + u_3\bar{\alpha}\bar{\beta}e^{i\omega_0\tau_k^{\pm}} + u_4\bar{\alpha}\bar{\beta}\Big)\Big], \\
K_{21} = {}& \bar{D}\tau_k^{\pm}\Big\{n_1\Big[W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big] \\
& + n_2\Big[\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0)\Big] \\
& + n_3\Big[\bar{\alpha}W_{20}^{(2)}(0) + 2\alpha W_{11}^{(2)}(0)\Big] \\
& + n_4\Big[\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(0)e^{i\omega_0\tau_k^{\pm}} + \frac{1}{2}W_{20}^{(2)}(0) + \alpha e^{-i\omega_0\tau_k^{\pm}}W_{11}^{(1)}(0) + W_{11}^{(2)}(0)\Big] \\
& + n_5\Big[\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_k^{\pm}} + \frac{1}{2}W_{20}^{(3)}(0) + \beta e^{-i\omega_0\tau_k^{\pm}}W_{11}^{(1)}(0) + W_{11}^{(3)}(0)\Big] \\
& + n_6\Big[\frac{1}{2}\bar{\beta}W_{20}^{(2)}(0)e^{i\omega_0\tau_k^{\pm}} + \frac{1}{2}\bar{\alpha}e^{i\omega_0\tau_k^{\pm}}W_{20}^{(3)}(0) \\
& \qquad + \beta e^{-i\omega_0\tau_k^{\pm}}W_{11}^{(2)}(0) + \alpha e^{-i\omega_0\tau_k^{\pm}}W_{11}^{(3)}(0)\Big] \\
& + n_7\Big[\bar{\alpha}e^{i\omega_0\tau_k^{\pm}}W_{20}^{(3)}(0) + 2\alpha W_{11}^{(2)}(0)\Big]
\end{aligned}
$$

$$+ n_8 \left[\frac{1}{2}\bar{\beta}W_{20}^{(2)}(0)e^{i\omega_0\tau_k^\pm} + \frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0) + \beta e^{-i\omega_0\tau_k^\pm}W_{11}^{(2)}(0) + \alpha W_{11}^{(3)}(0)\right]$$
$$+ l_1 \left[2\mathrm{Re}\{\bar{\alpha}\beta\} + \alpha\beta e^{-2i\omega_0\tau_k^\pm}\right] + l_1 \left[\alpha^2 e^{-i\omega_0\tau_k^\pm} + 2\mathrm{Re}\{\beta\}|\alpha|^2\bar{\beta}e^{-i\omega_0\tau_k^\pm}\right]$$
$$+ l_2 \left[\bar{\alpha}e^{i\omega_0\tau_k^\pm} + 2\alpha e^{-i\omega_0\tau_k^\pm}\right] + l_3 \left[\alpha^2 + |\alpha|^2 + 2\alpha e^{-2i\omega_0\tau_k^\pm}\right]$$
$$+ l_4 \left[2\alpha^2\bar{\alpha}e^{i\omega_0\tau_k^\pm} + \alpha^2\bar{\alpha}e^{-2i\omega_0\tau_k^\pm}\right] + l_5 \left[\bar{\beta}e^{i\omega_0\tau_k^\pm} + 2\beta e^{-i\omega_0\tau_k^\pm}\right]$$
$$+ l_7 \left[3 + 3\alpha^2\bar{\alpha}\right] + l_8 \left[\bar{\alpha} + 2\alpha + 2|\alpha|^2 + \alpha^2\right]$$
$$+ \bar{\alpha}^* \left[q_1 \left(\bar{\alpha}W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)\right)\right.$$
$$+ q_2 \left(\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0)\right)$$
$$+ q_3 \left(W_{20}^{(1)}(0) + 2W_{11}^{(2)}(0)\right)$$
$$+ q_4 \left(\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(-1) + \frac{1}{2}e^{i\omega_0\tau_k^\pm}W_{20}^{(2)}(0) + \alpha W_{11}^{(1)}(-1) + W_{11}^{(2)}(0)e^{-i\omega_0\tau_k^\pm}\right)$$
$$+ q_5 \left(\frac{1}{2}\bar{\beta}e^{i\omega_0\tau_k^\pm}W_{20}^{(2)}(0) + \frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0) + \beta e^{-i\omega_0\tau_k^\pm}W_{11}^{(2)}(0) + \alpha W_{11}^{(3)}(0)\right)$$
$$+ q_6 \left(\frac{1}{2}\bar{\beta}e^{i\omega_0\tau_k^\pm}W_{20}^{(1)}(-1) + \frac{1}{2}e^{i\omega_0\tau_k^\pm}W_{20}^{(3)}(-1)\right.$$
$$\left. + \beta e^{-i\omega_0\tau_k^\pm}W_{11}^{(1)}(-1) + e^{-i\omega_0\tau_k^\pm}W_{11}^{(3)}(-1)\right)$$
$$+ q_7 \left(\frac{1}{2}e^{i\omega_0\tau_k^\pm}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(-1) + e^{-i\omega_0\tau_k^\pm}W_{11}^{(1)}(0) + W_{11}^{(1)}(-1)\right)$$
$$+ q_8 \left(\frac{1}{2}\bar{\beta}e^{i\omega_0\tau_k^\pm}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \beta e^{-i\omega_0\tau_k^\pm}W_{11}^{(1)}(0) + W_{11}^{(2)}(0)\right)$$
$$+ s_1 \left(\alpha\bar{\beta} + 2\mathrm{Re}\{\bar{\alpha}\beta e^{i\omega_0\tau_k^\pm}\}\right) + s_1 \left(2\mathrm{Re}\{\beta\} + \beta e^{i\omega_0\tau_k^\pm}\right)$$
$$+ s_2 \left(\alpha^2 e^{i\omega_0\tau_k^\pm} + 2|\alpha|^2 e^{-i\omega_0\tau_k^\pm}\right) + s_3 \left(\bar{\alpha}e^{i\omega_0\tau_k^\pm} + 2\mathrm{Re}\{\alpha\}e^{-i\omega_0\tau_k^\pm}\right)$$
$$+ s_4 \left(\bar{\alpha}e^{i\omega_0\tau_k^\pm} + 2\alpha e^{-i\omega_0\tau_k^\pm}\right)$$
$$+ s_5 \left(\alpha^2\bar{\beta}e^{i\omega_0\tau_k^\pm} + |\alpha|^2 e^{-i\omega_0\tau_k^\pm} + |\alpha|^2\beta e^{-i\omega_0\tau_k^\pm}\right)$$
$$+ s_6 \left(\alpha\beta + 2\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^\pm}\}\right) + s_7 \left(3 + 3\alpha^2\bar{\alpha}\right)$$
$$\left. + s_8 \left(\alpha^2 + 3\alpha + |\alpha|^2 + \bar{\alpha}\right)\right]$$
$$+ \bar{\beta}^* \left[u_2 \left(\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \alpha W_{11}^{(1)}(0) + W_{11}^{(2)}(0)\right)\right.$$
$$+ u_3 \left(\frac{1}{2}\bar{\beta}e^{i\omega_0\tau_k^\pm}W_{20}^{(2)}(0) + \frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0) + \beta e^{-i\omega_0\tau_k^\pm}W_{11}^{(2)}(0) + \alpha W_{11}^{(3)}(0)\right)$$

$$+ u_4\left(\frac{1}{2}\bar{\beta}W_{20}^{(2)}(0)+\frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0)+\beta W_{11}^{(2)}(0)+\alpha W_{11}^{(3)}(0)\right)+v_1\left(\bar{\beta}+2\beta\right)$$
$$+ v_2\left(\alpha\beta+\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\}\right)+v_3\left(\alpha^2\bar{\beta}e^{i\omega_0\tau_k^{\pm}}+2|\alpha|^2\beta e^{-i\omega_0\tau_k^{\pm}}\right)$$
$$+v_4\left(\alpha\beta+\mathrm{Re}\{\bar{\alpha}\beta\}\right)+v_4\left(\alpha^2\beta e^{-i\omega_0\tau_k^{\pm}}+|\alpha|^2\beta e^{i\omega_0\tau_k^{\pm}}+|\alpha|^2\beta e^{-i\omega_0\tau_k^{\pm}}\right)\Big]\Bigg\}.$$

Then we obtain

$$g_{20}=2K_{20},\ g_{11}=K_{11},\ g_{02}=2K_{02},\ g_{21}=2K_{21}.$$

For unknown $W_{20}^{(i)}(0), W_{11}^{(i)}(0), W_{20}^{(i)}(-1), W_{11}^{(i)}(-1),\ (i=1,2,3)$ in g_{21}, we still need to compute them.

From (20) and (21), we have

(24)
$$W^{'}=\begin{cases} AW-2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1\le\theta<0,\\ AW-2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}+\bar{f}, & \theta=0\end{cases} \overset{\text{def}}{=} AW+H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \tag{25}$$

Comparing the coefficients, we obtain

$$(A-2i\tau_k^{\pm}\omega_0)W_{20}=-H_{20}(\theta), \tag{26}$$

$$AW_{11}(\theta)=-H_{11}(\theta), \tag{27}$$

$$\cdots\cdots.$$

And we know that for $\theta\in[-1,0)$,

$$H(z,\bar{z},\theta)=-\bar{q}^*(0)f_0q(\theta)-q^*(0)\bar{f}_0\bar{q}(\theta)=-g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta). \tag{28}$$

Comparing the coefficients of (28) with (25) gives that

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta), \tag{29}$$

$$H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta). \tag{30}$$

From (26), (29) and the definition of A, we get

$$\dot{W}_{20}(\theta)=2i\omega_0\tau_k^{\pm}W_{20}(\theta)+g_{20}q(\theta)+\bar{g_{02}}\bar{q}(\theta). \tag{31}$$

Noting that $q(\theta)=q(0)e^{i\omega_0\tau_k^{\pm}\theta}$, we have

$$W_{20}(\theta)=\frac{ig_{20}}{\omega_0\tau_k^{\pm}}q(0)e^{i\omega_0\tau_k^{\pm}\theta}+\frac{i\bar{g}_{02}}{3\omega_0\tau_k^{\pm}}\bar{q}(0)e^{-i\omega_0\tau_k^{\pm}\theta}+E_1e^{2i\omega_0\tau_k^{\pm}\theta}, \tag{32}$$

where $E_1=(E_1^{(1)},E_1^{(2)},E_1^{(3)})^T\in R^3$ is a constant vector.
Similarly, from (27), (30) and the definition of A, we have

$$\dot{W}_{11}(\theta)=g_{11}q(\theta)+\bar{g_{11}}\bar{q}(\theta), \tag{33}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k^{\pm}} q(0)e^{i\omega_0\tau_k^{\pm}\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k^{\pm}} \bar{q}(0)e^{-i\omega_0\tau_k^{\pm}\theta} + E_2, \tag{34}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (32) and (34), respectively. It follows from the definition of A, (29) and (30) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k^{\pm}W_{20}(0) - H_{20}(0) \tag{35}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{36}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (26), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_k^{\pm}(H_1, H_2, H_3)^T, \tag{37}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\tau_k^{\pm}(P_1, P_2, P_3)^T, \tag{38}$$

where

$$\begin{aligned}
H_1 &= n_1 + n_2\alpha + n_3\alpha^2 + n_4\alpha e^{-i\omega_0\tau_k^{\pm}} + n_5\beta e^{-i\omega_0\tau_k^{\pm}} \\
&\quad + n_6\alpha\beta e^{-2i\omega_0\tau_k^{\pm}} + n_7\alpha^2 e^{-i\omega_0\tau_k^{\pm}} + n_8\alpha\beta e^{-i\omega_0\tau_k^{\pm}}, \\
H_2 &= q_1\alpha^2 + q_2\alpha + q_3 + q_4\alpha e^{-i\omega_0\tau_k^{\pm}} + q_5\alpha\beta e^{-i\omega_0\tau_k^{\pm}} \\
&\quad + q_6\beta e^{-2i\omega_0\tau_k^{\pm}} + q_7 e^{-i\omega_0\tau_k^{\pm}} + q_8\beta e^{-i\omega_0\tau_k^{\pm}}, \\
H_3 &= u_2\beta + u_3\alpha\beta e^{-i\omega_0\tau_k^{\pm}} + u_4\alpha\beta, \\
P_1 &= 2n_1 + 2n_2\mathrm{Re}\{\alpha\} + 1n_3|\alpha|^2 + 2n_4\mathrm{Re}\{\bar{\alpha}e^{i\omega_0\tau_k^{\pm}}\} + 2n_5\mathrm{Re}\{\bar{\beta}e^{i\omega_0\tau_k^{\pm}}\} \\
&\quad + 2n_6\mathrm{Re}\{\bar{\beta}\beta\} + 2n_7|\alpha|^2 e^{i\omega_0\tau_k^{\pm}} + 2n_8\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\}, \\
P_2 &= 2|\alpha|^2 q_1 + 2q_2\mathrm{Re}\{\alpha\} + 2q_3 + 2q_4\mathrm{Re}\{\bar{\alpha}e^{-i\omega_0\tau_k^{\pm}}\} + q_5\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\} \\
&\quad + 2q_6\mathrm{Re}\{\beta\} + q_7(e^{i\omega_0\tau_k^{\pm}} + e^{-i\omega_0\tau_k^{\pm}} + q_8\mathrm{Re}\{\bar{\beta}e^{i\omega_0\tau_k^{\pm}}\}, \\
P_3 &= 2u_2\mathrm{Re}\{\beta\} + 2u_3\mathrm{Re}\{\bar{\alpha}\beta e^{-i\omega_0\tau_k^{\pm}}\} + 2u_4\mathrm{Re}\{\bar{\beta}\beta\}.
\end{aligned}$$

Noting that

$$\left(i\omega_0\tau_k^{\pm}I - \int_{-1}^{0} e^{i\omega_0\tau_k^{\pm}\theta}d\eta(\theta)\right)q(0) = 0,$$

$$\left(-i\omega_0\tau_k^{\pm}I - \int_{-1}^{0} e^{-i\omega_0\tau_k^{\pm}\theta}d\eta(\theta)\right)\bar{q}(0) = 0$$

and substituting (32) and (37) into (35), we have

$$\left(2i\omega_0\tau_k^{\pm}I - \int_{-1}^{0} e^{2i\omega_0\tau_k^{\pm}\theta}d\eta(\theta)\right)E_1 = 2\tau_k^{\pm}(H_1, H_2, H_3)^T.$$

That is

$$\begin{pmatrix} 2i\omega_0 - m_1 & -m_2 - m_3e^{i\omega_0\tau_k^{\pm}} & -m_4e^{i\omega_0\tau_k^{\pm}} \\ -p_2 - p_3 & i\omega_0 - p_1 & -p_4e^{i\omega_0\tau_k^{\pm}} \\ 0 & 0 & i\omega_0 - \mu_1 \end{pmatrix} E_1 = 2(H_1, H_2, H_3)^T.$$

It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \tag{39}$$

where

$$\Delta_1 = \det\begin{pmatrix} 2i\omega_0 - m_1 & -m_2 - m_3e^{i\omega_0\tau_k^{\pm}} & -m_4e^{i\omega_0\tau_k^{\pm}} \\ -p_2 - p_3 & i\omega_0 - p_1 & -p_4e^{i\omega_0\tau_k^{\pm}} \\ 0 & 0 & i\omega_0 - \mu_1 \end{pmatrix},$$

$$\Delta_{11} = 2\det\begin{pmatrix} H_1 & -m_4e^{i\omega_0\tau_k^{\pm}} & \\ H_2 & i\omega_0 - p_1 & -p_4e^{i\omega_0\tau_k^{\pm}} \\ H_3 & 0 & i\omega_0 - \mu_1 \end{pmatrix},$$

$$\Delta_{12} = 2\det\begin{pmatrix} 2i\omega_0 - m_1 & H_1 & -m_4e^{i\omega_0\tau_k^{\pm}} \\ -p_2 - p_3 & H_2 & -p_4e^{i\omega_0\tau_k^{\pm}} \\ 0 & H_3 & i\omega_0 - \mu_1 \end{pmatrix},$$

$$\Delta_{13} = 2\det\begin{pmatrix} 2i\omega_0 - m_1 & -m_2 - m_3e^{i\omega_0\tau_k^{\pm}} & H_1 \\ -p_2 - p_3 & i\omega_0 - p_1 & H_2 \\ 0 & 0 & H_3 \end{pmatrix}.$$

Similarly, substituting (33) and (38) into (36), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = 2\tau_k^{\pm}(P_1, P_2, P_3)^T.$$

That is

$$\begin{pmatrix} m_1 & m_2 + m_3 & m_4 \\ p_2 & p_1 & p_4 \\ 0 & 0 & \mu_1 \end{pmatrix} E_2 = 2(-P_1, -P_2, -P_3)^T.$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, \tag{40}$$

where

$$\Delta_2 = \det\begin{pmatrix} m_1 & m_2 + m_3 & m_4 \\ p_2 & p_1 & p_4 \\ 0 & 0 & \mu_1 \end{pmatrix},$$

$$\Delta_{21} = 2\det\begin{pmatrix} H_1 & m_2+m_3 & m_4 \\ H_2 & p_1 & p_4 \\ H_3 & 0 & \mu_1 \end{pmatrix},$$
$$\Delta_{22} = 2\det\begin{pmatrix} m_1 & H_1 & m_4 \\ p_2 & H_2 & p_4 \\ 0 & H_3 & \mu_1 \end{pmatrix},$$
$$\Delta_{23} = 2\det\begin{pmatrix} m_1 & m_2+m_3 & H_1 \\ p_2 & p_1 & H_2 \\ 0 & 0 & H_3 \end{pmatrix}.$$

From (32), (34), (39) and (40), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k^{\pm}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k^{\pm})\}}, \\ \beta_2 &= 2\mathrm{Re}(c_1(0)), \\ T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_k^{\pm})\}}{\omega_0\tau_k^{\pm}}. \end{aligned}$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (13) at $\tau = \tau_k^{\pm}$ $(k = 0, 2, 3, \ldots)$ on the center manifold. From the discussion above, we have the following result:

Theorem 3.1. *The periodic solution is supercritical* (*subcritical*) *if* $\mu_2 > 0$ $(\mu_2 < 0)$; *the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase* (*unstable*) *if* $\beta_2 < 0$ $(\beta_2 > 0)$; *the periods of the bifurcating periodic solutions increase* (*decrease*) *if* $T_2 > 0$ $(T_2 < 0)$.

Remark 3.2. A τT-periodic solution of (13) is a T-periodic solution of (5).

4. Numerical examples

In this section, we present some numerical results of system (3) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

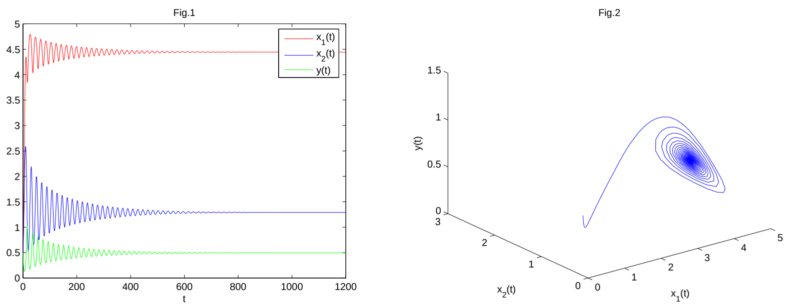
$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[0.5(1-0.2x_1) - \frac{0.5y(t-\tau)x_2(t-\tau)}{x_1+x_2}\right], \\ \dot{x}_2(t) = x_2(t)\left[0.5(1-0.3x_2) - \frac{0.8y(t-\tau)x_1(t-\tau)}{x_1+x_2}\right], \\ \dot{y}(t) = -0.5y + \frac{0.3x_1x_2(t-\tau)y}{x_1+x_2} + \frac{0.2x_1(t-\tau)x_2(t-\tau)y}{x_1+x_2} \end{cases} \tag{41}$$

which has a positive equilibrium $E_0(x_1^*, x_2^*, y^*) \approx (4.4437, 1.2904, 0.4941)$ and satisfies the conditions indicated in Theorem 2.2. When $\tau = 0$, the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$ is asymptotically stable. Take $k = 0$

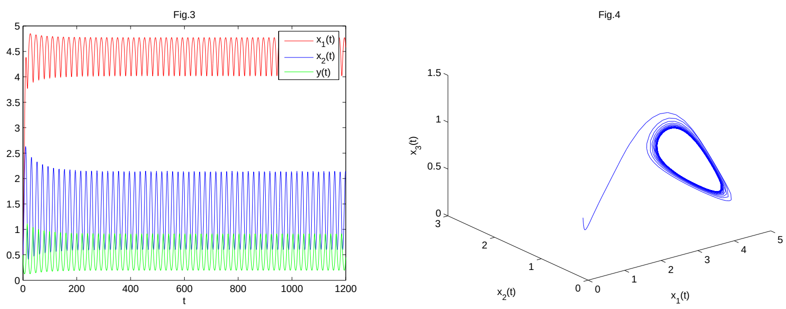
for example, by some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 1.0328, \tau_0 \approx 0.53, \lambda^{'}(\tau_0) \approx 0.7511 - 9.5401i$. Thus we can calculate the following values:

$$c_1(0) \approx -2.2241 - 6.4133i, \mu_2 \approx 2.9611, \beta_2 \approx -4.4482, T_2 \approx 63.3239.$$

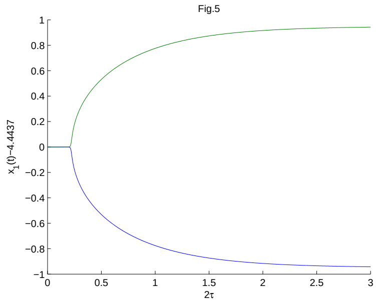
Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$ is stable when $\tau < \tau_0$ as is illustrated by the computer simulations (see Figs.1-2). When τ passes through the critical value τ_0, the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_0 \approx (4.4437, 1.2904, 0.4941)$ at τ_0 are stable, which are depicted in Figs.3-4. From the bifurcation diagrams (Figs.5-7), it is shown that the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$ is stable when $\tau < \tau_0 = 0.53$ and unstable when $\tau > \tau_0 = 0.53$.

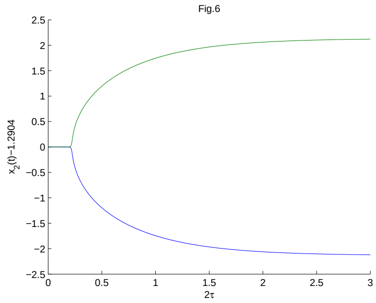


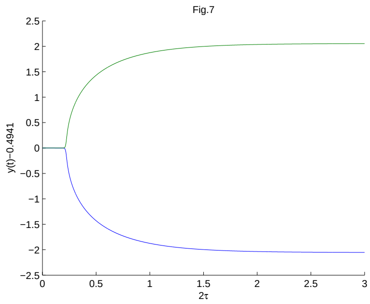
Figs.1-2 Behavior and phase portrait of system (41) with $\tau = 0.5 < \tau_0 \approx 0.53$. The positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$ is asymptotically stable. The initial value is (0.3,0.3,0.3).



Figs.3-4 Behavior and phase portrait of system (41) with $\tau = 0.6 > \tau_0 \approx 0.53$. Hopf bifurcation occurs from the positive equilibrium $E_0 \approx (4.4437, 1.2904, 0.4941)$. The initial value is (0.3,0.3,0.3).







Figs.5-7 Bifurcation diagrams of system (41) with initial value is (0.3,0.3,0.3).

5. Biological explanations and conclusions

5.1. Biological explanations

From the analysis in Section 2, we know that under the conditions (H1)-(H3), (i) if (K1) or (K2) holds, then the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ of system (3) is asymptotically stable for all $\tau \geq 0$. This shows that, in this case, the population density of prey species in two habits, the population density of predator species will tend to stabilization, that is, the population density of prey species in two habits, the population density of predator species will tend to x_1^*, x_2^*, y^*, respectively, and this fact is not influenced by the delay $\tau \geq 0$; (ii) If (K3) or (K4) or (K5) holds, then the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ of system (3) is asymptotically stable when $\tau \in [0, \tau_0)$. This shows that, in this case, the population density of prey species in two habits, the population density of predator species will tend to stabilization, that is, the population density of prey species in two habits, the population density of predator species will tend to x_1^*, x_2^*, y^*, respectively, and this fact is not influenced by the delay $\tau \in [0, \tau_0)$. When τ crosses through the critical value τ_0, the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ of system (3) loses stability and a Hopf bifurcation occurs. This shows that the population density of prey species in two habits, the population density of predator species may coexist and keep in an oscillatory mode near the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$.

5.2. Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ and local Hopf bifurcation in delayed predator-prey model of prey migration and predator switching. We have showed that if the conditions (H1)-(H3), (K1) or (H1)-(H3), (K2) hold, the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ of system (3) is asymptotically stable for all $\tau \geq 0$. Under the conditions (H1)-(H3), if the condition $(K3)$ or (K4) or (K5) holds, then the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$ of system (3) is asymptotically stable for all $\tau \in [0, \tau_0)$. As the delay τ increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$, i.e., a family of periodic orbits bifurcates from the positive equilibrium $E_0(x_1^*, x_2^*, y^*)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. A numerical example verifying our theoretical results is also correct.

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