

# Middle School Mathematics Teachers' Responses to a Student's Mistaken Mathematical Conjecture and Justification

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**ABSTRACT.** The purpose of the study was to investigate the reality of middle school mathematics teachers' subject matter knowledge for teaching mathematical conjecture and justification. Data in the study were collected through interviewing nine Chinese and ten Korean middle school mathematics teachers. The teachers responded to the question that was designed in the form of a scenario that presents a teaching task related to a geometrical topic. The teachers' oral responses were audiotaped and transcribed, and their written notes were collected. The results of the study were compared to the analysis of American and Chinese elementary and secondary teachers' responses to the same task in Ball (1988) and Ma (1999). The findings of the study suggested that teachers' approaches to explaining and demonstrating a mathematical topic were significantly influenced by their knowledge of learners and knowledge of the curriculum they teach. One of the practical implications of the study is that teachers should recognize the advantages of learning the conceptual structure of a mathematical topic. It allows the teachers to have the flexibility to come up with meaningful mathematical approaches to teaching the topic, which are comprehensible to the learners whatever the grade levels they teach, rather than rule-based algorithms.

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## I. Introduction

In the fall of 2002, I interviewed two pre-service teachers to collect data for the Beliefs of Pre-service Elementary Teachers (PSET) project at Indiana University. One of the interview questions I asked the pre-service teachers (PSTs) involves a scenario designed to examine teachers' responses to a student's mistaken claim that "as the perimeter of a closed figure increases, the area also increases." When I first read the scenario, I expected that pre-service teachers might have difficulty transferring their ideas about why the student's idea was wrong to the student. I was surprised to find that the PSTs had difficulty determining whether the student's idea was mathematically correct or not. One of them became anxious when she discovered it was a geometric problem. When I asked her the reason why she hesitated in solving the problem, she confessed that she forgot most of what she learned in geometry when she was a student. For this reason, she was afraid of encountering geometry problems. This pre-service teacher's unsatisfactory knowledge of geometry inhibited her teaching practice by causing her to avoid giving any response to the student's novel idea.

I researched the origin of the scenario problem, and I learned that it was excerpted from *Knowing and Teaching Elementary Mathematics* (Ma, 1999). This book is Liping Ma's dissertation, in which she studied differences between Chinese and American elementary teachers' understanding of mathematics using four different teaching tasks in the form of scenario problems in relation to specific mathematical topics. I found that the four teaching tasks, including the scenario problem described above, were designed to reveal teachers' deep knowledge of the mathematical topics, rather than their superficial knowledge of the topics. International researchers consider Ma's study a revolutionary that suggests that U.S. mathematics teachers' unsatisfactory knowledge of mathematics is one of the main reasons for the mathematical achievement gap between U.S. students and East Asian students.

The significant contributions of Ma's study to mathematics education impressed me deeply and stimulated me to know more about the features of mathematics teachers' understanding of mathematics.

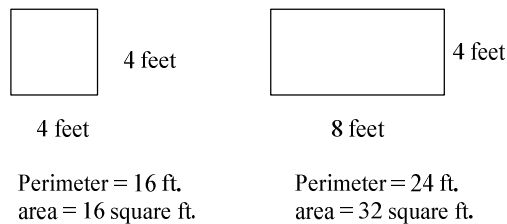
For the reasons, in this study, I investigated the Korean and Chinese

middle school mathematics teachers' responses to the same scenario problem which is about the relationship between the perimeter and area of a closed figure. Through investigating the Chinese and Korean teachers' responses to the student's erroneous mathematical statement and its supposed proof, this study reports the teachers' approaches to exploring the student's claim and their knowledge about mathematical generalization and its proof.

## II. Background

### Scenario Problem

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:



How would you respond to this student?

The student in the scenario problem above believed that she discovered a new mathematical fact about the relationship between the perimeter and area of a closed figure: as the perimeter of a closed figure increases, the area also increases. The student attempted to verify the mathematical claim with a specific supporting example: as the perimeter of a  $4 \times 4$  square increases into the shape of a  $4 \times 8$  rectangle, the area of the square also increases.

When we assume that the student generalized the finding from observing the relationships between the perimeters and areas of particular closed figures, squares and rectangles, to all closed figures, it becomes clear that she did not have an adequate understanding of mathematical

generalization and proof. In mathematics, an observed mathematical fact can be generalized when it is proved in every case without exception. There are an infinite number of closed figures in mathematics. To prove that the student's mathematical claim is true, she can check every type of closed figure or she could also attempt to provide a proof in the general case, and show that the mathematical statement is satisfied by every case. However, without performing this kind of systematic verification of the claim, the student believed that the relationship between perimeter and area would be the same for all cases of closed figures. Presumably, she believed this because she could not think of counterexamples to the generalization. Therefore, an appropriate response to the student's claim is for the student to realize the falsity of the claim by pointing out counterexamples, and to help the student understand the general process of mathematical generalization. Most teachers in the present study followed this protocol in responding to the student's claim.

Another possible response to the student's erroneous mathematical statement is to focus on the conditions under which the mathematical claim *is* satisfied. The teachers who have this perspective did not regard the claim as absolutely correct or absolutely wrong: it is conditionally correct. In Ma's (1999) study, 26 of the 72 Chinese elementary teachers evaluated the student's claim in this manner. Below is an example of a teacher response reported by Ma (1999).

So, now we can say that the student's claim is not absolutely wrong, but it is incomplete or conditional. Under certain conditions it is tenable, but under other conditions it does not necessarily hold. (Tr. J.) (Ma, 1999, p. 96).

From a strictly logical standpoint, this type of response may be problematic because it can confuse students about the nature of deductive logic and the truth values of propositions. In high school and college level mathematics, students learn the concept of a proposition, and they are asked to determine whether a given proposition is true or false. In this activity, there is no intermediate decision, such as conditionally true or conditionally false. For example, the student's argument that as the perimeter of a closed figure

increases, the area also increases is a *deductive argument*—“an argument of such a form that if its premises are true, the conclusion *must* be true, too” (Weston, 2000, p.40, italics added).

Premise 1: As the perimeter of a closed figure increases, the area of the figure also increases.

Premise 2: The perimeter of this closed figure has increased.

*Conclusion: Therefore, the area of this closed figure has increased.*

This argument is an instance of the form modus ponens as below, which is deductively valid.

Modus ponens is of the form:

Premise 1 : If A is true, then B is true.

Premise 2 : A is true.

Conclusion : B is true.

In a deductively valid argument, if all of the premises are true, then the conclusion must be true. However, it is possible for the student's conclusion to be false, if at least one of the premises is false. This is how we can demonstrate that the student's first premise—that as the perimeter of a closed figure increases, the area of the figure also increases—is false. In other words, by providing an example where increasing the perimeter of a closed figure does not also increase its area, we can show that the second premise is true, while the conclusion is false. Since the argument is deductively valid, the only way for the conclusion to be false is if one of the premises is false. The only premise that could be false, then, is the student's assumption that as the perimeter of a closed figure increases, the area of the figure also increases.

Despite the possibility that the student will be misled about the nature of propositions and deductive logic, there may be pedagogical reasons to respond in the intermediate manner. Teachers value praising and encouraging students' work, rather than simply rejecting it. Encouraging a student to revise and limit her claim is gentler than telling her that the claim is simply

false. Teachers prefer gentler methods of evaluating students' claims. The difference between the response of the logician and the response of the mathematics teacher reveals a feature of teachers' knowledge regarding students' motivation and performance.

### III. Methods and Procedures

#### 1. Subjects

This study investigated the features of 19 middle school mathematics teachers' subject matter knowledge for teaching. In China, 9 middle school mathematics teachers were interviewed from three urban middle schools in ChangSha, Hunan. The other 10 teachers came from nine Korean middle schools: four located in Seoul and five in southern Korea. Three female and six male Chinese mathematics teachers participated in the study. One of the Chinese teachers is a high school graduate and the other eight Chinese teachers have bachelor's degrees in mathematics education. Seven female and three male Korean mathematics teachers participated in the study. Four of the Korean teachers have bachelor's degrees in mathematics education, and five of the Korean teachers have master's degrees in mathematics education. The remaining Korean teacher has a bachelor's degree in mathematics and a master's degree in mathematics education. The Chinese teachers have an average of twelve years of school teaching experience, and the Korean teachers have an average of eight years of experience.

#### 2. Data Collection

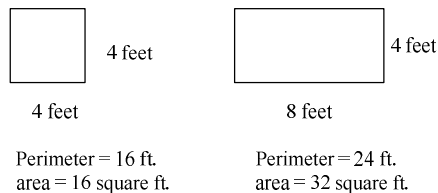
##### *Instrumentation*

To investigate the features of the participant teachers' knowledge of mathematical conjecture and justification, this study used one of the Teacher Education and Learning to Teach Study (TELT) mathematics interview questions developed by the National Center for Research on Teacher Education(NCRTE) at Michigan State University(Kennedy et al.,1993). The

teaching tasks in the TELT mathematics interview have been used to investigate in-service mathematics teachers' subject matter knowledge and its pedagogical aspects by influential qualitative researchers (Ball,1988; Ma,1996, 1999). Specifically, this study administered the teaching task in secondary mathematics that was used in Ball's(1988) study to reveal prospective secondary teachers' understanding of conjecture and justification as follows:

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Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:




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How would you respond to this student?

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### *Procedure*

Four university faculty members at mathematics education departments in China and Korea helped me recruit participants by convenience sampling. One faculty member was in ChangSha, and the other three faculty members were in Seoul and southern Korea. A brief summary of the study prepared for human subject approval was sent to the mathematics education professors to introduce and explain the study. The professors and I shared ideas about subject recruitment, the interview process, and potential interview locations via email for approximately three weeks. The professors then contacted potential participant teachers for the present study. The Chinese professor recruited nine participants who were his colleagues when he was a mathematics teacher or who were graduates from his university. The Korean professors recruited ten Korean subjects from among their graduate students and graduates from their universities.

The interviews were conducted outside of normal class time. Data collection in China was conducted with a Korean-Chinese translator who

works for a Korean Company as a translator for business meetings. The researcher explained the purpose of this study and the process of administrating the TELT interview to subjects prior to conducting interviews with subjects. Also, prior to conducting the Chinese interviews, the researcher ensured that the translator knew enough about the mathematical terms that were used in the interview questions.

Participants were not allowed to use any resources while completing the scenario question. Follow-up questions were an important component of the interview procedure. Some of the follow-up questions were standard, such as: “Why?”, “What do you mean by that?”, and “Can you give me an example?” There were also times when the follow-up questions were specific to the given situation. The interviews were audio taped and transcribed.

### **3. Data Analysis**

There are three major components of qualitative research: data, procedure for analyzing the data, and written and verbal reports (Strauss & Corbin, 1988). The data for this study are the Chinese and Korean teachers' responses to the structured task-based interview question. The participants' responses take the form of spoken and written materials. The first procedure for analyzing the interview data involved creating a conceptual framework for describing a teacher's conceptual understanding of a mathematical topic based on existing conceptual frameworks of teachers' subject matter knowledge, focusing especially on the framework developed by Ma (1999). The second step for analyzing the data involved predetermining which mathematical sub-topics and concepts are relevant for the teaching task, and then conducting a literature review about the topics and concepts.

In light of the creation of a conceptual framework and the literature review, the participants' verbal responses to the teaching task was analyzed through “microscopic examination” (Strauss & Corbin, 1988). This type of detailed line-by-line analysis involves very careful, often minute examination and interpretation of the data. The results from analyzing the participants' responses in this study were sometimes compared with those of Ball (1988) and Ma (1999).



## IV. Results

### Teachers' Reactions to the Student

Ball (1988) suggested that teachers' possible reactions to a new idea proposed by a student fell into three main categories: divert the student from pursuing ideas outside the scheduled curriculum, be responsible for evaluating the truth of the student's claim, and engage the student in exploring the truth of her claim (p.166). In terms of Ball's framework, Ma (1999) reported that the American and Chinese elementary teachers showed the second and third types of reactions. Additionally, most teachers responded that they would give a positive comment to the student (p. 89). Thus, Ma categorized the American and Chinese teachers' reactions to the student in terms of praise with explanations and praise with engagement in further exploration.

In the present study, the Chinese and Korean teachers' reactions to the student were very similar to the teachers' reactions in Ma's (1999) study. They first praised the student for attempting to discover such a mathematical fact on her own. Then the teachers took responsibility for evaluating the truth of the student's claim and explaining the result to her. However, in relation to the third type of reaction—engaging the student in exploration of the truth of her claim—there was a difference between the results of Ball's (1988) and Ma's (1999) studies and the present study. Ball and Ma reported that the teachers engaged the student in exploring the truth of her claim by herself, but in the present study, the teachers invited the student to participate in the teachers' own explorations for evaluating the truth of the claim. Therefore, the teachers' reactions to the student in the present study were summarized into two main protocols: *praise with teachers' own exploration/explanation* and *praise with exploration/explanation with the student*.

### Teachers' Explorations of the Student's Claim

The Chinese and Korean teachers' first reactions to the student's claim were similar to the American and Chinese elementary teachers'

reactions in Ma (1999). Most teachers did not immediately determine whether the claim is true or not, and they asked the interviewer for more time to evaluate whether the student's claim is true or false. Once they started to explore the claim, Korean teachers' behaviors were considerably different from those of the Chinese teachers. Most Chinese teachers kept quiet during the explorations, whereas most Korean teachers posed questions for the student to answer. The questions the Korean teachers posed for the students were intended not only to involve the student in exploring the truth of her claim, but also to clarify the student's claim. The questions most frequently posed by the Korean teachers are summarized below.

- Do you think that it works for every type of closed figure?
- You proved that it is true with the case of a square. Did you check whether it works for other figures?
- When the perimeter of the figure increases, does it matter whether the basic shape of the figure is changed, such as changing from a rectangle to a triangle, or from a triangle to a circle?

When the teachers finished exploring the student's claim, they first proclaimed the falsity of the claim: all Korean teachers evaluated the claim correctly, while two of the 9 Chinese teachers said, "I am not sure," and one of the unsure Chinese teachers suggested that the claim may be true. These two Chinese teachers tried to find a counterexample, but they failed. The results are summarized in Table 1.

Table 1

*Chinese and Korean Teachers' Evaluations of the Student's Claim*

	Chinese Teachers N=9	Korean Teachers N=10
The student's claim is false	7	10
I am not sure	1	-
I am not sure, but it seems true	1	-

*Teachers' Approaches for Disproving the Student's Claim*

All of the 17 Chinese and Korean teachers who said the student's claim is false attempted to explain why the claim is not valid, but some teachers failed in justifying their answers. Of the 7 Chinese teachers, 6 teachers justified their answers with counterexamples, but one teacher failed to find a counterexample. On the other hand, among the 10 Korean teachers, 6 teachers succeeded in finding counterexamples and one teacher could not find a counterexample. The remaining three teachers attempted to justify their answers in general symbolic reasoning by identifying the widths, heights, and areas of figures with two letters and their product, but they could not complete the reasoning. Table 2 presents the distribution of the teachers' approaches to disproving the student's claim and the results of their approaches.

Table 2

*Distribution of Teachers' Approaches to Disproving the Student's Claim and Results*

Approaches for disproving the claim	Chinese Teachers N=7	Korean Teachers N=10	Total N=17
Succeeded in finding counterexamples	6	6	12
Failed to find counterexample	1	1	2
General symbolic reasoning (incomplete)	-	3	3

Table 2 shows that the most frequent approach to disproving the student's claim was to find a counterexample. Epp (1998) explained that the basic method used to disprove most mathematical statements is to search for a counterexample, and it is just as important to teach students how to disprove statements as how to prove them. Watson (2001) also emphasized the importance of asking whether there are counterexamples when testing generalizations. He asserted that trying to think of counterexamples gives students a chance to correct overgeneralizations by themselves (p.18).

### *Types of Teachers' Disproof Schemes*

Sowder and Harel (1998) have developed a framework for classifying students' proof schemes that contains three categories: externally based proof schemes, empirical proof schemes, and analytic proof schemes, with subcategories for each. This study determined that some of Sowder and Harel's (1998) subcategories of proof schemes can serve as the framework for organizing the teachers' schemes for disproving the student's claim. The following four types of disproof scheme were identified in this study: *the examples-based disproof scheme, the symbolic disproof scheme, the perceptual disproof scheme, and the transformational disproof scheme*. These four types of disproof schemes were used in analyzing the disproof schemes of the teachers who provided correct solutions, and the distribution is presented in Table 3.

Table 3

*Distribution of Types of Teachers' Disproof Schemes*

Types of disproof Schemes	Chinese Teachers N=7	Korean Teachers N=10	Total N=17
Examples-based disproof schemes	3	5	8
Perceptual disproof schemes	4	2	6
Transformational disproof schemes	-	3	3

**Examples-based disproof schemes.** The teachers who provided particular figures, whose widths, heights, and areas were identified with specific numbers, can be classified into examples-based disproof schemes. They identified the length, width, and area of a new figure that was generated by increasing the perimeter of the original figure. With the original figure and the new figure, teachers tried to show that although the perimeter of the original figure increases, the area of the figure does not always increase: it can decrease or remain unchanged. Through the latter two types of examples, teachers can encourage the student to see that her claim breaks down under certain circumstances (see Table 3).

Table 4

*Possible Changes of the Area of a Closed Figure when the Perimeter of the Figure Increases*

Possible Cases	Original figures	New figures	Feasibility as a counterexample
Case 1	Perimeter = $a$ Area = $b$ ( $a, b$ : positive numbers)	Perimeter = $a + k$ ( $k$ : a positive number) Area = $c$ ( $c > b$ ) ( $c$ is a positive number)	Not feasible
Case 2	Perimeter = $a$ Area = $b$	Perimeter = $a + k$ Area = $b$	Feasible
Case 3	Perimeter = $a$ Area = $b$	Perimeter = $a + k$ ( $k$ is a positive number) Area = $c$ ( $c < b$ )	Feasible

As shown in Table 3, among the total 17 teachers who provided correct answers, three of the 7 Chinese teachers' and five of the 10 Korean teachers' disproof schemes were classified under examples-based proof schemes. The Chinese and Korean teachers preferred to use the particular  $4 \times 4$  square and  $4 \times 8$  rectangle mentioned by the student as the original figures. They identified the length, width, and area of a new figure by changing a pair of opposite sides or by changing both sets of opposite sides of the original figures, and then compared the perimeters and areas of the two figures. Of the 8 Chinese and Korean teachers' examples, seven examples coincided with Case 3 in Table 4: although the perimeter of the original figure increases, the area of the figure decreases. The response of one Korean teacher who belongs to this group is presented below.

I would explain it in this way...Let us think together. The perimeter of this square is 16 feet and its area is 16 square feet. As you said, when the perimeter of the square is increased to 24 feet, its area is increased to 32 feet. But, let us look for other rectangles whose perimeters are also 24 feet. Look at this rectangle [see below]. The perimeter is the same: 24 feet, but its area is 11 square feet. That is, the perimeter of the original square increased, but the area decreased. Do you still believe that what you said is true? If the student says, "I got it," I would engage the student in

thinking of “Are rectangles the only type of closed figure?” or I would say that there are many kinds of closed figures. If the student said, “how about circles?” I would say that your idea works with circles because the size of the perimeter and the area of a circle depend on the radius of the circle. To increase the perimeter of a circle, it is necessary to increase the radius of the circle, and it necessarily increases the area too. Think about the other closed figures under which your idea works. How about equilateral triangles? It works. That is, your claim works under the condition of symmetry.

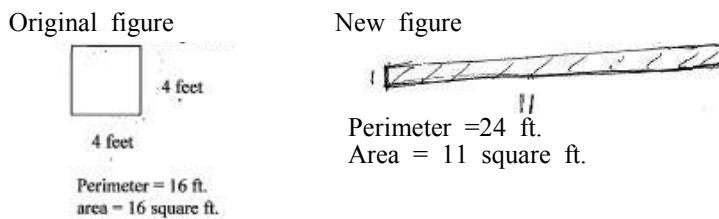


Figure 1. Examples-based disproof scheme

The remaining Chinese teacher provided an irrelevant counterexample that as the perimeter of the original figure remains unchanged, the area of the figure decreases: Perimeter =  $a$ , Area =  $b \Rightarrow$  Perimeter =  $a$ , Area =  $c$  ( $c < b$ ). The student’s claim is that as the perimeter of the figure *increases*, the area of the original figure *increases*. In order to reject the claim, teachers must prove that the area of the original figure remains *unchanged* or *decreases* when the perimeter of the figure *increases*. This Chinese teacher instead showed what can happen when the perimeter of the figure remains the same. This corresponds to the formal logical fallacy of denying the antecedent. Thus, this Chinese teacher’s justification is problematic, despite the fact that the teacher properly judged the student’s claim as false.

**Perceptual disproof schemes.** The teachers whose schemes are based on perceptual disproof schemes have arrived at conclusions on the basis of their perceptions of a single or several drawings. These teachers try

to convince others by showing them a drawing, so that it is not necessary to use numbers or symbolic letters in the process of disproof.

As shown in Table 3, four Chinese teachers and two Korean teachers revealed perceptual disproof schemes. One interesting finding is that these four teachers' common characteristic is that their conceptions of a closed figure were not restricted to particular figures, such as squares and rectangles. While most teachers' conceptions of a closed figure in the present study were limited to regular and irregular convex polygons and circles, these teachers' conceptions of a closed figure were extended to regular and irregular concave polygons and curves. In other words, the teachers whose schemes were based on the former two disproof schemes focused on discussion of the particular figures, squares and rectangles, but these teachers came up with the idea that the term "a closed figure" includes not only convex polygons, but it also includes concave polygons and concave curves.

The four Chinese and one Korean teachers' common strategy for disproving the claim was to perceptually compare the perimeters and areas of two closed figures. Four of the five teachers compared the perimeter and area of a convex polygon or circle with the perimeter and area of a concave polygon or a concave curve inscribed in the convex polygon or the circle (see Figure 2 and 3). One Chinese teacher compared the perimeters and areas of two separate concave curves (see Figure 4).

This claim is false. This student did not seem to understand the concept of a closed figure. She is talking with only these two figures [squares and rectangles]. In this case, the claim is valid, but she has to be careful about the fact that there are many kinds of closed figures like this [drawing a concave pentagon inscribed in the 4x4 square]. The perimeter of the square increases, but the area decreases. I would engage her in thinking of these varieties of closed figures on her own at home for herself to realize her errors.



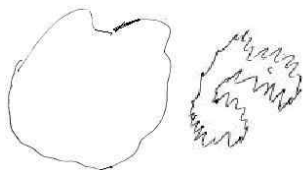
*Figure 2.* Comparison of the perimeter and area of a concave pentagon inscribed in a square

This student's claim is valid in this example, but she does not seem to understand the rigorous nature of mathematics. Her claim is valid under the condition of convex figures, but it is wrong with concave figures like this [drawing a figure]. As shown in this figure, although the perimeter of the original figure increases, its area decreases. The student probably thought of only convex figures because she is familiar with those figures.



*Figure 3.* Comparison of the perimeter and area of a concave curve inscribed in another concave curve

This claim is not true because she did not mention the specific figures she talked about. There are many shapes of closed figures having perimeters such as triangles, rectangles, circles, squares and so on. What if she said that as the perimeter of a square increases, the area also increases, it is true. However, she did not give the specific shapes of the figures except for telling a closed figure. Middle school students can focus on the discussion about particular figures without thinking of other different shapes of closed figures. Thus, teachers should show a variety of closed figures like this.



*Figure 4.* Comparison of the perimeters and areas of two concave curves

The remaining Korean teacher's counterexample derived from the perceptual disproof scheme was distinguished from those of the former four Chinese and one Korean teachers. This Korean teacher came up with a *Koch snowflake* that is the limit of an infinite construction. This Korean teacher's response is provided below.



After praising her, I would help her to find a counterexample like a Koch snowflake. In the Koch snowflake, while its perimeter infinitely increases, its area has the limit approaching a finite number so the area will be represented with a constant number at the end of the infinite construction.



*Figure 5.* A Koch snowflake

**Transformational disproof schemes.** Sowder and Harel (1998) regarded transformational proof schemes as a necessary precedent to their final proof schemes, axiomatic proof schemes. The axiomatic proof schemes are based on the idea that a body of mathematical knowledge is carefully organized by undefined terms (axioms), definitions, assumptions, and theorems, so that subsequent mathematical results are logical consequences of those components. On the other hand, the transformational proof schemes are concerned with the general aspects of a situation and involve reasoning rather than observing patterns of specific cases. Sowder and Harel (1998) believed that mathematics teachers regard these two analytic proof schemes as the ultimate types of justification in mathematics.

In the same manner, the present study identified teachers' disproof schemes, focusing on the general aspects of a mathematical statement as transformational disproof schemes. These teachers did not focus on rejecting the mathematical statement by searching for a particular counterexample; rather, they focused on testing whether the statement works for all cases in general symbolic reasoning. These teachers were aware of the advantage of algebraic reasoning in overcoming the fact that a mathematical statement concerning an infinite number of cases cannot be proved true by a finite number of examples, no matter how large that number is.

In the present study, three Korean teachers' disproof schemes were classified as transformational disproof schemes. They identified the width,

height, and area of a rectangle with two letters and their product, and then attempted to observe the change of the area of the figure by increasing the perimeter of the figure. The response of one Korean teacher who belongs to this group is presented below.

I thought about it this way, but I am not sure it is correct...if we suppose that the perimeter of a rectangle is constant ...the area of the figure is represented by a quadratic equation for  $x$ . The quadratic equation can show that the area of the figure increases as the perimeter of the rectangle increases by increasing the side  $b$ , but the increasing trend of the area turns to a decreasing trend at a certain point.




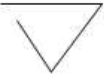


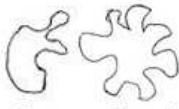

The figure shows handwritten mathematical work. On the left, a rectangle is drawn with height  $\frac{K-2b}{2}$  and width  $b$ . The perimeter is labeled as  $2(a+b) = K$  and the area as  $ab$ . To the right, the equations  $2a = K - 2b$  and  $a = \frac{K-2b}{2}$  are written. Further down, the area is expressed as a quadratic equation:  $\frac{K-2b}{2} b = \frac{bK-2b^2}{2} = -b^2 + \frac{bK}{2}$ . To the right of these equations is a hand-drawn downward-opening parabola, representing the quadratic area function.

Figure 6. Example of transformational disproof schemes

### Teachers' Perceptions of a Closed Figure

Examinations of the types of closed figures that served as counterexamples revealed the Chinese and Korean teachers' concepts of a closed figure. A closed figure is defined in terms of a figure that can be traced with the same starting and stopping points, and without crossing or retracing any part of the figure. Any polygon and any closed curve, including a circle, is an example of a closed figure. Table 5 below provides examples and non-examples of closed figures to help readers identify closed figures. Among the examples of a closed figure, figures (B), (C), (D), and (F) were excerpted from the Chinese and Korean middle school teachers' written responses.

Table 5  
*Examples and Nonexamples of a Closed Figure*

Examples			Nonexamples
(A)  Convex, Regular polygons	(B)  Convex, Irregular polygon	(C)  Concave Irregular polygon	 Not a polygon
(D)  Concave, Regular polygon	(E)  Circle	(F)  Concave, Closed curves	 Not a closed curve

Conducting interviews with the Chinese and Korean teachers made it clear that the concept of a closed figure played a key role in successfully responding to the student's claim. When the teachers were confronted with the interview question, their common first reactions were to identify the meaning of a closed figure. The teachers who came up with concave polygons or concave curves, such as examples (C), (D), and (F), immediately rejected the student's claim, while the teachers whose concepts of a closed figure were restricted to the particular figures, such as squares, rectangles, and triangles took a relatively long time to reject the student's claim, or they failed to find a counterexample to disprove the student's claim.

Ball (1988) and Ma (1999), who previously used the same interview question as the present study, however, were not concerned much with the teachers' conceptions of a closed figure as an important mathematical concept embedded in the interview question. Ball (1988) focused on exploring the preservice teachers' knowledge of proof, in addition to their awareness of the relationship between the perimeter and area of particular figures, squares and rectangles. Ma (1999) very briefly mentioned the elementary teachers' reactions to the term "a closed figure" as follows:

The term "a closed figure" used in the scenario was intended to invite the teachers to discuss various kinds of figures. However, during the interviews teachers talked exclusively about squares and rectangles. A

few Chinese teachers said that closed figure is a concept introduced at the secondary school level in China so they preferred to focus the discussion on the particular figure mentioned by the student(p.84).

I assume that the teachers' evaluations of the student's claim might be based on their chances of finding counterexamples, and their likelihood of finding counterexamples depends on their conceptions of a *closed figure*. But, Ball (1988) and Ma (1999) were not concerned with this fact. I believe that the fact that these researchers did not focus on conceptions of closed figures could lead to misinterpretations of the teachers' responses. For example, in Ma's (1999) study, of the 23 American elementary teachers, 18 teachers responded to the teaching task with the answer "not sure" (p. 92). In relation to these American teachers' not sure responses, Ma explained that "most U.S. teachers who held a 'notsure' opinion avoided a wrong answer"(p.91). However, my experience with the Chinese and Korean teachers in my study allowed a different point of view about those teachers' "notsure" answers. In the present study and in Ball's(1988) study, during the interviews, when the teachers looked at the student's claim, they assumed that the claim was false, and began searching for a counterexample to show this. But, when the teachers could not come up with a counterexample, the teachers' suspicion about the claim weakened. In such situations, the teachers tended to respond with "Iamnotsure" or "It seems true." In other words, the reason the teachers gave unclear answers was not simply to avoid the wrong answer; rather, these "not sure" responses may reflect the teachers' inability to obtain enough evidence to be able to reject the claim even though they still suspect the claim is false. If the "notsure" teachers had been aware of a variety of closed figures beyond the particular figures, they might have been able to easily find counterexamples, and under these circumstances, many of these teachers might no longer respond with "I am not sure."

Actually, many of the Korean teachers in the present study testified that they have little experience with concave polygons and curves in their everyday life, so they tend to only use the particular polygons or curves that are presented in mathematics textbooks when teaching geometric facts. One Korean teacher's testimony illustrates this point:

There are two types of polygons: convex and concave polygons. However, we usually have experience with convex polygons through mathematics textbooks and materials, so that once we are confronted with the term “ a closed figure,” we tend to just think of convex polygons, and thus tend to explain geometrical things with only those convex polygons. Some students think that a concave quadrilateral is not a quadrilateral. I believe that the teachers who can come up with those concave polygons or curves have more flexible thought than those who cannot come up with those figures.

### **Teachers' Understanding of Mathematical Proof**

Mathematical proof does not take the form of an “inductively valid argument,” which is an argument whose conclusion is proved by examples, analogy, and authority. Mathematical proof takes the form of a “deductively valid argument,” where the conclusion *must* be true if *all* of the premises of an argument are true and the logic of the argument is correct (Anderson,1985). Weston(2000) identified the former type of proof as “non deductive argument” and the latter type of proof, including mathematical proof, as “deductive argument.”

Martin and Harel (1989) emphasized the importance of teachers' understanding of what constitutes mathematical proof. They stated that because proof receives very limited attention in elementary school curriculum, the main source of children's experience with verification and proof is the classroom teacher. Thus, classroom teachers' understanding of mathematical proof is important, even though they do not directly teach that topic. They argued that teachers' frequent use of examples in verifying mathematical statements in early grades may reinforce their students' belief that a few-well chosen examples can serve as a legitimate process of mathematical proof, and it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students (p. 42).

In her 1988 study, Ball explored the prospective teachers' “justification of knowledge–proof” using the same interview task that is used in the present study. In Ball's (1988) study, the teachers did not concentrate on the fact that examples are not positive proof of mathematical generalizations,

although students take them to be so. Instead, the teachers focused on the substance of the student's claim—the specific concepts of perimeter and area and their relationship. The teachers were more concerned to teach the students about the relationship between perimeter and area, rather than teaching them about the nature of mathematical proof (p.170).

In the present study, most of the Chinese and Korean teachers urged the student to revisit her claim, which holds for specific closed figures, to check whether it will hold with all closed figures. Some of the teachers explained to the student why she has to check all closed figures based on their understanding of mathematical proof. Four Chinese teachers and one Korean teacher gave the student general guidelines regarding the nature of mathematical proof, but their explanations during the interview were superficial and incomplete. The responses of one Korean teacher and one Chinese teacher regarding the nature of mathematical proof are presented below.

Have you thought about any cases in which your idea does not work? When you think that you found a new mathematical theorem, you should check whether there are any exceptions in which the theorem does not work. Did you check whether there is a counterexample? In order to accept your theorem, we need to prove that the theorem works for every case, but before doing that, we can perceptually check whether there is a counterexample or not. Focus on searching for an example in which your theorem does not work, and then if you cannot find it, I will help you to find it. (Korean teacher)

Providing examples is not proof of mathematical generalizations. Mathematical theorems should be established through the precise and formalized process of mathematical proof showing that the theorem works for every case. (Chinese teacher)

The Korean teacher provided more explanations than that of the Chinese teacher, but both of the teachers' explanations were still superficial. These two teachers' responses regarding the nature of mathematical proof were

better than the responses of the other three teachers who addressed the nature of mathematical proof.

## V. Discussion

When the Chinese and Korean teachers were given the task of responding to the student's mistaken mathematical claim about the relationship between the perimeter and area of a closed figure, they initially focused on evaluating the truth of the claim. Most teachers' common approach to evaluating the truth of the claim was to check whether there is a counterexample that disproves the claim. These teachers' success in correctly evaluating the claim entirely depended on the likelihood of finding a counterexample. Those teachers whose concept of a closed figure included concave polygons and other curves beyond particular convex polygons were able to easily identify counterexamples to perceptually disprove the student's claim. These teachers urged the student to understand the nature of mathematical proof after proclaiming the falseness of the claim. By contrast, those teachers whose concept of a closed figure was restricted to particular convex polygons, such as squares and rectangles, experienced more difficulty identifying counterexamples to the student's claim. Some were able to identify a counterexample, but it took them a relatively long time; others failed to identify a counterexample to the student's claim. The common approach of those teachers who sought a counterexample is presented in Figure 7 below.

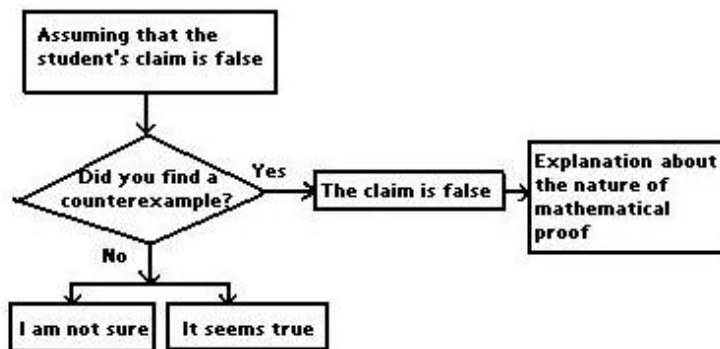


Figure 7. Approaches to disproof by counterexample

Three Korean teachers did not solely focus on looking for a counterexample. They concentrated on the nature of mathematical proof-thinking in terms of a “deductively valid argument,” where the conclusion of an argument *must* be true if *all* of the premises of the argument are true. These teachers attempted to test the truth of the student’s claim using general algebraic reasoning. They tried to examine the relationship of the perimeter and area of particular figures, such as rectangles, by identifying the width, height, and area of the figures with two random numbers and their product. However, these three teachers did not complete their proofs, so that they ultimately failed to explain why the student’s claim is false, despite their correct evaluations of the claim.

The approach using general algebraic reasoning is based on the transformational proof schemes that are desirable proof schemes for mathematics teachers. Sowder and Harel (1998) emphasized the transformational proof schemes as a necessary precedent to the ultimate proof schemes, axiomatic proof schemes in mathematics. However, the transformational proof schemes seemed to be inappropriate for testing the truth of the student’s claim. The student’s claim is about all closed figures, not about a particular closed figure, such as rectangles and triangles, whose perimeter and area can be calculated using a simple formulation. Even though the teachers succeeded in proving that the student’s claim is true for all rectangles, they failed to demonstrate whether the claim holds for every case of a closed figure. The three Korean teachers did not seem to realize that transformational proof schemes are inappropriate for testing the truth of the student’s claim; they attempted to use general algebraic reasoning to test the truth of the student’s claim based on the mathematical habits established in their everyday lives—habits that favor deductive reasoning.

These Korean teachers’ responses indicate that explaining specific relationships between the perimeter and area of particular closed figures is not the appropriate reaction to the students’ erroneous claim. The teachers in Ma’s (1999) study reacted similarly by examining the relationship between perimeter and area for specific closed figures, although Ma did not identify the fact that this type of approach is inappropriate. The existence of many different shapes of closed figures means that there exist many different relationships between perimeters and areas among the closed figures. It



would be impossible for a teacher to explain the enormous quantity of such relationships to the student. Thus, the key point of an appropriate response to the student's claim involves helping the student to realize that her claim is conditionally true, but that it cannot be generalized to every type of closed figure.

In addition to leading the student to the realization that her claim is only conditionally true, the teacher should teach the student to understand the nature of mathematical proof. Therefore, the common protocol of many of the Chinese and Korean teachers' can be considered a desirable response; these teachers reacted to the student's claim by providing a counterexample and explaining—based on the nature of mathematical proof—the insufficiency of a single example in establishing the claim.

Although the common protocol for reacting to the student's claim was appropriate, it was clear that most Chinese and Korean teachers' concepts of a closed figure were dominated by particular figures, such as convex polygons and curves. The teachers' problematic concept of a closed figure resulted in difficulty finding counterexamples and interrupted the teachers' otherwise legitimate reasoning. Petty and Jansson (1987) explained that the defining attributes of geometric figures were made more salient by presenting instances in rational sequences. The teachers' problematic concepts of a closed figure might be the result of their limited geometry learning experiences, which focused on particular instances of closed figures. Therefore, the findings of this study suggest that teachers need to understand the attributes of geometrical figures using rich examples and non-examples that go beyond the particular figures occupying traditional geometry textbooks. Even though geometry textbooks focus on explaining the concept of a figure using only particular instances, teachers should be encouraged to provide even more abundant instances in rational sequences. This type of instruction, using abundant examples of various kinds presented in a rational sequence, can play an important role in facilitating students' proper conceptions of geometric figures.

Another finding from this study was that the Chinese and Korean teachers did not provide clear explanations about why a single example, or even an infinite number of examples, cannot constitute positive proof in mathematics. To explain the nature of mathematical proof, the teachers must

understand the difference between inductively valid reasoning and deductively valid reasoning. In addition, they should understand the functions of proof in mathematics. In the present study, it is difficult to conclude that the Chinese and Korean teachers' unclear and superficial explanations about mathematical proof resulted from their insufficient understanding of the properties of proof in mathematics because the teachers mostly focused on determining the truth of the student's claim during the interview; they did not speak extensively about their understandings of mathematical proof. In spite of this limitation, assuming that the teachers' unsatisfactory explanations were based on their beliefs that it is too hard for young students to formulate a mathematical conjecture and test it, the teachers should be encouraged to help the students to understand the nature of mathematical proof in a way that is comprehensible to them. But prior to looking for an explanation that will be comprehensible to young students, teachers should understand that the functions of proof in mathematics go beyond the role of verifying that ready-made theorems are true (Battista & Clements, 1995). By being led through process of mathematical proof, students have the opportunity to make new discoveries, provide their own insight into why the discoveries may not be true, and systematize their findings. These capabilities are not only needed for solving mathematical problems, but they are also critical skills for logically solving real-life problems.

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