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ELLIPTIC EQUATIONS WITH COMPACTLY SUPPORTED SOLUTIONS

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Abstract. For any $p \in (1,2)$ and arbitrary $f \in L^p(\mathbb{R}^2)$ with compact support, it is proved that there exists a pair (L,u), with L second order uniformly elliptic operator and $u \in W_0^{2,p}(\mathbb{R}^2)$ such that $Lu = f$ a.e. in $\mathbb{R}^2$.

1. Introduction

Let L be a second order uniformly elliptic operator with bounded measurable coefficients in $\mathbb{R}^2$ of the form

$$L := a^{11}\partial_{xx} + 2a^{12}\partial_{xy} + a^{22}\partial_{yy}. \tag{1.1}$$

When $u \in W^{2,2}(\mathbb{R}^2)$ is a solution of the equation $Lu = f$ for a compactly supported function f, in general, one cannot expect that u also has compact support.

On the other hand, for the case when p is small enough so that the a priori bounds of K. Astala, T. Iwaniec, G. Martin [1] do not hold, Buonocore and Manselli proved in [3] that there exists an operator L (of the above form and with first order terms) and a non trivial $u \in W^{2,p}(\mathbb{R}^2)$ with compact support satisfying the equation $Lu = 0$ a.e. (see [3]). A similar example in $\mathbb{R}^3$ has been constructed in [4].

In this paper, we consider the corresponding question on compactly supported solutions for the non-homogeneous equation $Lu = f$ and prove that:
For any given $p \in (1,2)$ and $f \in L^p(\mathbb{R}^2)$ with compact support, there exist an operator L of the form (1.1) *and a function $u \in W_0^{2,p}(\mathbb{R}^2)$ satisfying $Lu = f$ a.e. in $\mathbb{R}^2$.*

The proof basically follows arguments in [6]. In that paper, the authors considered the homogeneous equation $Lu = 0$ and proved that, given two arbitrary functions $f^{(0)}$ and $f^{(1)}$ on the boundary ∂D of the unit disk $D \subset \mathbb{R}^2$, there exists a function u and a second order uniformly elliptic operator L of the form (1.1) so that $Lu = 0$ in D, $u|_{\partial D} = f^{(0)}$ and $\frac{\partial u}{\partial n}\big|_{\partial D} = f^{(1)}$.

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Here, the construction starts by taking a function u satisfying $\Delta u = f$ a.e. in the unit disk D of the form $u = w + \tilde{u}$, where w is the solution to the Dirichlet problem $\Delta w = f$ in D, $w = 0$ on ∂D and $\tilde{u}$ is the sum of a series of Green functions with poles in a countable subset $\mathcal{N}$ of D with no accumulation points in D. Such a function u assumes the boundary conditions $u|_{\partial D} = 0$, $\frac{\partial u}{\partial n}\big|_{\partial D} = 0$ in a suitably generalized sense and its existence follows from a result in [5]. After this, following the method used in [6], we modify the function u and the operator Δ in suitably chosen disks centered at the points of $\mathcal{N}$ and we obtain an elliptic operator L of the form (1.1) and a function $u' \in W^{2,p}(D)$ satisfying all required properties.

The paper is organized as follows: In §2, we recall notations and results of [6] and determine the previously described function $u = w + \tilde{u}$ (Lemma 2.2). In §3 we outline the modifying procedure for u and Δ and in §4, we prove the main result.

2. Notations and preliminary results

In what follows, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and, for any $r > 0$, we denote by $D(\mathfrak{a}, r)$ the open disk centered at $\mathfrak{a} \in \mathbb{C}$ with radius r. The unit disk $D(0,1)$ will be simply denoted by D.

$\mathcal{L}_\alpha$ is the family of linear second order uniformly elliptic operators with bounded measurable coefficients in D of the form (1.1) with lower ellipticity constant $\alpha > 0$ and upper ellipticity constant $1/\alpha$.

Given $W^{2,p}(D)$, the Sobolev space of functions in $L^p(D)$ with second derivatives in $L^p(D)$, $p > 1$, for any $v \in W^{2,p}(D)$, we denote by $v|_{\partial D}$ and $\frac{\partial v}{\partial n}\big|_{\partial D}$ the traces on ∂D of v and of $\frac{\partial v}{\partial r}$, respectively. We also denote by $W_0^{2,p}(D)$ the closure of $C_0^\infty(D)$ in $W^{2,p}(D)$, i.e., the class of $v \in W^{2,p}(D)$ such that $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial n}\big|_{\partial D} = 0$.

Let us recall some notations and definitions from [6].

For any real number $\gamma \geq 0$, we denote by $\mathbb{A}^{(\gamma)}$ the Banach space of real-valued functions, defined on ∂D, of the form

$$f(e^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{2.1}$$

with Fourier coefficients a_n, b_n such that

$$\{n^\gamma a_n\},\ \{n^\gamma b_n\}\ \in \ell^1 \tag{2.2}$$

and norm defined by $\|f\|_{\mathbb{A}^{(\gamma)}} = \|\{n^\gamma a_n\}\|_{\ell^1} + \|\{n^\gamma b_n\}\|_{\ell^1}$.

Also, for any $\gamma \geq 0$, let σ be a fixed constant, $0 < \sigma < \frac{1}{2}$, depending on γ such that

$$\sum_{p=1}^{\infty} (2p+1)^\gamma\, \sigma^{2p} < 1 \tag{2.3}$$

and let $\zeta^{(j)}_{2n,\,l}$ ($n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, $j = 1, 2$) be the $4n$-th roots of σ^4 ordered as follows:

$$\zeta^{(1)}_{2n,\,l} = \sigma^{\frac{1}{n}} e^{-\frac{\pi}{n}\,l\,i}, \quad l = 0, \dots, 2n-1, \tag{2.4}$$

$$\zeta^{(2)}_{2n,\,l} = \sigma^{\frac{1}{n}} e^{-(-\frac{\pi}{2n}+\frac{\pi}{n}\,l)\,i}, \quad l = 0, \dots, 2n-1. \tag{2.5}$$

Let us denote by $\mathcal{N} := \{\mathfrak{a}_\nu\}_{\nu \geq 0}$ the sequence given by 0 and the points $\zeta^{(j)}_{2n,\,l}$ ordered in the following way:

$$\mathfrak{a}_0 = 0, \quad \mathfrak{a}_\nu = \zeta^{(j)}_{2n,\,l} \quad \text{if } \nu = 1 + 2(n-1)n + 2(j-1)n + l$$

for any $n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, $j = 1, 2$. Notice that $\mathcal{N}$ has no limit points in D.

We also set

$$m_\nu := \frac{1}{2} \min_{\mu \neq \nu} |\mathfrak{a}_\mu - \mathfrak{a}_\nu| \tag{2.6}$$

(indeed, one may alternatively consider constants $m_\nu := \epsilon \min_{\mu\neq\nu} |\mathfrak{a}_\mu - \mathfrak{a}_\nu|$ for any other fixed $\epsilon \in (0, \frac{1}{2}])$. Notice that as $1 + 2n(n-1) \leq \nu \leq 2n^2 + 2n$, for any fixed value of ν, the corresponding value of n satisfies the inequalities

$$\frac{-1 + \sqrt{1 + 2\nu}}{2} \leq n \leq \frac{1 + \sqrt{2\nu - 1}}{2}. \tag{2.7}$$

Since $|\mathfrak{a}_\nu| = \sigma^{\frac{1}{n}}$, it follows that $m_\nu \geq \frac{\sigma^{\frac{1}{n+1}} - \sigma^{\frac{1}{n}}}{2} = \frac{1}{2} e^{\tilde{x}} \frac{\ln(1/\sigma)}{n(n+1)}$ for some $\tilde{x} \in (\frac{1}{n}\ln\sigma, \frac{1}{n+1}\ln\sigma)$ and hence

$$m_\nu \geq \frac{\sigma \ln(1/\sigma)}{2(n+1)^2} \geq \frac{2\sigma\ln(1/\sigma)}{(3 + \sqrt{2\nu - 1})^2} \geq \frac{C(\sigma)}{\nu}. \tag{2.8}$$

Notice that the set

$$D_o = D \setminus \cup_{\nu=0}^{\infty} \overline{D}(\mathfrak{a}_\nu, \frac{2}{3} m_\nu)$$

is an open non-empty subset of D.

Let $G(z, \zeta)$ be the Green function for the Laplace operator in D with pole ζ:

$$G(z, \zeta) = -\frac{1}{2\pi} \ln \left| \frac{z - \zeta}{1 - z\overline{\zeta}} \right|, \qquad z \neq \zeta.$$

The following result is proved in [5].

Fact 2.1. Let $\gamma > 0$ and σ be as above and $1 < p < 2$. Given $f^{(1)} \in \mathbb{A}^{(\gamma)}$, there exist $a_0 \in \mathbb{R}$, two sequences $\{\alpha_n\}$, $\{\beta_n\}$ and a constant $K > 0$ (depending on γ, p only) such that:

(a) $|a_0| + ||(\cdot)^\gamma \alpha.||_{\ell^1} + ||(\cdot)^\gamma \beta.||_{\ell^1} \leq K \left\| f^{(1)} \right\|_{\mathbb{A}^{(\gamma)}}$;

(b) The function

$$\widetilde{u}(z) = -\pi a_0 G(z,0) - \pi \sum_{n=1}^{\infty} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} \left[G\left(z, \zeta^{(1)}_{2n,\,2p}\right) - G\left(z, \zeta^{(1)}_{2n,\,2p+1}\right)\right] - \pi \sum_{n=1}^{\infty} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} \left[G\left(z, \zeta^{(2)}_{2n,\,2p}\right) - G\left(z, \zeta^{(2)}_{2n,\,2p+1}\right)\right] \tag{2.9}$$

is harmonic in $D \setminus \mathcal{N}$; $\widetilde{u}$ belongs to $L^p(D)$ with its first derivatives and

$$||\widetilde{u}||_{L^p(D)} + ||D\widetilde{u}||_{L^p(D)} \leq K \|f^{(1)}\|_{\mathbb{A}^{(\gamma)}};$$

(c) For $N \in \mathbb{N}$, the partial sums of $\widetilde{u}$ defined as

$$\widetilde{u}^{(N)}(z) = -\pi a_0 G(z,0) - \pi \sum_{n=1}^{N} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} \left[G\left(z, \zeta^{(1)}_{2n,\,2p}\right) - G\left(z, \zeta^{(1)}_{2n,\,2p+1}\right)\right] - \pi \sum_{n=1}^{N} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} \left[G\left(z, \zeta^{(2)}_{2n,\,2p}\right) - G\left(z, \zeta^{(2)}_{2n,\,2p+1}\right)\right] \tag{2.10}$$

have the boundary properties:

i) $\widetilde{u}^{(N)}$ is of class C^2 in a neighbourhood of ∂D, $\widetilde{u}^{(N)}\big|_{\partial D} = 0$, $\frac{\partial \widetilde{u}^{(N)}}{\partial n}\Big|_{\partial D} \in \mathbb{A}^{(\gamma)}$;

ii) $\widetilde{u}^{(N)}$ converges to $\widetilde{u}$ uniformly on any compact subset of $D \setminus \mathcal{N}$ and

$$\left\| \frac{\partial \widetilde{u}^{(N)}}{\partial n}\Big|_{\partial D} - f^{(1)} \right\|_{\mathbb{A}^{(\gamma)}} \to 0. \tag{2.11}$$

Now let $f \in L^p(\mathbb{R}^2)$, $p > 1$, and denote by $supp\, f$ the support of f, i.e., the complement of the greatest open set in which $f = 0$ a.e.. For the moment, assume that

$$supp\, f \subset\subset D_o = D \setminus \cup_{\nu=0}^{\infty} \overline{D}(\mathfrak{a}_\nu, \frac{2}{3} m_\nu)$$

and let $w \in W^{2,p}(D)$ be the solution to the Dirichlet problem

$$\begin{cases} \Delta w = f & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases} \tag{2.12}$$

Notice that w is harmonic in a neighbourhood of ∂D and hence $\frac{\partial w}{\partial n}\big|_{\partial D}$ belongs to $\mathbb{A}^{(\gamma)}$ for all $\gamma \geq 0$. Thus, the following is an immediate consequence of Fact 2.1, applied with $f^{(1)} = -\frac{\partial w}{\partial n}\big|_{\partial D}$.

Lemma 2.2. *Let γ, σ, p, f, w be as before and $\widetilde{u}$ the function associated to $f^{(1)} := -\frac{\partial w}{\partial n}\big|_{\partial D}$ as in Fact 2.1. Then*

(a) *the function*

$$u = w + \widetilde{u} \tag{2.13}$$

is in $W^{2,p}_{loc}(D \setminus \mathcal{N}) \cap W^{1,p}(D)$ *and* $\Delta u = f$ *a.e. in* D;

(b) *The partial sums* $u^{(N)}(z) = w(z) + \widetilde{u}^{(N)}(z)$ *converge to* u *uniformly on compact subsets of* $D \setminus \mathcal{N}$, *are harmonic in a neighbourhood of* ∂D *with* $u^{(N)}\big|_{\partial D} = 0$, *and*

$$\left\| \frac{\partial u^{(N)}}{\partial n}\bigg|_{\partial D} \right\|_{\mathbb{A}^{(\gamma)}} \to 0. \tag{2.14}$$

Let us write u in the form

$$u(z) = w(z) + \pi \sum_{\nu=0}^{\infty} \omega_\nu G(z, \mathfrak{a}_\nu), \tag{2.15}$$

where the coefficients ω_ν are: $\omega_0 = -a_0$,

$$\omega_\nu = \begin{cases} (-1)^{l+1} \frac{\alpha_n}{2n\sigma} & \text{if } \nu = 1 + 2(n-1)n + l, \\ \\ (-1)^{l+1} \frac{\beta_n}{2n\sigma} & \text{if } \nu = 1 + 2(n-1)n + 2n + l \end{cases}$$

and satisfy $\{\nu^{\frac{\gamma}{2}} \omega_\nu\} \in \ell^1$.

Moreover, one can write u as the sum $u = u_1 + u_2$, where

$$u_1(z) = -\frac{1}{2} \sum_{\nu=0}^{\infty} \omega_\nu \ln|z - \mathfrak{a}_\nu|, \tag{2.16}$$

and

$$u_2(z) = w + \frac{1}{2} \sum_{\nu=1}^{\infty} \omega_\nu \left(\ln|\mathfrak{a}_\nu| + \ln|z - \frac{1}{\overline{\mathfrak{a}}_\nu}| \right). \tag{2.17}$$

Given $\nu_0 \in \mathbb{N} \cup \{0\}$, let us define

$$l^{(\nu_0)}(z) := -\frac{1}{2} \omega_{\nu_0} \ln|z - \mathfrak{a}_{\nu_0}| \tag{2.18}$$

and

$$u_1^{(\nu_0)}(z) := u_1(z) - l^{(\nu_0)}(z) = -\frac{1}{2} \sum_{\nu \neq \nu_0} \omega_\nu \ln|z - \mathfrak{a}_\nu|. \tag{2.19}$$

The following lemma states some properties of these functions.

Lemma 2.3. *Let* $f \in L^p(D)$ *with* $supp\, f \subset\subset D_o$.

a) u_1 *is harmonic in* $D \setminus \mathcal{N}$ *and* $u_1^{(\nu_0)}$ *in* $(D \setminus \mathcal{N}) \cup \{\mathfrak{a}_{\nu_0}\}$; u_2 *is harmonic in* $D \setminus supp\, f$ *and* $\Delta u_2 = f$ *a.e. in* D.

b) *If* $\gamma > 2 - \frac{2}{p}$, *then* $u_2 \in W^{2,p}(D)$.

c) *Let* $\nu_0 \in \mathbb{N} \cup \{0\}$ *and* m_{ν_0} *be as in* (2.6). *Then there exists a positive constant* A *such that*

$$\max_{D(\mathfrak{a}_{\nu_0}, m_{\nu_0}/2)} |D^2 u_2| \le \frac{A}{m_{\nu_0}^2} \Big(\sum_{\nu=1}^{\infty} |\omega_\nu| + \|f\|_{L^p(D)} \Big). \tag{2.20}$$

Proof. It is sufficient to prove only (b) and (c). Clearly,

$$\|w\|_{W^{2,p}(D)} \le C\,\|f\|_{L^p(D)}$$

for some constant C. Now take a point $z \in \overline{D}$ and observe that

$$\left| z - \frac{1}{\overline{\mathfrak{a}}_\nu} \right| \ge \frac{1}{|\mathfrak{a}_\nu|} - 1 = \frac{1 - |\mathfrak{a}_\nu|}{|\mathfrak{a}_\nu|}.$$

Since $1 - |\mathfrak{a}_\nu| = 1 - \sigma^{\frac{1}{n}}$ for some $n \in \mathbb{N}$, we have that $1 - |\mathfrak{a}_\nu| = e^{\tilde{x}} \frac{1}{n} \ln \frac{1}{\sigma}$ for some $\tilde{x}$ in the interval $(\frac{1}{n} \ln \sigma, 0)$ and hence, by (2.7),

$$1 - |\mathfrak{a}_\nu| \ge \frac{\sigma \ln 1/\sigma}{n} \ge \frac{\widetilde{K}}{\sqrt{\nu}},$$

where $\widetilde{K}$ is a constant depending on γ. Then from (2.17), one has

$$\max_D |u_2 - w| \le C \sum_{\nu=1}^{\infty} |\omega_\nu| (\frac{1}{2} |\log \sigma| + |\log \widetilde{K}| + \frac{1}{2} \log \nu) < +\infty.$$

Moreover, it is not difficult to check that

$$\begin{aligned} \left\| D^2(u_2 - w) \right\|_{L^p(D)} &\le C \sum_{\nu=1}^{\infty} |\omega_\nu| \left\| | \cdot - \frac{1}{\overline{\mathfrak{a}}_\nu} |^{-2} \right\|_{L^p(D)} \\ &\le C \sum_{\nu=1}^{\infty} |\omega_\nu| + C \sum_{\nu=N+1}^{\infty} \nu^{\gamma/2} |\omega_\nu| \left\| | \cdot - \frac{1}{\overline{\mathfrak{a}}_\nu} |^{\gamma-2} \right\|_{L^p(D(\frac{1}{\overline{\mathfrak{a}}_\nu}, 3))}, \end{aligned}$$

where N is chosen sufficiently large such that $\frac{1}{|\overline{\mathfrak{a}}_\nu|} \le 2$ for $\nu > N$.

Since $2 + p(\gamma - 2) > 0$ and

$$\left\| | \cdot - \frac{1}{\overline{\mathfrak{a}}_{\ \nu}} |^{\gamma-2} \right\|_{L^p(D(\frac{1}{\overline{\mathfrak{a}}_\nu}, 3))} \le (2\pi)^{1/p} \Big(\frac{3^{2+p(\gamma-2))}}{2 + p(\gamma - 2))} \Big)^{1/p} < +\infty,$$

(b) follows.

Let us prove (c). As $supp\, f \subset\subset D_o$ and

$$\left| D_z^2 G(z, \zeta) \right| \le \frac{C}{|z - \zeta|^2}$$

for some constant C, we get

$$\max_{D(\mathfrak{a}_{\nu_0}, m_{\nu_0}/2)} |D^2 w| \le \max_{D(\mathfrak{a}_{\nu_0}, m_{\nu_0}/2)} \left| \int_{D_o} D_z^2 G(z, \zeta) f(\zeta)\, d\zeta \right| \le \frac{C}{m_{\nu_0}^2} ||f||_{L^p(D)}.$$

Moreover,

$$\max_{D(\mathfrak{a}_{\nu_0},m_{\nu_0})}|D^2(u_2-w)|\le \frac{C}{m_{\nu_0}^2}\sum_{\nu=1}^{\infty}|\omega_\nu|$$

and the bound (2.20) follows. □

3. Modifying u and Δ in neighbourhoods of $\mathfrak{a}_\nu \in \mathcal{N}$

From the results of the previous section, it turns out that we need to modify the function u and the operator Δ in neighbourhoods of the points $\mathfrak{a}_\nu \in \mathcal{N}$ in order to obtain a new function v and a new second order uniformly elliptic operator L of the form (1.1) with the following properties:

i) $v \in C^{1,1}(D\setminus\mathcal{N})$;
ii) $v \in W_0^{2,p}(D)$;
iii) $Lv = f$ a.e. in D.

For a fixed $\nu_0 \in \mathbb{N}\cup\{0\}$, let

$$r_{\nu_0} = \lambda_{\nu_0} m_{\nu_0}$$

with m_{ν_0} as in (2.6) and λ_{ν_0} a constant in (0,1/3) to be fixed later. Of course, $\overline{D}(\mathfrak{a}_{\nu_0}, r_{\nu_0}) \subset D$ and $\overline{D}(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}) \cap \overline{D}(\mathfrak{a}_\nu, 2r_\nu) = \emptyset$ if $\nu \ne \nu_0$.

To modify the function u inside the disk $D(\mathfrak{a}_{\nu_0}, r_{\nu_0})$, let us replace the term $l^{(\nu_0)}$ with a smoother function as suggested by the following lemma from [6].

Lemma 3.1. *Let* $\nu_0 \in \mathbb{N}\cup\{0\}$, $1<p<2, 0<2-\frac{2}{p}<h<1$ *and consider the function in* $D(\mathfrak{a}_{\nu_0}, r_{\nu_0})$ *defined by*

$$s^{(\nu_0)}(r) = H_0^{(\nu_0)} + H_1^{(\nu_0)} r^h, \qquad r = |z-\mathfrak{a}_{\nu_0}|, \tag{3.1}$$

where

$$H_0^{(\nu_0)} = -\frac{\omega_{\nu_0}}{2}\ln r_{\nu_0} + \frac{\omega_{\nu_0}}{2h}, \qquad H_1^{(\nu_0)} = -\frac{\omega_{\nu_0}}{2h} r_{\nu_0}^{-h}. \tag{3.2}$$

Then

i) $s^{(\nu_0)}(r_{\nu_0}) = l^{(\nu_0)}(r_{\nu_0})$ *and* $\frac{\partial s^{(\nu_0)}}{\partial r}(r_{\nu_0}) = \frac{\partial l^{(\nu_0)}}{\partial r}(r_{\nu_0})$;
ii) $s^{(\nu_0)}(|\cdot - \mathfrak{a}_{\nu_0}|)$ *belongs to* $W^{2,p}(D(\mathfrak{a}_{\nu_0}, r_{\nu_0}))$ *and*

$$||s^{(\nu_0)}||_{L^p(D(\mathfrak{a}_{\nu_0},r_{\nu_0}))} \le C|\omega_{\nu_0}| r_{\nu_0}^{2p}(1+|\ln r_{\nu_0}|), \tag{3.3}$$

$$||\Delta s^{(\nu_0)}||_{L^p(D(\mathfrak{a}_{\nu_0},r_{\nu_0}))} = C'|\omega_{\nu_0}| r_{\nu_0}^{\frac{2}{p}-2}, \tag{3.4}$$

where C *and* C' *are constants, both depending only on* p *and on* h.

For reader's convenience, we recall its short proof.

Proof. It is enough to prove the last two formulas. By means of (3.1) and (3.2), we have

$$||s^{(\nu_0)}||_{L^p(D(\mathfrak{a}_{\nu_0},r_{\nu_0}))} \le |H_0^{(\nu_0)}|\pi^{1/p} r_{\nu_0}^{2/p} + |H_1^{(\nu_0)}|(2\pi)^{1/p}\left(\int_0^{r_{\nu_0}} r^{ph+1}\,dr\right)^{\frac{1}{p}}$$

$$= \pi^{1/p} \frac{|\omega_{\nu_0}|}{2} r_{\nu_0}^{2/p} (|\ln r_{\nu_0}| + \frac{1}{h} + \frac{2^{1/p}}{h(ph+2)^{1/p}})$$
$$\leq C |\omega_{\nu_0}| r_{\nu_0}^{2/p} (1 + |\ln r_{\nu_0}|).$$

Moreover, $\Delta s^{(\nu_0)} = (s^{(\nu_0)})_{rr} + \frac{(s^{(\nu_0)})_r}{r} = -\frac{h}{2}\omega_{\nu_0} r_{\nu_0}^{-h} r^{h-2}$ so that

$$||\Delta s^{(\nu_0)}||_{L^p(D(\mathfrak{a}_{\nu_0}, r_{\nu_0}))} = (2\pi)^{1/p} |\omega_{\nu_0}| r_{\nu_0}^{-h} \frac{h}{2} \left(\int_0^{r_{\nu_0}} r^{p(h-2)+1}\, dr \right)^{\frac{1}{p}}$$
$$= \frac{(2\pi)^{1/p} |\omega_{\nu_0}| (h/2)}{(p(h-2)+2)^{1/p}} r_{\nu_0}^{\frac{2}{p}-2}.$$

□

Then we have:

Lemma 3.2. *Let p, h, ν_0 be as in the previous lemma. Set $\Omega = \sum_{\nu=0}^{\infty} |\omega_\nu| + ||f||_{L^p(D)}$ and*

$$\lambda_{\nu_0} = \min \left\{ \frac{1}{4},\ \sqrt{\frac{(1-h)|\omega_{\nu_0}|}{4(4A+1)\Omega}} \right\}, \tag{3.5}$$

where A is the constant in the estimate (2.20)*.*

Consider the following function $v^{(\nu_0)}$ on $D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0})$:

$$v^{(\nu_0)} = \begin{cases} s^{(\nu_0)}(|\cdot - \mathfrak{a}_{\nu_0}|) + u_1^{(\nu_0)} + u_2 & \text{in } D(\mathfrak{a}_{\nu_0}, r_{\nu_0}), \\ u & \text{in } D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}) \setminus D(\mathfrak{a}_{\nu_0}, r_{\nu_0}). \end{cases}$$

It turns out that:

(a) $v^{(\nu_0)} \in C^{1,1}(D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}) \setminus \{\mathfrak{a}_{\nu_0}\})$;
(b) *$v^{(\nu_0)}$ is harmonic in $D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}) \setminus D(\mathfrak{a}_{\nu_0}, r_{\nu_0})$;*
(c) $v^{(\nu_0)} \in W^{2,p}(D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}))$.

Moreover, $v^{(\nu_0)}$ satisfies a second order, uniformly elliptic equation $Lv^{(\nu_0)} = 0$ with bounded measurable coefficients in $D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0})$ and lower ellipticity constant $\frac{1}{2}(1-h)$.

Proof. Statement (a) follows by Lemma 3.1(i) and from the fact that $v^{(\nu_0)}$ has second order derivatives bounded in every compact subset of $D(\mathfrak{a}_{\nu_0}, 2r_{\nu_0}) \setminus \{\mathfrak{a}_{\nu_0}\}$. Statement (b) is clear and in regard to (c), it is enough to use Lemma 3.1(ii).

As far as the last claim is concerned, by known facts (see e.g. [2], Ch. 6), one needs to verify the existence of a number $q \in (0,1)$ such that

$$\left| \frac{(v^{(\nu_0)})_{z\bar{z}}}{(v^{(\nu_0)})_{zz}} \right| \leq q \quad \text{in } D(\mathfrak{a}_{\nu_0}, r_{\nu_0}). \tag{3.6}$$

Let us prove that (3.6) holds true with $q = h$. Indeed, recalling that $u_1^{(\nu_0)}$ and u_2 are harmonic in $D(\mathfrak{a}_{\nu_0}, r_{\nu_0})$, one may write

$$\left|\frac{(v^{(\nu_0)})_{z\overline{z}}}{(v^{(\nu_0)})_{zz}}\right| = \left|\frac{(s^{(\nu_0)}(|\cdot - \mathfrak{a}_{\nu_0}|))_{z\overline{z}}}{(s^{(\nu_0)}(|\cdot - \mathfrak{a}_{\nu_0}|) + u_1^{(\nu_0)} + u_2)_{zz}}\right|.$$

On the other hand, by (3.1), (3.2) and using polar coordinates with origin $\mathfrak{a}_{\nu_0}$,

$$(s^{(\nu_0)}(|\cdot - \mathfrak{a}_{\nu_0}|))_{zz} = \frac{\omega_{\nu_0}}{4}(1 - \frac{h}{2})\left(\frac{r}{r_{\nu_0}}\right)^h (z - \mathfrak{a}_{\nu_0})^{-2},$$

$$(s_{\nu_0}(|\cdot - \mathfrak{a}_{\nu_0}|))_{z\overline{z}} = -\frac{\omega_{\nu_0}}{8} h \left(\frac{r}{r_{\nu_0}}\right)^h r^{-2}.$$

Then by easy calculations,

$$\left|\frac{(v^{(\nu_0)})_{z\overline{z}}}{(v^{(\nu_0)})_{zz}}\right| = \left|\frac{\omega_{\nu_0} h r^{-2}(z - \mathfrak{a}_{\nu_0})^2}{2\omega_{\nu_0}(1 - \frac{h}{2}) + 8(u_1^{(\nu_0)} + u_2)_{zz}\left(\frac{r}{r_{\nu_0}}\right)^{-h}(z - \mathfrak{a}_{\nu_0})^2}\right|$$
$$= \frac{|\omega_{\nu_0}| h}{\left|\omega_{\nu_0}(2 - h) + 8(u_1^{(\nu_0)} + u_2)_{zz}\left(\frac{r}{r_{\nu_0}}\right)^{-h}(z - \mathfrak{a}_{\nu_0})^2\right|}.$$

Now, it is clear that

$$\left|8(u_1^{(\nu_0)} + u_2)_{zz}\left(\frac{r}{r_{\nu_0}}\right)^{-h}(z - \mathfrak{a}_{\nu_0})^2\right| \le 8r_{\nu_0}^2 \max_{\overline{D}(\mathfrak{a}_{\nu_0}, r_{\nu_0})} \left|(u_1^{(\nu_0)} + u_2)_{zz}\right|.$$

Moreover, since for any $z \in \overline{D}(\mathfrak{a}_{\nu_0}, r_{\nu_0})$ and $\nu \neq \nu_0$,

$$|z - \mathfrak{a}_\nu| \ge |\mathfrak{a}_{\nu_0} - \mathfrak{a}_\nu| - |z - \mathfrak{a}_{\nu_0}| \ge m_{\nu_0} - r_{\nu_0} = (1 - \lambda_{\nu_0}) m_{\nu_0},$$

we have

$$\left|\frac{\partial^2 u_1^{(\nu_0)}}{\partial z^2}\right| \le \frac{1}{4}\sum_{\nu \neq \nu_0} \frac{|\omega_\nu|}{|z - \mathfrak{a}_\nu|^2} \le \frac{1}{4}\frac{\Omega}{(1 - \lambda_{\nu_0})^2 m_{\nu_0}^2}$$

and, recalling the bound (2.20) and using (3.5), we get

$$8r_{\nu_0}^2 \max_{\overline{D}(\mathfrak{a}_{\nu_0}, r_{\nu_0})} \left|(u_1^{(\nu_0)} + u_2)_{zz}\right| \le 8r_{\nu_0}^2 \left\{\frac{A\Omega}{m_{\nu_0}^2} + \frac{\Omega}{4(1 - \lambda_{\nu_0})^2 m_{\nu_0}^2}\right\} \le (1 - h)|\omega_{\nu_0}|.$$

Therefore,

$$\left|\frac{(v^{(\nu_0)})_{z\overline{z}}}{(v^{(\nu_0)})_{zz}}\right| \le \frac{|\omega_{\nu_0}| h}{|\omega_{\nu_0}|(2 - h) - 8(r_{\nu_0})^2 \max_{\overline{D}(\mathfrak{a}_{\nu_0}, r_{\nu_0})} |(u_1^{(\nu_0)} + u_2)_{zz}|} \le h$$

and (3.6) holds true for $q = h$. □

4. The main theorem

Now we are ready to prove our main result under the hypothesis that $suppf \subset\subset D_o = D \setminus \cup_{\nu=0}^{\infty} \overline{D}(\mathfrak{a}_\nu, \frac{2}{3}m_\nu)$.

Theorem 4.1. *Let $1 < p < 2$, $2 - \frac{2}{p} < h < 1$ and $f \in L^p(D)$ with $suppf \subset\subset D_o$. Then there exist a function $v \in W_0^{2,p}(D)$ and a uniformly elliptic operator L of the form* (1.1) *with bounded measurable coefficients in D and lower ellipticity constant $\frac{1-h}{2}$ such that $Lv = f$ a.e. in D.*

Proof. In what follows, we will denote by the same letter C different constants. Choose $\gamma > 4(p-1)$ and notice that since $\gamma > 2 - \frac{2}{p}$, Lemma 2.3 holds true. For any $\nu \in \mathbb{N} \cup \{0\}$, denote by D_ν the disk of center $\mathfrak{a}_\nu$ and radius $r_\nu = \lambda_\nu m_\nu$, where λ_ν is given by (3.5). Let

$$v := \begin{cases} u & \text{in } D \setminus \bigcup_{\nu=0}^{\infty} D_\nu, \\ v^{(\nu)} = s^{(\nu)} + u_1^{(\nu)} + u_2 & \text{in } D_\nu \quad \text{for all} \quad \nu = 0, 1 \dots . \end{cases} \tag{4.1}$$

Then v satisfies $\Delta v = f$ in $D \setminus \bigcup_{\nu=0}^{\infty} D_\nu$ and, by Lemma 3.2, it solves an elliptic equation $Lv = 0$, i.e., $Lv = f$, in each D_ν with ellipticity constant $\frac{1}{2}(1-h)$. By the same lemma, it is also in $W_{loc}^{2,p}(D)$. To prove that $v \in W^{2,p}(D)$, it is sufficient to show that $v \in L^p(D)$ and $\Delta v \in L^p(D)$. First of all, one has

$$\|v\|_{L^p(D)} \le \|u\|_{L^p(D)} + \frac{1}{2}\sum_{\nu=0}^{\infty} |\omega_\nu| \|\log(|\cdot - \mathfrak{a}_\nu|)\|_{L^p(D_\nu)} + \sum_{\nu=0}^{\infty} \|s_\nu\|_{L^p(D_\nu)}.$$

Now by Lemma 2.3, $\|u\|_{L^p(D)} < +\infty$; moreover,

$$\sum_{\nu=0}^{\infty} |\omega_\nu| \|\log(|\cdot - \mathfrak{a}_\nu|)\|_{L^p(D_\nu)} \le C \sum_{\nu=0}^{\infty} |\omega_\nu| < +\infty.$$

In addition, by (3.3) of Lemma 3.1,

$$\sum_{\nu=0}^{\infty} \|s_\nu\|_{L^p(D_\nu)} \le C \sum_{\nu=0}^{\infty} |\omega_\nu| r_\nu^{2p} (1 + |\log(r_\nu)|) < +\infty$$

since $\{\omega_\nu\} \in \ell^1$ and $r_\nu^{2p}(1 + |\log(r_\nu)|)$ tends to zero as $\nu \to \infty$. Hence $v \in L^p(D)$.

On the other hand, since $\Delta v = f$ in $D \setminus \bigcup_{\nu=0}^{\infty} D_\nu$ and by (3.4) of Lemma 3.1,

$$\begin{aligned} \|\Delta v\|_{L^p(D)}^p &\le \|f\|_{L^p(D)}^p + \sum_{\nu=0}^{\infty} \|\Delta s_\nu\|_{L^p(D_\nu)}^p \\ &\le \|f\|_{L^p(D)}^p + C \sum_{\nu=0}^{\infty} |\omega_\nu|^p r_\nu^{2(1-p)}. \end{aligned}$$

Now from (3.5), we may write $\lambda_\nu^2 \le C|\omega_\nu|$ and using (2.8) for sufficiently large ν, we have

$$|\omega_\nu|^p r_\nu^{2(1-p)} = |\omega_\nu|^p (\lambda_\nu^2 m_\nu^2)^{(1-p)} \le \mathrm{C}\frac{|\omega_\nu|}{\nu^{2(1-p)}} = \mathrm{C}(|\omega_\nu|\nu^{\frac{\gamma}{2}})\nu^{-\frac{\gamma}{2}+2(p-1)}.$$

Since $\{\nu^{\gamma/2}\omega_\nu\} \in \ell^1$ and $2(p-1) - \gamma/2 \le 0$, it follows that the series

$$\sum_{\nu=0}^{\infty} |\omega_\nu|^p r_\nu^{2(1-p)}$$

is convergent and hence that $\|\Delta v\|_{L^p(D)}^p < +\infty$.

To conclude, we need to prove that $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial n}|_{\partial D} = 0$.

For any $N = 0, 1, \ldots,$ we set

$$v^{(N)} := \begin{cases} u^{(N)} & \text{in } D \setminus \bigcup_{\nu=0}^N D_\nu, \\ u^{(N)} + s^{(\nu)} - l^{(\nu)} & \text{in } D_\nu,\ \ \nu = 0, 1, \ldots, N, \end{cases}$$

where $u^{(N)}$ is the partial sum defined in Lemma 2.3. The function $v^{(N)}$ coincides with $u^{(N)}$ in a neighbourhood of ∂D and hence it is regular up to the boundary. In particular,

$$v^{(N)}\Big|_{\partial D} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\| \frac{\partial v^{(N)}}{\partial n}\Big|_{\partial D} \right\|_{\mathbb{A}(\gamma)} = 0.$$

Moreover, $v^{(N)}$ converges to v in $L^p(D)$: In fact,

$$\|v^{(N)} - v\|_{L^p(D)}^p \le \|v^{(N)} - v\|_{L^p(D\setminus\bigcup_{\nu=N+1}^\infty D_\nu)}^p + \sum_{\nu=N+1}^{\infty} \|v^{(N)} - v\|_{L^p(D_\nu)}^p,$$

$$\begin{aligned} \|v^{(N)} - v\|_{L^p(D\setminus\bigcup_{\nu=N+1}^\infty D_\nu)} &\le \pi \sum_{\nu=N+1}^{\infty} |\omega_\nu| \|G(\cdot, \mathfrak{a}_\nu)\|_{L^p(D)} \\ &\le C(p) \sum_{\nu=N+1}^{\infty} |\omega_\nu| \overset{N\to\infty}{\longrightarrow} 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{\nu=N+1}^{\infty} \|v^{(N)} - v\|_{L^p(D_\nu)} \le \pi &\sum_{\nu=N+1}^{\infty} |\omega_\nu| \|G(\cdot, \mathfrak{a}_\nu)\|_{L^p(D)} + \sum_{\nu=N+1}^{\infty} \|s_\nu\|_{L^p(D_\nu)} + \\ &+ \frac{1}{2} \sum_{\nu=N+1}^{\infty} |\omega_\nu| \| \ln|z - \mathfrak{a}_\nu| \|_{L^p(D)} \\ &\le \mathrm{C} \sum_{\nu=N+1}^{\infty} |\omega_\nu| + \sum_{\nu=N+1}^{\infty} \|s^\nu\|_{L^p(D_\nu)} \overset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

In addition,

$$\|\Delta v^{(N)} - \Delta v\|_{L^p(D)} \leq \sum_{\nu=N+1}^{\infty} \|\Delta s^\nu\|_{L^p(D_\nu)} \overset{N\to\infty}{\longrightarrow} 0$$

since the series $\sum_{\nu=0}^{\infty} \|\Delta s_\nu\|_{L^p(D_\nu)}$ converge.

From this it follows that $v^{(N)}$ converges to v in $W^{2,p}(D)$, and hence that $v^{(N)}\big|_{\partial D}$ tends to $v|_{\partial D}$ in $W^{2-1/p,p}(\partial D)$ and $\frac{\partial v^{(N)}}{\partial n}\Big|_{\partial D}$ tends to $\frac{\partial v}{\partial n}\big|_{\partial D}$ in $W^{1-1/p,p}(\partial D)$. This implies that $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial n}\big|_{\partial D} = 0$. □

Finally, let us remove the previous restriction on the support of f.

Theorem 4.2. *Let $1 < p < 2$, $2 - \frac{2}{p} < h < 1$ and $f \in L^p(\mathbb{R}^2)$ with compact support. Then there exist a function $v \in W_0^{2,p}(\mathbb{R}^2)$ and a uniformly elliptic operator L with bounded and measurable coefficients and lower ellipticity constant $\frac{1-h}{2}$ such that $Lv = f$ a.e. in $\mathbb{R}^2$.*

Proof. Assume $supp\, f \subset D(0,R)$ and let $z_o \in D_o$ and $\rho > 0$ be such that $\overline{D}(z_o,\rho) \subset D_o$. Then $\widetilde{f}(z') := (\frac{\rho}{R})^2 f(\frac{R}{\rho}(z'-z_o))$ satisfies $supp\, \widetilde{f} \subset D(z_o,\rho)$ and by Theorem 4.1, there exist a function $\widetilde{v} \in W_0^{2,p}(D)$ and a uniformly elliptic operator $\widetilde{L}$ of the form $\widetilde{L} := \widetilde{a}^{11}(z')\partial_{x'x'} + 2\widetilde{a}^{12}(z')\partial_{x'y'} + \widetilde{a}^{22}(z')\partial_{y'y'}$ with lower ellipticity constant $\frac{1-h}{2}$ such that $\widetilde{L}\widetilde{v} = \widetilde{f}$ a.e. in D.

Now, let $v(z) := \widetilde{v}(z') = \widetilde{v}(z_o + \frac{\rho}{R}z)$ and $L := a^{11}(z)\partial_{xx} + 2a^{12}(z)\partial_{xy} + a^{22}(z)\partial_{yy}$ with $a^{ij}(z) := \widetilde{a}^{ij}(z') = \widetilde{a}^{ij}(z_o + \frac{\rho}{R}z)$ in $D(-(R/\rho)z_o, R/\rho)$ and $L = \Delta$, otherwise. Then $v \in W_0^{2,p}(\mathbb{R}^2)$, L is uniformly elliptic with the same ellipticity constant of $\widetilde{L}$ and $Lv = f$ a.e. in $\mathbb{R}^2$. □

Remark 4.3. Let p, f, v, L be as in Theorem 4.2. By classical results on second order elliptic equations and elliptic first order system (see e.g. [2]), one has that the function $w := v_x - iv_y$ belongs to $W_0^{1,p}(\mathbb{C})$ and satisfies a complex uniformly elliptic first order system of the form

$$w_{\bar{z}} = \mu w_z + \nu \bar{w}_{\bar{z}} + \gamma \quad \text{ in } \mathbb{C}$$

with $\mu = \mu(z)$, $\nu = \nu(z)$ and $\gamma = \gamma(z)$, complex-valued functions, such that $|\mu| + |\nu| \leq k < 1$ and $|\gamma| \leq k'$.

References

[1] K. Astala, T. Iwaniec, and G. Martin, *Pucci's conjecture and the Alexandrov inequality for elliptic PDEs in the plane*, J. Reine Angew. Math. **591** (2006), 49–74.

[2] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, Interscience, 1964.

[3] P. Buonocore and P. Manselli, *Solutions to two dimensional, uniformly elliptic equations, that lie in Sobolev spaces and have compact support*, Rend. Circ. Mat. Palermo (2) **51** (2002), no. 3, 476–484.

[4] C. Giannotti, *A compactly supported solution to a three-dimensional uniformly elliptic equation without zero order term*, J. Differential Equations **201** (2004), no. 2, 234–249.

[5] C. Giannotti and P. Manselli, *Expansions with Poisson kernels and related topics*, Proc. Edinb. Math. Soc. (2) **53** (2010), no. 1, 153–173.

[6] ———, *On elliptic extensions in the disk*, Potential Anal. **33** (2010), no. 3, 249–262.

[7] T. H. Wolff, *Some constructions with solutions of variable coefficient elliptic equations*, J. Geom. Anal. **3** (1993), no. 5, 423–511.

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