16-QAM Periodic Complementary Sequence Mates Based on Interleaving Technique and Quadriphase Periodic Complementary Sequence Mates

Fanxin Zeng, Xiaoping Zeng, Lingna Xiao, Zhenyu Zhang, and Guixin Xuan

Abstract: **Based on an interleaving technique and quadriphase periodic complementary sequence (CS) mates, this paper presents a method for constructing a family of 16-quadrature amplitude modulation (QAM) periodic CS mates. The resulting mates arise from the conversion of quadriphase periodic CS mates, and the period of the former is twice as long as that of the latter. In addition, based on the existing binary periodic CS mates, a table on the existence of the proposed 16-QAM periodic CS mates is given. Furthermore, the proposed method can also transform a mutually orthogonal (MO) quadriphase CS set into an MO 16-QAM CS set. Finally, three examples are given to demonstrate the validity of the proposed method.**

Index Terms: **Aperiodic and periodic correlation, complementary sequence set, quadrature amplitude modulation (QAM) constellation, quaternary sequence.**

I. INTRODUCTION

As multimedia services, which require supporting devices with a high transmission data rate (TDR), become increasingly popular in communications, the design of high-TDR communication systems has assumed critical importance. In many potential candidates, spread-spectrum code-division multiple-access (CDMA) communication systems, employing sequences over a quadrature amplitude modulation (QAM) constellation as their spreading sequences, are significant owing to their inherent advantages. More clearly, in comparison with the systems using traditional spreading sequences of the same length, the TDR of the former is a multiple of that of the latter [1]–[5]. In addition, there also are other advantages related to sequences over the QAM constellation, such as the fact that QAM Golay complementary sequence (CS) sets can be applied to an orthogonal frequency division multiplexing (OFDM) communication system so as to reduce the peak-to-mean envelope power ratio (PMEPR) in such a system [6]–[16], and zero correlation zone (ZCZ) QAM sequences can be employed in an approximately synchronous CDMA communication system so as to remove multiple access interference (MAI) and multi-path interference (MPI) synchronously and completely [17], [18].

This paper will focus on construction of QAM periodic CS sets. The technique invariably used to investigate CSs is to divide them into periodic and aperiodic cases, respectively; for example, QAM Golay CS sets, which were referred to above, belong to the aperiodic case. Tarokh and Sadjadpour derived a method for producing QAM CSs with a low PMEPR by quadriphase Golay CSs [6]. Davis and Jedwab found the relationship between Golay CSs and Reed-Muller codes [7]. Sadjadpour described a family of non-square M-QAM sequences [8]. Lee and Golomb discussed 64-QAM Golay CSs [10], and so did Chang, Li, and Hirata [9]. Li investigated both 16-QAM and 64-QAM Golay CSs [12], [13], and Zeng *et al.* discussed 16-QAM Golay CSs as well [14], [15]. In addition, Fiedler, Jedwab, and Parker proposed a framework for constructing Golay CSs [16]. On the other hand, although periodic and aperiodic sequences are equally important in communications, all aperiodic CSs must be periodic ones, whereas the reverse may not hold [19, p. 332]. Hence, investigation of QAM periodic CS sets is valuable. For instance, QAM sequences with low correlation [1]–[5], ZCZ QAM sequences [17], [18], 8-QAM+ periodic CSs [20], and almost perfect or perfect QAM sequences/arrays [21]–[24] are periodic sequences. In this paper, a family of 16-QAM periodic CS mates will be presented that results from an interleaving technique and quadriphase periodic CS mates.

This paper is organized as follows. In Section II, the relevant definitions referred to in this paper are recapitulated. In Section III, the basic properties of some 16-QAM sequences are discussed. In Section IV, the proposed 16-QAM periodic CS mates are described, and three examples are included. Finally, the concluding remarks are made in Section V.

II. PRELIMINARIES

Throughout this paper, $B^l = (\underline{b}_0^l, \underline{b}_1^l, \cdots, \underline{b}_{M-1}^l)$ denotes a sequence set that consists of M subsequences, where $\underline{b}_d^l = \{b_d^l(t)\} = (b_d^l(0), b_d^l(1), b_d^l(2), \cdots, b_d^l(N-1))$ $(0 \le d \le M-1)$ denotes a sequence comprising complex values, having length N, with $\overline{b_d^l(t)}$ being the complex conjugate of $b_d^l(t)$ and the symbol j being an imaginary unit; that is, $j^2 = -1$.

Definition 1: For $\forall\ \underline{b}_d^l \in B^l$ and $\forall\ \underline{b}_k^h \in B^h$, for a time shift τ, we refer to

Manuscript received June 1, 2012; approved for publication by Shin, Dong-Joon, Division I Editor, June 23, 2013.

This work was supported by the National Natural Science Foundation of China (NSFC) under Grants 60872164, 61002034, 61171089, and 61271003, and the Ministry of Industry and Information Technology of China (No. Equipment[2010]307).

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Digital object identifier 10.1109/JCN.2013.000107

$$R_{b_d^l, b_k^h}(\tau) = \sum_{t=0}^{N-1} b_d^l(t)\overline{b_k^h(t+\tau)} \qquad (1)$$

as a periodic correlation function (CF) between $\underline{b}_d^l$ and $\underline{b}_k^h$, where the sum $t+\tau$ is calculated modulo N. If $l = h$ and $d = k$, $R_{b_d^l, b_d^l}(\tau)$ is called a periodic autocorrelation function (ACF); otherwise, it is called a periodic cross-correlation function (CCF).

Definition 2: Let $B^l = (\underline{b}_0^l, \underline{b}_1^l, \cdots, \underline{b}_{M-1}^l)$ consist of M subsequences, each of length N. If we have

$$\sum_{k=0}^{M-1} R_{b_k^l, b_k^l}(\tau) = \begin{cases} > 0, & \tau \equiv 0 \pmod N \\ 0 \quad , & \tau \not\equiv 0 \pmod N \end{cases} \qquad (2)$$

we refer to the sequence set B^l as a periodic CS set, denoted by $PCSS_H(B^l, M, N)$, where $H = 4$ and $H =$ 16-QAM imply that the corresponding sequences are quaternary and 16-QAM ones, respectively. When $M = 2$, the sequence set B^l is called a periodic CS pair.

Notes: In (2), when $\tau \equiv 0 \pmod N$, the sum in the traditional CS sets equals MN, whereas it equals a multiple of MN in the 16-QAM CS sets (see Section IV for more details).

Definition 3: For $PCSS_H(B^l, M, N)$ and $PCSS_H(B^h, M, N)$, if we have

$$\sum_{k=0}^{M-1} R_{b_k^l, b_k^h}(\tau) = 0 \quad (\forall\, \tau) \qquad (3)$$

we say that the sequence sets B^l and B^h are the mates to each other.

Definition 4: Let A consist of T periodic CS sets $PCSS_H(B^l, M, N)$ $(1 \le l \le T)$. For $\forall\ l,\ h$ $(1 \le l, h \le T$ and $l \ne h)$, if $PCSS_H(B^l, M, N)$ and $PCSS_H(B^h, M, N)$ are the mates to each other, we refer to A as a mutually orthogonal (MO) CS set.

Definition 5: Let the symbol L denote a left cyclic shift operator. This is, for a sequence $\underline{b}_d^l \in B^l$ and an integer ζ, $L^\zeta \underline{b}_d^l = (b_d^l(\zeta), b_d^l(\zeta+1), \cdots, b_d^l(\zeta+N-1))$, in which the addition $\zeta + t$ $(t \in [0, N-1])$ is performed modulo N. For two sequences $\underline{b}_d^l$ and $\underline{b}_k^h$, if there exists an integer ζ such that $\underline{b}_d^l = L^\zeta \underline{b}_k^h$, these two sequences are said to be equivalent; otherwise they are said to be distinct. For $PCSS_H(B^l, M, N)$ and $PCSS_H(B^h, M, N)$, if there is an integer ζ such that $\underline{b}_k^l = L^\zeta \underline{b}_k^h$ $(0 \le k \le M-1)$, these two periodic CS sets are said to be equivalent; otherwise, they are said to be distinct.

A. 16-QAM Symbols

The 16-QAM symbols can be driven by the QPSK symbols [6]. This is, the 16-QAM symbols can be expressed by [1], [2].

Expression 1:

$$\{(1+j)(j^{a_0} + 2j^{a_1}) | a_0, a_1 \in Z_4\} \qquad (4)$$

Expression 2:

$$\{(1-j)(j^{a_0} - 2j^{a_1}) | a_0, a_1 \in Z_4\} \qquad (5)$$

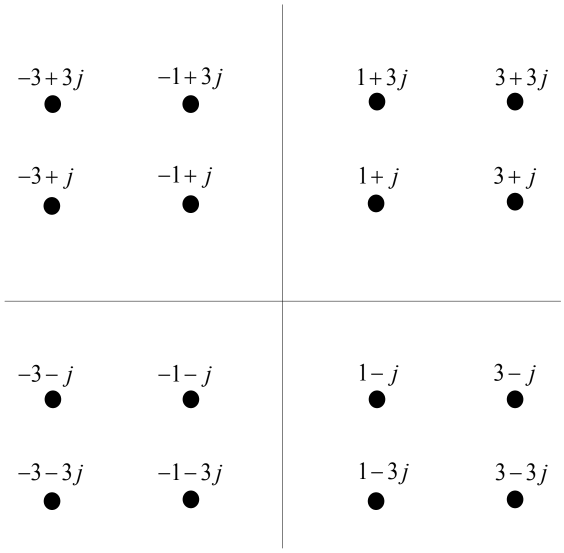

Fig. 1. 16-QAM constellation.

both of which are mainly employed by [1]–[8], where $Z_4 = \{0, 1, 2, 3\}$.

Apart from those aforementioned two expressions, the authors find that the 16-QAM symbols can be equivalently described as follows

Expression 3:

$$\{-j(1+j)(j^{a_0} + 2j^{a_1}) | a_0, a_1 \in Z_4\} \qquad (6)$$

Expression 4:

$$\{j(1-j)(j^{a_0} - 2j^{a_1}) | a_0, a_1 \in Z_4\}. \qquad (7)$$

In fact, Expression 3 makes four rotations of 16-QAM symbols such as $1+j \to 1-j \to -1-j \to -1+j$ or $1+3j \to 3-j \to -1-3j \to -3+j$ and so on. Similarly, Expression 4 also makes four rotations. In particular, it is exactly by utilizing this property of four expressions that we construct the 16-QAM periodic CS mates. The 16-QAM constellation is shown in Fig.1.

B. An Interleaved Sequence

For the sequences $\{b_d^l(t)\}$ and $\{b_k^h(t)\}$, each of period N, we construct a new sequence called an interleaved sequence denoted by $\{b_d^l(t')\} \odot \{b_k^h(t')\}$ or $\underline{b}_d^l \odot \underline{b}_k^h$, as follows

$$\underline{b}_d^l \odot \underline{b}_k^h = (b_d^l(0), b_k^h(0), b_d^l(1), b_k^h(1), \cdots, b_d^l(N-1), b_k^h(N-1)) \qquad (8)$$

which implies that the interleaved sequence has a length of $2N$.

III. NEW 16-QAM SEQUENCES AND THEIR PROPERTIES

Before we present the main results in this paper, some basic correlation properties of the new 16-QAM sequences are discussed. For the sake of convenience, throughout this section, let

$U^l = (\underline{u}_0^l, \underline{u}_1^l, \cdots, \underline{u}_{M-1}^l)$ and $U^h = (\underline{u}_0^h, \underline{u}_1^h, \cdots, \underline{u}_{M-1}^h)$ consist of M quadriphase subsequences of length N and the nonnegative integer δ satisfy $0 \le \delta \le N-1$.

For $\forall\, k$ $(0 \le k \le M-1)$, the new 16-QAM sequences based on four expressions in (4)–(7) are defined as follows

$$\begin{aligned} v_k^{+,l,h}(t) &= (1+j)(j^{u_k^l(t)} + 2j^{u_k^h(t+\delta)}),\\ v_k^{-,l,h}(t) &= (1-j)(j^{u_k^h(t+\delta)} - 2j^{u_k^l(t)}),\\ w_k^{+,l,h}(t) &= -jv_k^{+,l,h}(t),\\ w_k^{-,l,h}(t) &= jv_k^{-,l,h}(t) \end{aligned} \tag{9}$$

whose correlation properties are as follows.

Lemma 1: The autocorrelation functions of the 16-QAM sequences in (9) can be calculated with

$$\begin{aligned} &R_{v_k^{+,l,h},v_k^{+,l,h}}(\tau) = R_{w_k^{+,l,h},w_k^{+,l,h}}(\tau)\\ &= 2[R_{u_k^l,u_k^l}(\tau) + 2R_{u_k^l,u_k^h}(\tau+\delta) + 2R_{u_k^h,u_k^l}(\tau-\delta) + 4R_{u_k^h,u_k^h}(\tau)], \end{aligned} \tag{10}$$

$$\begin{aligned} &R_{v_k^{-,l,h},v_k^{-,l,h}}(\tau) = R_{w_k^{-,l,h},w_k^{-,l,h}}(\tau)\\ &= 2[R_{u_k^h,u_k^h}(\tau) - 2R_{u_k^h,u_k^l}(\tau-\delta) - 2R_{u_k^l,u_k^h}(\tau+\delta) + 4R_{u_k^l,u_k^l}(\tau)]. \end{aligned} \tag{11}$$

Proof: Owing to limitations of sapce, only $R_{v_k^{+,l,h},v_k^{+,l,h}}(\tau)$ is derived; the other derivations are similar. For a time shift τ, we have

$$\begin{aligned} &R_{v_k^{+,l,h},v_k^{+,l,h}}(\tau) =\\ &(1+j)(1-j)\sum_{t=0}^{N-1}[j^{u_k^l(t)} + 2j^{u_k^h(t+\delta)}][j^{-u_k^l(t+\tau)} + 2j^{-u_k^h(t+\delta+\tau)}]\\ &= 2[R_{u_k^l,u_k^l}(\tau) + 2R_{u_k^l,u_k^h}(\tau+\delta) + 2R_{u_k^h,u_k^l}(\tau-\delta) + 4R_{u_k^h,u_k^h}(\tau)]. \end{aligned} \tag{12}$$

This completes our proof. □

Lemma 2: The cross-correlation functions of the 16-QAM sequences in (9) can be calculated by

$$\begin{aligned} R_{v_k^{+,l,h},v_k^{-,l,h}}(\tau) = 2j[&R_{u_k^l,u_k^h}(\tau+\delta) - 2R_{u_k^l,u_k^l}(\tau) +\\ &2R_{u_k^h,u_k^h}(\tau) - 4R_{u_k^h,u_k^l}(\tau-\delta)], \end{aligned} \tag{13}$$

$$\begin{aligned} R_{v_k^{-,l,h},v_k^{+,l,h}}(\tau) = -2j[&R_{u_k^h,u_k^l}(\tau-\delta) + 2R_{u_k^h,u_k^h}(\tau)\\ &-2R_{u_k^l,u_k^l}(\tau) - 4R_{u_k^l,u_k^h}(\tau+\delta)], \end{aligned} \tag{14}$$

$$R_{v_k^{+,l,h},w_k^{+,l,h}}(\tau) = jR_{v_k^{+,l,h},v_k^{+,l,h}}(\tau), \tag{15}$$

$$R_{v_k^{-,l,h},w_k^{-,l,h}}(\tau) = -jR_{v_k^{-,l,h},v_k^{-,l,h}}(\tau), \tag{16}$$

$$R_{v_k^{+,l,h},w_k^{-,l,h}}(\tau) = -jR_{v_k^{+,l,h},v_k^{-,l,h}}(\tau), \tag{17}$$

$$R_{v_k^{-,l,h},w_k^{+,l,h}}(\tau) = jR_{v_k^{-,l,h},v_k^{+,l,h}}(\tau). \tag{18}$$

Proof: Again only $R_{v_k^{+,l,h},v_k^{-,l,h}}(\tau)$ is derived, for the reasons mentioned earlier. Hence, we have

$$\begin{aligned} &R_{v_k^{+,l,h},v_k^{-,l,h}}(\tau) =\\ &(1+j)(1+j)\sum_{t=0}^{N-1}[j^{u_k^l(t)} + 2j^{u_k^h(t+\delta)}][j^{-u_k^h(t+\delta+\tau)} - 2j^{-u_k^l(t+\tau)}]\\ &= 2j[R_{u_k^l,u_k^h}(\tau+\delta) - 2R_{u_k^l,u_k^l}(\tau) + 2R_{u_k^h,u_k^h}(\tau) - 4R_{u_k^h,u_k^l}(\tau-\delta)]. \end{aligned} \tag{19}$$

Hence, this lemma is proved. □

IV. NEW 16-QAM PERIODIC COMPLEMENTARY SEQUENCE MATES

Consider the quadriphase periodic CS mates $PCSS_4(U^l, M, N)$ and $PCSS_4(U^h, M, N)$. By making use of the interleaving technique, we construct two classes of 16-QAM sequences with length $2N$ as follows

$$\underline{x}_k^{l,h} = \underline{v}_k^{+,l,h} \odot \underline{v}_k^{-,l,h}, \tag{20}$$

$$\underline{y}_k^{l,h} = \underline{w}_k^{+,l,h} \odot \underline{w}_k^{-,l,h} \tag{21}$$

whose properties are as follows.

Theorem 1: The 16-QAM sequence sets $X^{l,h} = (\underline{x}_0^{l,h}, \underline{x}_1^{l,h}, \cdots, \underline{x}_{M-1}^{l,h})$ and $Y^{l,h} = (\underline{y}_0^{l,h}, \underline{y}_1^{l,h}, \cdots, \underline{y}_{M-1}^{l,h})$ are $PCSS_{\text{16-QAM}}(X^{l,h}, M, 2N)$ and $PCSS_{\text{16-QAM}}(Y^{l,h}, M, 2N)$, respectively.

Proof: Only $PCSS_{\text{16-QAM}}(X^{l,h}, M, 2N)$ is derived, for the reasons mentioned earlier. For the sake of convenience, we consider the odd and even time shifts.

Case 1: The even time shift $\tau = 2\eta$ $(0 \le \eta \le N-1)$.

The relationship between the 16-QAM sequence $\underline{x}_k^{l,h}$ and its cyclical shift version with the time shift $\tau = 2\eta$ is

$$\begin{aligned} &v_k^{+,l,h}(0),\ v_k^{-,l,h}(0),\ v_k^{+,l,h}(1),\ v_k^{-,l,h}(1),\ \cdots,\\ &v_k^{+,l,h}(\eta),\ v_k^{-,l,h}(\eta),\ v_k^{+,l,h}(\eta+1),\ v_k^{-,l,h}(\eta+1),\ \cdots. \end{aligned} \tag{22}$$

Hence, we have

$$\begin{aligned} R_{x_k^{l,h},x_k^{l,h}}(2\eta) = &\sum_{t'=0}^{N-1} v_k^{+,l,h}(t')\overline{v_k^{+,l,h}(t'+\eta)} +\\ &\sum_{t'=0}^{N-1} v_k^{-,l,h}(t')\overline{v_k^{-,l,h}(t'+\eta)}\\ = &R_{v_k^{+,l,h},v_k^{+,l,h}}(\eta) + R_{v_k^{-,l,h},v_k^{-,l,h}}(\eta). \end{aligned} \tag{23}$$

After (10) and (11) are substituted into (23), (23) reduces to

$$R_{x_k^{l,h},x_k^{l,h}}(2\eta) = 10[R_{u_k^l,u_k^l}(\eta) + R_{u_k^h,u_k^h}(\eta)]. \tag{24}$$

Furthermore, we have

$$\begin{aligned} &\sum_{k=0}^{M-1} R_{x_k^{l,h},x_k^{l,h}}(2\eta) = 10[\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\eta) + \sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\eta)]\\ &= \begin{cases} 20MN, & \eta \equiv 0 \pmod N \text{ (i.e., } \tau \equiv 0 \pmod{2N})\\ 0\quad, & \eta \not\equiv 0 \pmod N \end{cases} \end{aligned} \tag{25}$$

which results from Definition 2.

Case 2: The odd time shift $\tau = 2\eta + 1$ ($0 \leq \eta \leq N-1$).

The relationship between the 16-QAM sequence $\underline{x}_k^{l,h}$ and its cyclical shift version with the time shift $\tau = 2\eta+1$ is as follows

$$v_k^{+,l,h}(0),\ v_k^{-,l,h}(0),\ v_k^{+,l,h}(1),\ v_k^{-,l,h}(1),\ \cdots,$$
$$v_k^{-,l,h}(\eta),\ v_k^{+,l,h}(\eta+1),\ v_k^{-,l,h}(\eta+1),\ v_k^{+,l,h}(\eta+2),\ \cdots. \quad (26)$$

Therefore, we have

$$\begin{aligned} R_{x_k^{l,h},x_k^{l,h}}(2\eta+1) = & \sum_{t'=0}^{N-1} v_k^{+,l,h}(t')\overline{v_k^{-,l,h}(t'+\eta)} + \\ & \sum_{t'=0}^{N-1} v_k^{-,l,h}(t')\overline{v_k^{+,l,h}(t'+\eta+1)} \\ = & R_{v_k^{+,l,h},v_k^{-,l,h}}(\eta) + R_{v_k^{-,l,h},v_k^{+,l,h}}(\eta+1). \end{aligned} \quad (27)$$

In accordance with (13) and (14), we obtain

$$\sum_{k=0}^{M-1} R_{x_k^{l,h},x_k^{l,h}}(2\eta+1) = 2j\Bigg[\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\eta+\delta) -$$
$$2\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\eta) + 2\sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\eta) - 4\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\eta-\delta)\Bigg]$$
$$-2j\Bigg[\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\eta-\delta+1) + 2\sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\eta+1) -$$
$$2\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\eta+1) - 4\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\eta+\delta+1)\Bigg] = 0 \quad (28)$$

which is because

$$\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\zeta) = \sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\zeta) \quad (\forall\, \zeta), \quad (29)$$

$$\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\zeta) = 0\ (\forall\, \zeta), \quad (30)$$

$$\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\zeta) = 0\ (\forall\, \zeta). \quad (31)$$

Summarizing Cases 1 and 2, this theorem follows immediately. □

Theorem 2: Two periodic CS sets $PCSS_{\text{16-QAM}}(X^{l,h}, M, 2N)$ and $PCSS_{\text{16-QAM}}(Y^{l,h}, M, 2N)$ in Theorem 1 are the mates to each other.

Proof: According to Definition 3, we only need to show that those two sequence sets satisfy (3). Similarly, we consider the odd and even time shifts as well.

Case 1: The even time shift $\tau = 2\eta$ ($0 \leq \eta \leq N-1$).

The relationship between the 16-QAM sequence $\underline{x}_k^{l,h}$ and the cyclical shift version of $\underline{y}_k^{l,h}$ with the time shift $\tau = 2\eta$ is

$$v_k^{+,l,h}(0),\ v_k^{-,l,h}(0),\ v_k^{+,l,h}(1),\ v_k^{-,l,h}(1),\ \cdots,$$
$$w_k^{+,l,h}(\eta),\ w_k^{-,l,h}(\eta),\ w_k^{+,l,h}(\eta+1),\ w_k^{-,l,h}(\eta+1),\ \cdots. \quad (32)$$

As a consequence, we have

$$R_{x_k^{l,h},y_k^{l,h}}(2\eta) = R_{v_k^{+,l,h},w_k^{+,l,h}}(\eta) + R_{v_k^{-,l,h},w_k^{-,l,h}}(\eta). \quad (33)$$

Utilizing (15) and (16), we get

$$\sum_{k=0}^{M-1} R_{x_k^{l,h},y_k^{l,h}}(2\eta) = 6j\sum_{k=0}^{M-1}[R_{u_k^h,u_k^h}(\eta) - R_{u_k^l,u_k^l}(\eta)] +$$
$$8j\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\eta+\delta) + 8j\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\eta-\delta) = 0 \quad (34)$$

which follows from (29)–(31).

Case 2: The odd time shift $\tau = 2\eta + 1$ ($0 \leq \eta \leq N-1$).

The relationship between the 16-QAM sequence $\underline{x}_k^{l,h}$ and the cyclical shift version of $\underline{y}_k^{l,h}$ with the time shift $\tau = 2\eta + 1$ is as follows

$$v_k^{+,l,h}(0),\ v_k^{-,l,h}(0),\ v_k^{+,l,h}(1),\ v_k^{-,l,h}(1),\ \cdots$$
$$w_k^{-,l,h}(\eta),\ w_k^{+,l,h}(\eta+1),\ w_k^{-,l,h}(\eta+1),\ w_k^{+,l,h}(\eta+2),\ \cdots. \quad (35)$$

Consequently, we have

$$R_{x_k^{l,h},y_k^{l,h}}(2\eta+1) = R_{v_k^{+,l,h},w_k^{-,l,h}}(\eta) + R_{v_k^{-,l,h},w_k^{+,l,h}}(\eta+1). \quad (36)$$

Then, by (17) and (18), we can calculate

$$\sum_{k=0}^{M-1} R_{x_k^{l,h},y_k^{l,h}}(2\eta+1) = 2\Bigg[\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\eta+\delta) - 2\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\eta)$$
$$+2\sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\eta) - 4\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\eta-\delta)\Bigg] + 2\Bigg[\sum_{k=0}^{M-1} R_{u_k^h,u_k^l}(\eta-\delta+1)$$
$$+2\sum_{k=0}^{M-1} R_{u_k^h,u_k^h}(\eta+1) - 2\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(\eta+1)$$
$$-4\sum_{k=0}^{M-1} R_{u_k^l,u_k^h}(\eta+\delta+1)\Bigg] = 0 \quad (37)$$

which follows from (29)–(31) as well.

Clearly, this theorem is true. □

Theorem 3: Let $\{PCSS_4(U^l, M, N) | 1 \leq l \leq T\}$ be an MO quadriphase CS set. Again, let $X^{2(l-1)+1,2(l-1)+2}$ and $Y^{2(l-1)+1,2(l-1)+2}$ be constructed in accordance with (20) and (21) from the mates $PCSS_4(U^{(2(l-1)+1}, M, N)$ and $PCSS_4(U^{(2(l-1)+2}, M, N)$, where $1 \leq l \leq \lfloor T/2 \rfloor$. Then, we obtain an MO 16-QAM CS set

$$\{PCSS_{\text{16-QAM}}(X^{2(l-1)+1,2(l-1)+2}, M, 2N),$$
$$PCSS_{\text{16-QAM}}(Y^{2(l-1)+1,2(l-1)+2}, M, 2N) | 1 \leq l \leq \lfloor T/2 \rfloor\}$$

Table 1. Autocorrelation values of $PCSS_{\text{16-QAM}}(X^{1,2}, 4, 12)$ in (38).

auto corr.	τ											
	0	1	2	3	4	5	6	7	8	9	10	11
$R_{x_0^{1,2},x_0^{1,2}}(\tau)$	120	$12j$	0	$-12j$	-40	$-36j$	0	$36j$	-40	$12j$	0	$-12j$
$R_{x_1^{1,2},x_1^{1,2}}(\tau)$	120	$-12j$	0	$12j$	-40	$36j$	0	$-36j$	-40	$-12j$	0	$12j$
$R_{x_2^{1,2},x_2^{1,2}}(\tau)$	120	$12j$	0	$20j$	40	$-20j$	0	$20j$	40	$-20j$	0	$-12j$
$R_{x_3^{1,2},x_3^{1,2}}(\tau)$	120	$-12j$	0	$-20j$	40	$20j$	0	$-20j$	40	$20j$	0	$12j$
sum	480	0	0	0	0	0	0	0	0	0	0	0

Table 2. Cross-correlation values between $PCSS_{\text{16-QAM}}(X^{1,2}, 4, 12)$ and $PCSS_{\text{16-QAM}}(Y^{1,2}, 4, 12)$ in (38) and (39), respectively.

cross corr.	τ											
	0	1	2	3	4	5	6	7	8	9	10	11
$R_{x_0^{1,2},y_0^{1,2}}(\tau)$	0	-12	$32j$	-12	0	36	$-96j$	36	0	-12	$32j$	-12
$R_{x_1^{1,2},y_1^{1,2}}(\tau)$	0	12	$-32j$	12	0	-36	$96j$	-36	0	12	-32	12
$R_{x_2^{1,2},y_2^{1,2}}(\tau)$	0	-12	$32j$	-44	$-48j$	-44	$32j$	-44	$-48j$	-44	$32j$	-12
$R_{x_3^{1,2},y_3^{1,2}}(\tau)$	0	12	$-32j$	44	$48j$	44	$-32j$	44	$48j$	44	$-32j$	12
sum	0	0	0	0	0	0	0	0	0	0	0	0

where $\lfloor a \rfloor$ denotes the largest integer not exceeding a.

Proof: For convenience, let $t_1 = 2(l_1 - 1) + 1$ and $t_2 = 2(l_2 - 1) + 1$ $(1 \le l_1, l_2 \le \lfloor T/2 \rfloor)$. By Theorem 1, X^{t_1,t_1+1} and Y^{t_1,t_1+1} are $PCSS_{\text{16-QAM}}(X^{t_1,t_1+1}, M, 2N)$ and $PCSS_{\text{16-QAM}}(Y^{t_1,t_1+1}, M, 2N)$, respectively. By employing the method used to derive Theorem 2, $PCSS_{\text{16-QAM}}(X^{t_1,t_1+1}, M, 2N)$ and $PCSS_{\text{16-QAM}}(X^{t_2,t_2+1}, M, 2N)$ $(l_1 \neq l_2)$, $PCSS_{\text{16-QAM}}$ $(Y^{t_1,t_1+1}, M, 2N)$ and $PCSS_{\text{16-QAM}}(Y^{t_2,t_2+1}, M, 2N)$ $(l_1 \neq l_2)$, and $PCSS_{\text{16-QAM}}$ $(X^{t_1,t_1+1}, M, 2N)$ and $PCSS_{\text{16-QAM}}$ $(Y^{t_2,t_2+1}, M, 2N)$ ($\forall$ l_1 and $\forall$ l_2) are the mates of each other, respectively. □

Here are two examples to help the reader understand.

Example 1: Consider the quadriphase periodic CS mate $\{PCSS_4(U^l, 4, 6) | l = 1, 2\}$, copied from [26].

$$\begin{bmatrix} U^1 \\ U^2 \end{bmatrix} = \begin{bmatrix} \underline{u}_0^1 & \underline{u}_1^1 & \underline{u}_2^1 & \underline{u}_3^1 \\ \underline{u}_0^2 & \underline{u}_1^2 & \underline{u}_2^2 & \underline{u}_3^2 \end{bmatrix} = \begin{bmatrix} 113002 & 203312 & 010101 & 300211 \\ 122033 & 331220 & 021130 & 232323 \end{bmatrix}.$$

Let $\delta = 1$. Then, by Theorem 2, the resulting 16-QAM periodic CS mate is as follows

$$X^{1,2} = \begin{bmatrix} \underline{x}_0^{1,2} \\ \underline{x}_1^{1,2} \\ \underline{x}_2^{1,2} \\ \underline{x}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -3 & -3 & -3 & -3 & 3 & 3 & 3 & -3 & 3 & -3 & -3 & 3 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -3 & -3 & 3 & 3 & -3 & 3 & -3 & 3 & 3 & -3 & -3 & -3 \end{pmatrix} \\ \begin{pmatrix} -1 & -3 & -3 & -1 & -1 & -1 & 1 & -3 & 3 & -1 & 1 & -1 \\ -1 & 3 & 3 & -1 & 3 & 3 & -1 & -3 & 3 & 1 & 3 & -3 \end{pmatrix} \\ \begin{pmatrix} 3 & 1 & -1 & -3 & 3 & -3 & -3 & 1 & 1 & -3 & -3 & -3 \\ -3 & 1 & -1 & 3 & -1 & 1 & -3 & -1 & -1 & -3 & -1 & -1 \end{pmatrix} \end{bmatrix}, \quad (38)$$

$$Y^{1,2} = \begin{bmatrix} \underline{y}_0^{1,2} \\ \underline{y}_1^{1,2} \\ \underline{y}_2^{1,2} \\ \underline{y}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 3 & -3 & 3 & -3 & -3 & 3 & -3 & -3 & -3 & -3 & 3 & 3 \end{pmatrix} \\ \begin{pmatrix} -3 & 3 & 3 & -3 & -3 & -3 & -3 & -3 & 3 & 3 & -3 & 3 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & -3 & 3 & 1 & 3 & -3 & -1 & 3 & 3 & -1 & 3 & 3 \\ 1 & -3 & 3 & -1 & 1 & -1 & -1 & -3 & -3 & -1 & -1 & -1 \end{pmatrix} \\ \begin{pmatrix} -3 & -1 & -1 & -3 & -1 & -1 & -3 & 1 & -1 & 3 & -1 & 1 \\ -3 & 1 & 1 & -3 & -3 & -3 & 3 & 1 & -1 & -3 & 3 & -3 \end{pmatrix} \end{bmatrix} \quad (39)$$

where $\begin{pmatrix} \cdots a \cdots \\ \cdots b \cdots \end{pmatrix}$ expresses the element $a + bj$ in the 16-QAM constellation, and $\begin{pmatrix} \cdots a\ c \cdots \\ \cdots b\ d \cdots \end{pmatrix}$ denotes a 16-QAM sequence $(\cdots, a + bj, c + dj, \cdots)$.

For the sake of clarity, the autocorrelation values of (38) and the cross-correlation values between (38) and (39) are listed in Tables 1 and 2, respectively.

Example 2: Consider an MO quaternary CS set $\{PCSS_4(U^l, 4, 4) | 1 \le l \le 4\}$, copied from [27], as follows

$$\begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{bmatrix} = \begin{bmatrix} \underline{u}_0^1 & \underline{u}_1^1 & \underline{u}_2^1 & \underline{u}_3^1 \\ \underline{u}_0^2 & \underline{u}_1^2 & \underline{u}_2^2 & \underline{u}_3^2 \\ \underline{u}_0^3 & \underline{u}_1^3 & \underline{u}_2^3 & \underline{u}_3^3 \\ \underline{u}_0^4 & \underline{u}_1^4 & \underline{u}_2^4 & \underline{u}_3^4 \end{bmatrix} = \begin{bmatrix} 1232 & 0323 & 1232 & 0323 \\ 3032 & 2123 & 3032 & 2123 \\ 3212 & 2303 & 1030 & 0121 \\ 1012 & 0103 & 3230 & 2321 \end{bmatrix}.$$

Let $\delta = 1$. Thus, by utilizing Theorem 3 we can produce an MO 16-QAM CS set as follows

$$X^{1,2} = \begin{bmatrix} \underline{x}_0^{1,2} \\ \underline{x}_1^{1,2} \\ \underline{x}_2^{1,2} \\ \underline{x}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \end{pmatrix} \end{bmatrix}, \quad (40)$$

$$Y^{1,2} = \begin{bmatrix} \underline{y}_0^{1,2} \\ \underline{y}_1^{1,2} \\ \underline{y}_2^{1,2} \\ \underline{y}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix} \end{bmatrix}, \quad (41)$$

$$X^{3,4} = \begin{bmatrix} \underline{x}_0^{3,4} \\ \underline{x}_1^{3,4} \\ \underline{x}_2^{3,4} \\ \underline{x}_3^{3,4} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \end{pmatrix} \\ \begin{pmatrix} -3 & 3 & 3 & 3 & 3 & -3 & 3 & 3 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} -3 & -3 & 3 & -3 & 3 & 3 & 3 & -3 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix} \end{bmatrix}, \quad (42)$$

$$Y^{3,4} = \begin{bmatrix} \underline{y}_0^{3,4} \\ \underline{y}_1^{3,4} \\ \underline{y}_2^{3,4} \\ \underline{y}_3^{3,4} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -3 & 3 & 3 & 3 & 3 & -3 & 3 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 3 & 3 & -3 & 3 & -3 & -3 & -3 & 3 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 3 & -3 & -3 & -3 & -3 & 3 & -3 & -3 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -3 & -3 & 3 & -3 & 3 & 3 & 3 & -3 \end{pmatrix} \end{bmatrix} \quad (43)$$

whose correlation values are omitted because of limitations of space.

Because it is well known that all aperiodic CSs must be periodic ones [19, p. 332], the quaternary aperiodic MO CS sets given in [27] can be treated as periodic ones. Hence, we have $\{PCSS_4(U^l, 2^{r+1}, 2^r n) | 1 \le l \le 2^{r+1}\}$, where r and

Table 3. Existing 16-QAM periodic CS mates come from binary periodic CS mates up to $M = 12$ and $N = 50$.

Binary CS sets [29]-[32]	Binary CS mates	Quaternary CS mates (Theorem 5 [27])	Theorem 2 here
$M \mid N$	$M \mid N$	$M \mid N$	$M \mid 2N$
$2 \mid$ 2, 4, 8, 10, 16, 20, 26, 32, 34, 40, 50	(44)	$2 \mid$ 2, 4, 8, 10, 16, 20, 26, 32, 34, 40, 50	$2 \mid$ 4, 8, 16, 20, 32, 40, 52, 64, 68, 80, 100
$3 \mid$ 4, 8, 12, 16, 24, 28, 32, 36, 40, 44, 48	unknown	unknown	unknown
$4 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	(44)	$4 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	$4 \mid$ 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 87, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100
$5 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48	unknown	unknown	unknown
$6 \mid$ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50	(44)	$6 \mid$ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50	$6 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100
$7 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48	unknown	unknown	unknown
$8 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	(44)	$8 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	$8 \mid$ 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 87, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100
$9 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48	unknown	unknown	unknown
$10 \mid$ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50	(44)	$10 \mid$ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50	$10 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100
$11 \mid$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48	unknown	unknown	unknown
$12 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	(44)	$12 \mid$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50	$12 \mid$ 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 87, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100

n are two positive integers (see [27] and [28] for their meanings), which provides the rich raw materials from Theorem 3. Hence, we have $\{PCSS_{\text{16-QAM}}(Z^l, 2^{r+1}, 2^{r+1}n) | 1 \leq l \leq 2^r\}$ by Theorem 3. Apart from the quaternary periodic CSs mentioned above, [25]–[27] can also provide such inputs. This is, for any binary periodic CS set $PCSS_2(U^l, 2M', N)$ consisting of $2M'$ subsequences that are each of length N, where $U^l = (\underline{u}_0^l, \underline{u}_1^l, \cdots, \underline{u}_{2M'-1}^l)$, one of its mates is [19, p. 333]

$$U^h = (\underline{\widetilde{u}}_1^l, -\underline{\widetilde{u}}_0^l, \cdots, \underline{\widetilde{u}}_{2M'-1}^l, -\underline{\widetilde{u}}_{2M-2}^l) \tag{44}$$

where $\underline{\widetilde{u}}_r^l$ denotes the reversal of the sequence $\underline{u}_r^l$ (for its definition, see [19, p. 32]. In accordance with Theorem 5 in [27], the aforementioned mates $PCSS_2(U^l, 2M', N)$ and $PCSS_2(U^h, 2M', N)$ must result in the quaternary periodic CS mates $PCSS_4(X^l, 2M', N)$ and $PCSS_4(X^h, 2M', N)$ (for X^l and X^h, see [27]), or $PCSS_4(Z^l, 2M', 2N)$ and $PCSS_4(Z^h, 2M', 2N)$ (for Z^l and Z^h, please see [26, p. 240], where N is an odd integer), respectively. As a consequence, we obtain 16-QAM periodic CS mates $PCSS_{\text{16-QAM}}(X^{l,h}, 2M', 2N)$ and $PCSS_{\text{16-QAM}}(Y^{l,h}, 2M', 2N)$ by employing Theorem 2 in this paper. In conclusion, in accordance with the existing binary periodic CS mates up to $M = 12$ and $N = 50$ [29]–[32], the 16-QAM periodic CS mates that result from Theorem 2 in this paper are listed in Table 3.

The following theorem gives the distinct 16-QAM periodic CS sets from Theorem 1.

Theorem 4: For the quaternary periodic CS mates $PCSS_4(U^l, M, N)$ and $PCSS_4(U^h, M, N)$, let two integers δ_1 and δ_2 satisfy $\delta_1 \neq \delta_2$ $(0 \leq \delta_1, \delta_2 \leq N-1)$. For $\forall\ k$ $(0 \leq k \leq M-1)$ and $i = 1, 2$,

$$\begin{aligned} v_{k,i}^{+,l,h}(t) &= (1+j)(j^{u_k^l(t)} + 2j^{u_k^h(t+\delta_i)}), \\ v_{k,i}^{-,l,h}(t) &= (1-j)(j^{u_k^h(t+\delta_i)} - 2j^{u_k^l(t)}). \end{aligned} \tag{45}$$

By applying these relations to (20), we obtain two 16-QAM periodic CS sets $PCSS_{16-\text{QAM}}(X_1^{l,h}, M, 2N)$ and $PCSS_{16-\text{QAM}}(X_2^{l,h}, M, 2N)$, where

$$\underline{x}_{k,1}^{l,h} = \underline{v}_{k,1}^{+,l,h} \odot \underline{v}_{k,1}^{-,l,h} \text{ and } \underline{x}_{k,2}^{l,h} = \underline{v}_{k,2}^{+,l,h} \odot \underline{v}_{k,2}^{-,l,h}. \tag{46}$$

Then, $PCSS_{16-\text{QAM}}(X_1^{l,h}, M, 2N)$ and $PCSS_{16-\text{QAM}}(X_2^{l,h}, M, 2N)$ are distinct. Further, employing (20) yields N distinct 16-QAM periodic CS sets.

Proof: Provided that for $\delta_1 \neq \delta_2$, $PCSS_{16-\text{QAM}}(X_1^{l,h}, M, 2N)$ and $PCSS_{16-\text{QAM}}(X_2^{l,h}, M, 2N)$ are equivalent to each other; that is, there exists an integer ζ so as to satisfy $\underline{x}_{k,1}^{l,h} = L^\zeta \underline{x}_{k,2}^{l,h}$ $(0 \leq k \leq M-1)$. We can, then, conclude that a contradiction must appear. In fact, for $\forall k$ we consider two cases: $\zeta = 2\pi$ and $\zeta = 2\pi + 1$ $(0 \leq \pi \leq N-1)$.

Case 1: $\zeta = 2\pi$.

Clearly, the relationship between the 16-QAM sequences $\underline{x}_{k,1}^{l,h}$ and $L^{2\pi}\underline{x}_{k,2}^{l,h}$ is

$$\begin{aligned} &v_{k,1}^{+,l,h}(0),\ v_{k,1}^{-,l,h}(0),\ v_{k,1}^{+,l,h}(1),\ v_{k,1}^{-,l,h}(1),\ \cdots, \\ &v_{k,2}^{+,l,h}(\pi),\ v_{k,2}^{-,l,h}(\pi),\ v_{k,2}^{+,l,h}(\pi+1),\ v_{k,2}^{-,l,h}(\pi+1),\ \cdots. \end{aligned} \tag{47}$$

As a consequence, from Definition 5 we have

$$\begin{cases} \underline{v}_{k,1}^{+,l,h} = L^\pi \underline{v}_{k,2}^{+,l,h}, \\ \underline{v}_{k,1}^{-,l,h} = L^\pi \underline{v}_{k,2}^{-,l,h} \end{cases} \tag{48}$$

which reduces to

$$\begin{cases} j^{u_k^l(t)} - j^{u_k^l(t+\pi)} = 2[j^{u_k^h(t+\delta_2+\pi)} - j^{u_k^h(t+\delta_1)}], \\ 2[j^{u_k^l(t)} - j^{u_k^l(t+\pi)}] = j^{u_k^h(t+\delta_1)} - j^{u_k^h(t+\delta_2+\pi)} \end{cases} \tag{49}$$

with the application of (45). Further, (49) is simplified to

$$\begin{cases} j^{u_k^l(t)} = j^{u_k^l(t+\pi)} \ \forall\, t, \\ j^{u_k^h(t+\delta_2+\pi)} = j^{u_k^h(t+\delta_1)} \ \forall\, t \end{cases} \tag{50}$$

which implies both that $\pi = 0$ and $\delta_1 = \delta_2$. Obviously, this is a contradiction.

Case 2: $\zeta = 2\pi + 1$.

Similarly, the relationship between the 16-QAM sequences $\underline{x}_{k,1}^{l,h}$ and $L^{2\pi+1}\underline{x}_{k,2}^{l,h}$ is given by

$$v_{k,1}^{+,l,h}(0),\ v_{k,1}^{-,l,h}(0),\ v_{k,1}^{+,l,h}(1),\ v_{k,1}^{-,l,h}(1),\ \cdots \tag{51}$$

$$v_{k,2}^{-,l,h}(\pi),\ v_{k,2}^{+,l,h}(\pi+1),\ v_{k,2}^{-,l,h}(\pi+1),\ v_{k,2}^{+,l,h}(\pi+2),\ \cdots. \tag{52}$$

Hence, by Definition 5 we have

$$\begin{cases} \underline{v}_{k,1}^{+,l,h} = L^{\pi}\underline{v}_{k,2}^{-,l,h}, \\ \underline{v}_{k,1}^{-,l,h} = L^{\pi+1}\underline{v}_{k,2}^{+,l,h} \end{cases} \tag{53}$$

Substituting (45) into (52) yields

$$j^{u_k^l(t)} + 2j^{u_k^h(t+\delta_1)} = -j[j^{u_k^h(t+\delta_2+\pi)} - 2j^{u_k^l(t+\pi)}], \tag{54a}$$

$$j^{u_k^h(t+\delta_1)} - 2j^{u_k^l(t)} = j[j^{u_k^l(t+\pi+1)} + 2j^{u_k^h(t+\delta_2+\pi+1)}]. \tag{54b}$$

Multiplying $\overline{j^{u_k^l(t)}}$ to the right-hand side in (53a), and applying to (1), yields

$$R_{u_k^l,u_k^l}(0) + 2R_{u_k^h,u_k^l}(-\delta_1) = -j[R_{u_k^h,u_k^l}(-\delta_2-\pi) - 2R_{u_k^l,u_k^l}(-\pi)] \tag{55}$$

which reduces to

$$\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(0) = 2j\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(-\pi) \tag{56}$$

from (2) and (3).

Similarly, multiplying $\overline{j^{u_k^l(t+1)}}$ to the right-hand side in (53b) yields

$$R_{u_k^h,u_k^l}(1-\delta_1) - 2R_{u_k^l,u_k^l}(1) = j[R_{u_k^l,u_k^l}(-\pi) + 2R_{u_k^h,u_k^l}(-\delta_2-\pi)] \tag{57}$$

which reduces to

$$j\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(-\pi) = -2\sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(1) = 0 \tag{58}$$

from (2) and (3).

Hence, according to (55) and (57) we have

$$MN = \sum_{k=0}^{M-1} R_{u_k^l,u_k^l}(0) = 0 \tag{59}$$

which is clearly impossible.

In addition, when the parameter δ varies over the range 0 to $N-1$, the foregoing derivation guarantees that the N resultant 16-QAM CS sets from Theorem 1 are distinct from one another. □

Example 3: In Example 1, let $\delta = 2$. Therefore, we have

$$X^{1,2} = \begin{bmatrix} \underline{x}_0^{1,2} \\ \underline{x}_1^{1,2} \\ \underline{x}_2^{1,2} \\ \underline{x}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -3 & -3 & 1 & -1 & 3 & 1 & 3 & -3 & -1 & -1 & -3 & 1 \\ -1 & -1 & 3 & -3 & -3 & 1 & -1 & 1 & 3 & 3 & -3 & -1 \end{pmatrix} \\ \begin{pmatrix} -3 & 3 & -1 & -3 & -1 & 1 & 3 & 3 & 1 & -3 & 1 & 1 \\ 1 & -1 & -1 & 3 & -3 & 3 & 1 & 1 & -1 & -3 & -3 & -3 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 & -3 & -1 & 3 & -3 & 1 & -1 & 3 & -1 & -3 & -3 \\ 3 & 3 & 3 & -1 & -1 & 1 & 3 & -3 & 3 & 1 & -1 & -1 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 & 3 & -3 & -1 & -3 & 1 & 1 & -3 & -3 & 1 & -3 \\ -3 & 3 & -1 & 1 & -1 & 3 & -3 & -3 & -1 & -1 & -1 & -3 \end{pmatrix} \end{bmatrix}, \tag{60}$$

$$Y^{1,2} = \begin{bmatrix} \underline{y}_0^{1,2} \\ \underline{y}_1^{1,2} \\ \underline{y}_2^{1,2} \\ \underline{y}_3^{1,2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -1 & 1 & 3 & 3 & -3 & -1 & -1 & -1 & 3 & -3 & -3 & 1 \\ 3 & -3 & -1 & -1 & -3 & 1 & -3 & -3 & 1 & -1 & 3 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & -1 & -3 & -3 & -3 & 1 & -1 & -1 & 3 & -3 & 3 \\ 3 & 3 & 1 & -3 & 1 & 1 & -3 & 3 & -1 & -3 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & -3 & 3 & 1 & -1 & -1 & 3 & 3 & 3 & -1 & -1 & 1 \\ 1 & -1 & 3 & -1 & -3 & -3 & -1 & -1 & -3 & -1 & 3 & -3 \end{pmatrix} \\ \begin{pmatrix} -3 & -3 & -1 & -1 & -1 & -3 & -3 & 3 & -1 & 1 & -1 & 3 \\ 1 & 1 & -3 & -3 & 1 & -3 & -1 & 1 & 3 & -3 & -1 & -3 \end{pmatrix} \end{bmatrix}. \tag{61}$$

Clearly, the resultant 16-QAM CS sets in (38), (39), (59), and (60) are cyclically distinct from one another.

V. CONCLUSION

In this paper, we construct a family of 16-QAM periodic CS mates, and expand the resultant mates to a class of MO 16-QAM CS sets. The proposed methods are applicable to conversion of all currently known quadriphase CS mates to 16-QAM ones.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for helpful suggestions that greatly improved the presentation quality of this paper.

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