J. Korean Math. Soc. **50** (2013), No. 2, pp. 393–410
http://dx.doi.org/10.4134/JKMS.2013.50.2.393

TWO GENERALIZATIONS OF LCM-STABLE EXTENSIONS

Gyu Whan Chang, Hwankoo Kim, and Jung Wook Lim

Abstract. Let $R \subseteq T$ be an extension of integral domains, X be an indeterminate over T, and $R[X]$ and $T[X]$ be polynomial rings. Then $R \subseteq T$ is said to be LCM-stable if $(aR \cap bR)T = aT \cap bT$ for all $0 \neq a, b \in R$. Let w_A be the so-called w-operation on an integral domain A. In this paper, we introduce the notions of $w(e)$- and w-LCM stable extensions: (i) $R \subseteq T$ is $w(e)$-LCM-stable if $((aR \cap bR)T)_{w_T} = aT \cap bT$ for all $0 \neq a, b \in R$ and (ii) $R \subseteq T$ is w-LCM-stable if $((aR \cap bR)T)_{w_R} = (aT \cap bT)_{w_R}$ for all $0 \neq a, b \in R$. We prove that LCM-stable extensions are both $w(e)$-LCM-stable and w-LCM-stable. We also generalize some results on LCM-stable extensions. Among other things, we show that if R is a Krull domain (resp., PvMD), then $R \subseteq T$ is $w(e)$-LCM-stable (resp., w-LCM-stable) if and only if $R[X] \subseteq T[X]$ is $w(e)$-LCM-stable (resp., w-LCM-stable).

0. Introduction

Let $R \subseteq T$ be an extension of integral domains, X be an indeterminate over T, and $R[X]$ and $T[X]$ be polynomial rings. As in [10], we say that $R \subseteq T$ is *LCM-stable* if $(aR \cap bR)T = aT \cap bT$ for all $0 \neq a, b \in R$. Clearly, if $T = R_S$ for a multiplicative subset S of R, then $R \subseteq T$ is LCM-stable. Also, $R \subseteq R[X]$ is LCM-stable. This concept was first introduced by Gilmer [10] and has been studied by many authors [1, 6, 18, 19, 20, 21]. It is known that R is a Prüfer domain if and only if $R \subseteq T$ is LCM-stable for any domain T containing R [20, Corollary 1.8]; if R is a GCD-domain, then $R \subseteq T$ is LCM-stable if and only if T is t-linked over R, if and only if $R[X] \subseteq T[X]$ is LCM-stable [20, Corollary 3.7]; and if R is a Krull domain, then $R \subseteq T$ is LCM-stable if and only if $R[X] \subseteq T[X]$ is LCM-stable [21, Theorem 11]. Also, it was noted that R is a Prüfer domain if and only if $R[X] \subseteq T[X]$ is LCM-stable for each domain T containing R as a subring [6]. For the case of power series rings, Condo proved that R is a Dedekind domain if and only if $R[\![X]\!] \subseteq T[\![X]\!]$ is LCM-stable for any domain T containing R as a subring [6, Theorem 11].

Received May 2, 2012; Revised September 24, 2012.
2010 *Mathematics Subject Classification.* 13A15, 13G05.
Key words and phrases. star-operation, LCM-stable, w-LCM-stable, $w(e)$-LCM-stable, PvMD, Krull domain.

In modern multiplicative ideal theory, star-operations are essential and important tools for characterizing and investigating several classes of integral domains (Definitions related to star-operations will be reviewed in Section 1). Among these, the w-operation can be used to characterize GCD-domains, Prüfer v-multiplication domains (PvMD) and Krull domains. So it is natural and reasonable to study the w-operation version of LCM-stable extensions. Let w_R and w_T be the w-operations on R and T, respectively. In this paper, for an extension $R \subseteq T$ of integral domains, we introduce the concepts of w_R-LCM-stableness and $w_T(e)$-LCM-stableness and investigate some properties of them.

In Section 1, we review some notations and basic facts on star-operations, then we define the notions of $*(e)$- and $*$-LCM-stable extensions. Let $*_T$ be a star-operation on T and $*_R$ be a star-operation on R with $(*_R)_{w_R} = *_R$. We show that LCM-stable extensions are both $*_T(e)$-LCM-stable and $*_R$-LCM-stable. In Section 2, we study $w(e)$-LCM-stable extensions: $R \subseteq T$ is $w(e)$-*LCM-stable* if $((aR \cap bR)T)_{w_T} = aT \cap bT$ for all $0 \neq a, b \in R$. Among other things, we show that $R \subseteq T$ is $w(e)$-LCM-stable if and only if $R \subseteq T_M$ is LCM-stable for all maximal t-ideals M of T. We also prove that if R is a Krull domain, then $R \subseteq T$ is $w(e)$-LCM-stable if and only if $R[X] \subseteq T[X]$ is $w(e)$-LCM-stable. Moreover, if T is an overring of a Krull domain R, then $R \subseteq T$ is $w(e)$-LCM-stable if and only if T is t-linked over R. Finally in Section 3, we study w-LCM-stable extensions: $R \subseteq T$ is w-*LCM-stable* if $((aR \cap bR)T)_{w_R} = (aT \cap bT)_{w_R}$ for all $0 \neq a, b \in R$. We show that $R \subseteq T$ is w-LCM-stable if and only if $R_P \subseteq T_P$ is LCM-stable for all maximal t-ideals P of R. In particular, if T is t-linked over R, then w-LCM-stable extensions are $w(e)$-LCM-stable. We finally show that if R is a PvMD, then $R \subseteq T$ is w-LCM-stable if and only if $R[X] \subseteq T[X]$ is w-LCM-stable. As a corollary, we have that R is a PvMD if and only if $R[X] \subseteq T[X]$ is w-LCM-stable for each overring T of R.

1. Star-operations and LCM-stableness

Let R be an integral domain and $qf(R)$ be the quotient field of R. Let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R. A mapping $* : \mathbf{F}(R) \to \mathbf{F}(R)$, $I \mapsto I_*$, is called a *star-operation on* R if the following three conditions are satisfied for all $0 \neq a \in qf(R)$ and $I, J \in \mathbf{F}(R)$: (i) $(aR)_* = aR$ and $(aI)_* = aI_*$, (ii) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$, and (iii) $(I_*)_* = I_*$.

Let $\mathbf{f}(R)$ be the set of nonzero finitely generated fractional ideals of R; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. Given a star-operation $*$ on R, we can construct two new star-operations $*_f$ and $*_w$ on R as follows: $I_{*_f} = \bigcup\{J_* \mid J \subseteq I \text{ and } J \in \mathbf{f}(R)\}$ and $I_{*_w} = \{x \in qf(R) \mid xJ \subseteq I \text{ for some } J \in \mathbf{f}(R) \text{ with } J_* = R\}$ for all $I \in \mathbf{F}(R)$. We say that $*$ is of *finite character* if $*_f = *$. Clearly, $(*_f)_f = *_f$ and $(*_w)_f = *_w = (*_f)_w$, and hence $*_f$ and $*_w$ are of finite character. We say that $I \in \mathbf{F}(R)$ is a $*$-*ideal* if $I_* = I$. A $*$-ideal of R is called a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals of R. It is known that if R is not a field, then a maximal $*_f$-ideal of R always exists. Let $*$-Max(R) be the set of maximal $*$-ideals of R. It is known that $*_f$-Max$(R) = *_w$-Max(R) and $I_{*_w} = \bigcap_{P \in *_f\text{-Max}(R)} IR_P$ for all $I \in \mathbf{F}(R)$ [2, Corollary 2.10], hence $(I_{*_w})R_P = IR_P$ for all $P \in *_f$-Max(R).

The most well-known examples of star-operations are the d-, v-, t-, and w-operations. The d-operation is just the identity function on $\mathbf{F}(R)$, *i.e.*, $I_d = I$ for all $I \in \mathbf{F}(R)$. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in qf(R) \mid xI \subseteq R\}$ for all $I \in \mathbf{F}(R)$. The t-operation (resp., w-operation) is given by $t = v_f$ (resp., $w = v_w$). It is clear that $d = d_f = d_w$, $t_f = t$ and $w = w_f = t_w = w_w$. Let $*_1$ and $*_2$ be star-operations on R. We mean by $*_1 \leq *_2$ that $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathbf{F}(R)$. It is well-known that if $*_1 \leq *_2$, then $(*_1)_f \leq (*_2)_f$ and $(*_1)_w \leq (*_2)_w$. Also, $d \leq *_w \leq *_f \leq * \leq v$, $*_f \leq t$, and $*_w \leq w$ for any star-operation $*$ on R. For details on more basic properties of star operations, the reader may consult [11, Sections 32 and 34] (A more appropriate reference, companion to [10], may be [24, Section 1]).

We first give the definition of $*_T(e)$-LCM-stable extensions, which is a natural generalization of LCM-stable extensions.

Definition 1.1. Let $R \subseteq T$ be an extension of integral domains, and let $*_T$ be a star-operation on T. We say that $R \subseteq T$ is $*_T(e)$*-LCM-stable* if $((aR \cap bR)T)_{*_T} = (aT \cap bT)_{*_T}$ for all $0 \neq a, b \in R$.

It is clear that $\alpha T \cap \beta T$ is a v-ideal of T, and thus $(\alpha T \cap \beta T)_{*_T} = \alpha T \cap \beta T$ for any $0 \neq \alpha, \beta \in T$. Hence $R \subseteq T$ is $*_T(e)$-LCM-stable if and only if $((aR \cap bR)T)_{*_T} = aT \cap bT$ for all $0 \neq a, b \in R$. Note that if d_T is the d-operation on T, then $((aR \cap bR)T)_{d_T} = (aR \cap bR)T$ for all $0 \neq a, b \in R$; hence the $d_T(e)$-LCM-stable extension is just the LCM-stable extension. Note also that if $*$ is a star-operation on T, then $* \leq v_T$, where v_T is the v-operation on T; so if $R \subseteq T$ is $*(e)$-LCM-stable, then $((aR \cap bR)T)_* = ((aR \cap bR)T)_{v_T}$ for all $0 \neq a, b \in R$.

Lemma 1.2. *Let $*_1 \leq *_2$ be star-operations on T.*

(1) *If $R \subseteq T$ is $*_1(e)$-LCM-stable, then $R \subseteq T$ is $*_2(e)$-LCM-stable.*
(2) *If $R \subseteq T$ is LCM-stable, then $R \subseteq T$ is $*_1(e)$-LCM-stable.*
(3) *If $R \subseteq T$ is $*_1(e)$-LCM-stable, then $R \subseteq T$ is $v_T(e)$-LCM-stable.*

Proof. For (1), let $0 \neq a, b \in R$. Since $R \subseteq T$ is $*_1(e)$-LCM-stable, $((aR \cap bR)T)_{*_1} = aT \cap bT$, and hence $((aR \cap bR)T)_{*_2} = aT \cap bT$ because $*_1 \leq *_2$ and $aT \cap bT$ is a v-ideal. Thus $R \subseteq T$ is $*_2(e)$-LCM-stable. (2) and (3) follow directly from (1) because LCM-stable extensions are $d_T(e)$-LCM-stable and $d_T \leq *_1 \leq v_T$. □

Let X be an indeterminate over T and $T[X]$ be the polynomial ring over T. For any $f \in T[X]$, we denote by $c_T(f)$ the fractional ideal of T generated by the coefficients of f.

Let $*$ be a star-operation of finite type on R, and let M be an R-module with $M \subseteq qf(R)$. Then since each finitely generated R-submodule of M is a fractional ideal of R, we can define M_* as follows: $M_* = \bigcup\{N_* \mid N \subseteq M$ and N is a nonzero finitely generated R-module$\}$. What happens if $M \nsubseteq qf(R)$? In general, there is no way to define M_*, but we can define M_* if $* = *_w$ by setting $M_* = \{\frac{a}{b} \mid a, b \in M, b \neq 0$ and $\frac{a}{b}J \subseteq M$ for some $J \in \mathbf{f}(R)$ with $J_* = R\}$.

Lemma 1.3 (*cf.* [5, Lemma 2.3])**.** *Let $R \subseteq T$ be an extension of integral domains. If $*$ is a star-operation on R and $N_* = \{f \in R[X] \mid c_R(f)_* = R\}$, then $A_{*_w} = A[X]_{N_*} \cap qf(T)$, and hence $A_{*_w}[X]_{N_*} = A[X]_{N_*}$ and $(A_{*_w})_{*_w} = A_{*_w}$ for all nonzero fractional ideals A of T.*

Proof. If $u \in A_{*_w}$, then there is a nonzero finitely generated ideal J of R such that $J_* = R$ and $uJ \subseteq A$. So if we choose a polynomial $f \in R[X]$ with $c_R(f) = J$, then $f \in N_*$, and hence $u = \frac{uf}{f} \in A[X]_{N_*} \cap qf(T)$. Thus $A_{*_w} \subseteq A[X]_{N_*} \cap qf(T)$. For the reverse containment, let $a = \frac{g}{h} \in A[X]_{N_*} \cap qf(T)$, where $g \in A[X]$ and $h \in N_*$. Then $ah = g$ and $c_R(h)_* = R$, and since $ac_R(h) = c_R(ah) \subseteq c_T(ah) = c_T(g) \subseteq A$, we have $a \in A_{*_w}$. Thus $A[X]_{N_*} \cap qf(T) \subseteq A_{*_w}$. □

We next give another generalization of LCM-stable extensions.

Definition 1.4. Let $R \subseteq T$ be an extension of integral domains, and let $*_R$ be a star-operation on R such that $(*_R)_w = *_R$. We say that $R \subseteq T$ is *$*_R$-LCM-stable* if $((aR \cap bR)T)_{*_R} = (aT \cap bT)_{*_R}$ for all $0 \neq a, b \in R$.

Note that if d_R is the d-operation on R, then $(d_R)_w = d_R$ and $A_{d_R} = A$ for all nonzero fractional ideals A of T. Hence LCM-stable extensions are also just the d_R-LCM-stable extensions. Thus $R \subseteq T$ is a d_R-LCM-stable extension if and only if $R \subseteq T$ is an LCM-stable extension, if and only if $R \subseteq T$ is a $d_T(e)$-LCM-stable extension.

Lemma 1.5. *Let $*_1 \leq *_2$ be star-operations on R such that $(*_i)_w = *_i$ for $i = 1, 2$.*

1. *If $R \subseteq T$ is $*_1$-LCM-stable, then $R \subseteq T$ is $*_2$-LCM-stable.*
2. *An LCM-stable extension is a $*_1$-LCM-stable extension.*
3. *Every $*_1$-LCM-stable extension is a w_R-LCM-stable extension.*

Proof. (1) Let A be a nonzero fractional ideal of T. Let $N_{*_i} = \{f \in R[X] \mid c_R(f)_{*_i} = R\}$ for $i = 1, 2$. Since $*_1 \leq *_2$, then $N_{*_1} \subseteq N_{*_2}$. Hence by Lemma 1.3, $A_{*_1} = A[X]_{N_{*_1}} \cap qf(T) \subseteq A[X]_{N_{*_2}} \cap qf(T) = A_{*_2}$; so $(A_{*_1})_{*_2} = A_{*_2}$. Thus $R \subseteq T$ is $*_2$-LCM-stable.

(2) This follows from (1) because an LCM-stable extension is just the d_R-LCM-stable extension and $d_R \leq *_1$.

(3) This also follows from (1) because $*_1 \leq w_R$. □

It is well-known that the w-operation can be defined on any integral domain. So we use the terms "w- and $w(e)$-LCM-stable" instead of "w_R- and $w_T(e)$-LCM-stable". Also, the w-operation has many properties similar to those of the d-operation. For example, if I is a nonzero fractional ideal of R, then $(I_{w_R})R_P = IR_P$ for all maximal t-ideals P of R. So in this paper (Sections 2 and 3), we are mainly interested in $w(e)$- and w-LCM-stable extensions.

2. $w(e)$-LCM stable extensions

Let $R \subseteq T$ be an extension of integral domains, and let v_T and w_T be the v- and w-operations on T, respectively (when it is clear, we will use the notations v and w instead of v_T and w_T). Let X be an indeterminate over T and let $R[X]$ and $T[X]$ be polynomial rings over R and T, respectively. In this section, we study some properties of $w(e)$-LCM-stable extensions.

Recall that $R \subseteq T$ is *$w(e)$-LCM-stable* if $((aR \cap bR)T)_{w_T} = aT \cap bT$ for all $0 \neq a, b \in R$. In Lemma 1.2, we noted that LCM-stable extensions are $w(e)$-LCM-stable. We begin this section with an example of $w(e)$-LCM-stable extensions that is not LCM-stable.

Example 2.1. We first recall that an integral domain R is a *Krull domain* if (i) $R = \bigcap_{P \in X^{(1)}(R)} R_P$, where $X^{(1)}(R)$ is the set of height-one prime ideals of R, (ii) R_P is a rank-one DVR for all $P \in X^{(1)}(R)$, and (iii) each nonzero nonunit of R is contained in only a finite number of height-one prime ideals.

Let R be a Krull domain, $Q \in X^{(1)}(R)$ and $T = \bigcap_{P \in X^{(1)}(R) \setminus \{Q\}} R_P$. Then T is a Krull domain and $X^{(1)}(T) = \{PR_P \cap T \mid P \in X^{(1)}(R) \text{ and } P \neq Q\}$. Note that if $0 \neq a, b \in R$, then

$$\begin{aligned} ((aR \cap bR)T)_{w_T} &= \bigcap_{P \in X^{(1)}(R) \setminus \{Q\}} ((aR \cap bR)T)R_P \\ &= \bigcap_{P \in X^{(1)}(R) \setminus \{Q\}} (aR_P \cap bR_P) \\ &= \left(\bigcap_{P \in X^{(1)}(R) \setminus \{Q\}} aR_P \right) \cap \left(\bigcap_{P \in X^{(1)}(R) \setminus \{Q\}} bR_P \right) \\ &= aT \cap bT. \end{aligned}$$

Thus $R \subseteq T$ is $w(e)$-LCM-stable.

Next, since R is a Krull domain, we can choose $0 \neq x, y \in R$ such that $Q = (1, \frac{y}{x})^{-1} = \frac{1}{y}(xR \cap yR)$. So if $R \subseteq T$ is LCM-stable, then $QT = \frac{1}{y}(xR \cap yR)T = \frac{1}{y}(xT \cap yT)$. Note that $(QT)_{v_T} = T$; hence $QT = \frac{1}{y}(xT \cap yT) = (QT)_{v_T} = T$. Thus if $QT \neq T$, then $R \subseteq T$ is not LCM-stable.

Our next result is a characterization of $w(e)$-LCM-stable extensions, which relates $w(e)$-LCM-stable extensions to LCM-stable extensions so that we can predict the properties of $w(e)$-LCM-stable extensions.

Theorem 2.2. *$R \subseteq T$ is $w(e)$-LCM-stable if and only if $R \subseteq T_M$ is LCM-stable for each maximal t-ideal M of T.*

Proof. ($\Rightarrow$) For $0 \neq a, b \in R$, we have $((aR \cap bR)T)_{w_T} = aT \cap bT$. So if M is a maximal t-ideal of T, then

$$\begin{aligned}(aR \cap bR)T_M &= (((aR \cap bR)T)_{w_T})_M \\ &= (aT \cap bT)_M \\ &= aT_M \cap bT_M.\end{aligned}$$

Thus $R \subseteq T_M$ is LCM-stable.

($\Leftarrow$) Let $0 \neq a, b \in R$. Then $(aR \cap bR)T_M = aT_M \cap bT_M$ for all maximal t-ideals M of T, and hence we have

$$\begin{aligned}((aR \cap bR)T)_{w_T} &= \bigcap_{M \in t\text{-Max}(T)} (aR \cap bR)T_M \\ &= \bigcap_{M \in t\text{-Max}(T)} (aT_M \cap bT_M) \\ &= \left(\bigcap_{M \in t\text{-Max}(T)} aT_M\right) \cap \left(\bigcap_{M \in t\text{-Max}(T)} bT_M\right) \\ &= aT \cap bT.\end{aligned}$$

Thus $R \subseteq T$ is $w(e)$-LCM-stable. □

Corollary 2.3. *If $R \subseteq T$ is $w(e)$-LCM-stable, then $R \subseteq T_S$ is $w(e)$-LCM-stable for each multiplicative subset S of T.*

Proof. Let Q be a maximal t-ideal of T_S. Then there is a prime t-ideal P of T such that $Q = PT_S$ (*cf.* [15, Lemma 3.17]). So if M is a maximal t-ideal of T containing P, then $(T_S)_Q = (T_S)_{PT_S} = T_P = (T_M)_{P_M}$, and since $R \subseteq T_M$ is LCM-stable by Theorem 2.2, $R \subseteq T_Q$ is also LCM-stable. Thus, again by Theorem 2.2, $R \subseteq T_S$ is $w(e)$-LCM stable. □

Following [8], we say that T is *t-linked over* R if for I a nonzero finitely generated ideal of R, $I^{-1} = R$ implies $(IT)^{-1} = T$. Equivalently, if M is a maximal t-ideal of T with $M \cap R \neq (0)$, then $(M \cap R)_t \subsetneq R$ [3, Proposition 2.1].

Let T be an overring of R. As in [16], we say that T is *t-flat* over R if $T_Q = R_{Q \cap R}$ for all maximal t-ideal Q of T. Clearly, if T is flat over R, then T is t-flat over R. It is known that if $R \subseteq T$ is flat, then $R \subseteq T$ is LCM-stable [20, Proposition 1.1]. Our next result is the t-flat analog of this result.

Corollary 2.4. *Let T be an overring of R.*

(1) *If T is t-flat over R, then $R \subseteq T$ is $w(e)$-LCM-stable.*
(2) *If T is t-linked over R, then $R \subseteq T$ is $w(e)$-LCM-stable if and only if T is t-flat over R.*

Proof. (1) Let Q be a maximal t-ideal of T. Then $T_Q = R_{Q\cap R}$ and hence $R \subseteq T_Q$ is LCM-stable. Thus $R \subseteq T$ is $w(e)$-LCM-stable by Theorem 2.2.

(2) Suppose that $R \subseteq T$ is $w(e)$-LCM-stable. Then by [16, Proposition 2.5], it suffices to show that $((y :_R x)T)_{w_T} = T$ for each $0 \neq x, y \in R$ with $\frac{x}{y} \in T$. Let $\frac{x}{y} \in T$, where $x, y \in R$ and $y \neq 0$. Then since $R \subseteq T$ is $w(e)$-LCM-stable, we have $((y :_R x)T)_{w_T} = y :_T x = T$. The converse always holds by (1). □

Recall that R is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal I of R is t-invertible, *i.e.*, $(II^{-1})_t = R$. We know that R is a Prüfer domain if and only if $R \subseteq T$ is LCM-stable for any integral domain T containing R, if and only if $R \subseteq R[u]$ is LCM-stable for each $u \in qf(R)$ [20, Corollary 1.8]. Now we give the PvMD analog of this fact.

Corollary 2.5. *R is a PvMD if and only if $R \subseteq T$ is $w(e)$-LCM-stable for any t-linked overring T of R.*

Proof. It is well-known that R is a PvMD if and only if every t-linked overring of R is t-flat over R [16, Proposition 2.10]. Thus the result is an immediate consequence of Corollary 2.4. □

By Lemma 1.2, LCM-stable extensions are $w(e)$-LCM-stable extensions. We next give some integral domains in which $w(e)$-LCM-stable extensions are LCM-stable.

Example 2.6. $R \subseteq T$ is LCM-stable if (and only if) $R \subseteq T$ is $w(e)$-LCM-stable in any of the cases below.

(1) Each maximal ideal of T is a t-ideal.
(2) R is a GCD-domain.
(3) R is a UFD.
(4) T is a Prüfer domain.
(5) T is an integral domain of (Krull) dimension one.

Proof. (1) Recall that $R \subseteq T$ is LCM-stable if and only if $R_P \subseteq T_Q$ is LCM-stable for each maximal ideal Q of T with $Q \cap R = P$ [20, Proposition 1.6], which implies that $R_{S_1} \subseteq T_{S_2}$ is LCM-stable for any multiplicative subsets S_1 and S_2 of R and T, respectively, with $S_1 \subseteq S_2$ [20, Corollary 1.5]. Thus the result follows from Theorem 2.2.

(2) If $0 \neq a, b \in R$, then $aR \cap bR = cR$ for some $c \in R$ because R is a GCD-domain. Since $R \subseteq T$ is $w(e)$-LCM-stable, we have $aT \cap bT = ((aR \cap bR)T)_{w_T}$. Thus we obtain

$$aT \cap bT = ((aR \cap bR)T)_{w_T} = ((cR)T)_{w_T} = (cR)T = (aR \cap bR)T,$$

which indicates that $R \subseteq T$ is LCM-stable.

(3) This follows from (2) because UFDs are GCD-domains.

(4) and (5) These follow from (1) because each maximal ideal of a Prüfer domain and an integral domain of Krull dimension one is a t-ideal. □

In [20], Uda introduced the notions of $\mathcal{R}_2$-stableness and G_2-stableness. The G_2-stableness is just the t-linkedness [20, page 363]. As in [20], we say that $R \subseteq T$ is *$\mathcal{R}_2$-stable* if $aR \cap bR = cR$ with $a, b, c \in R$ implies $aT \cap bT = cT$. It is known that T is t-linked over R if and only if $T[X]$ is t-linked over $R[X]$, if and only if $R[X] \subseteq T[X]$ is $\mathcal{R}_2$-stable [20, Theorem 3.5]. Also, it was shown that if R is a GCD-domain, then $R \subseteq T$ is LCM-stable if and only if T is t-linked over R, if and only if $R \subseteq T$ is $\mathcal{R}_2$-stable, if and only if $R[X] \subseteq T[X]$ is LCM-stable [20, Corollary 3.7].

Proposition 2.7. *Let $R \subseteq T$ be an extension of integral domains.*

(1) *If $R \subseteq T$ is $w(e)$-LCM-stable, then $R \subseteq T$ is $\mathcal{R}_2$-stable.*
(2) *$R \subseteq T$ is $\mathcal{R}_2$-stable if and only if $(a, b)^{-1} = R$ for $0 \neq a, b \in R$ implies $((a, b)T)^{-1} = T$.*
(3) *If T is t-linked over R, then $R \subseteq T$ is $\mathcal{R}_2$-stable.*

Proof. (1) This implication is clear.

(2) ($\Rightarrow$) Let $0 \neq a, b \in R$ be such that $(a, b)^{-1} = R$. Then $aR \cap bR = abR$ because $(a, b)^{-1} = \frac{1}{ab}(aR \cap bR)$. Thus $((a, b)T)^{-1} = \frac{1}{ab}(aT \cap bT) = \frac{1}{ab}abT = T$.

($\Leftarrow$) Assume $aR \cap bR = cR$ with $0 \neq a, b, c \in R$. Then $(a, b)^{-1} = \frac{1}{ab}(aR \cap bR) = \frac{c}{ab}R$, and hence $(\frac{c}{b}, \frac{c}{b})^{-1} = R$. Therefore, we have

$$T = ((\frac{c}{b}, \frac{c}{a})T)^{-1} = \frac{ab}{c^2}(\frac{c}{b}T \cap \frac{c}{a}T) = \frac{a}{c}T \cap \frac{b}{c}T.$$

Thus $aT \cap bT = cT$.

(3) This is an immediate consequence of (2) above. □

We say that R is of *finite t-character* if each nonzero nonunit of R is contained in only a finite number of maximal t-ideals of R. For example, Krull domains and Noetherian domains are of finite t-character. If R is of finite t-character, then the converse of Proposition 2.7(3) holds.

Corollary 2.8. *Suppose that R is of finite t-character.*

(1) *$R \subseteq T$ is $\mathcal{R}_2$-stable if and only if T is t-linked over R.*
(2) *If $R \subseteq T$ is $w(e)$-LCM-stable, then T is t-linked over R.*

Proof. (1) By Proposition 2.7(2) and (3), it suffices to show that if I is a nonzero finitely generated ideal of R with $I^{-1} = R$, then there are some $a, b \in I$ such that $(a, b)^{-1} = R$. Choose a nonzero $a \in I$. Since R is of finite t-character, there are only finitely many maximal t-ideals of R containing a, say, $P_1, \ldots, P_n$. Choose another $b \in I \setminus \bigcup_{i=1}^{n} P_i$. Then $(a, b)^{-1} = R$.

(2) This follows directly from (1) above and Proposition 2.7(1). □

We know that if R is a GCD-domain, then $R \subseteq T$ is LCM-stable if and only if $R[X] \subseteq T[X]$ is LCM-stable. Thus by Example 2.6, if R is a GCD-domain, then $R \subseteq T$ is $w(e)$-LCM-stable if and only if $R[X] \subseteq T[X]$ is $w(e)$-LCM-stable. It was shown in [21, Theorem 11] that if R is a Krull domain, then $R \subseteq T$ is LCM-stable if and only if $R[X] \subseteq T[X]$ is LCM-stable. We next give the $w(e)$-LCM-stable extension analog of this result in Theorem 2.11. Before proving the theorem, we need a couple of lemmas.

Lemma 2.9 (*cf.* [21, Lemma 9])**.** *Let R be a Krull domain. Assume that $R \subseteq T$ is $w(e)$-LCM-stable. If I is a v-ideal of R, then $(IT)_w$ is a v-ideal of T.*

Proof. Since I is a v-ideal of R, there are nonzero $a, b \in qf(R)$ such that $I = aR \cap bR$ [11, Corollary 44.6]. Since $R \subseteq T$ is $w(e)$-LCM-stable, we have

$$(IT)_w = ((aR \cap bR)T)_w = aT \cap bT.$$

Thus $(IT)_w$ is a v-ideal. □

Lemma 2.10. *If $R \subseteq T$ is $w(e)$-LCM-stable, then $R \subseteq T[X]$ is $w(e)$-LCM-stable.*

Proof. Let $0 \neq a, b \in R$. Then $((aR \cap bR)T)_{w_T} = aT \cap bT$, and hence

$$\begin{aligned} ((aR \cap bR)T[X])_{w_{T[X]}} &= ((aR \cap bR)T)_{w_T}T[X] \\ &= (aT \cap bT)T[X] \\ &= aT[X] \cap bT[X], \end{aligned}$$

where the first equality follows from [12, Proposition 4.3]. Thus $R \subseteq T[X]$ is $w(e)$-LCM-stable. □

Let $N_v(R) = \{f \in R[X] \mid c_R(f)_v = R\}$. Then $N_v(R)$ is a multiplicative subset of $R[X]$, and the quotient ring $R[X]_{N_v(R)}$ is called the *t-Nagata ring* of R (To the best of our knowledge, this notion was first considered implicitly by Gilmer in [9] and then systemically by Kang in [14, 15]).

Theorem 2.11. *The following statements are equivalent for a Krull domain R.*

(1) *$R \subseteq T$ is $w(e)$-LCM-stable.*
(2) *$((a :_R b)T)_{w_T} = a :_T b$ for all $0 \neq a, b \in R$.*
(3) *$R[X] \subseteq T[X]$ is $w(e)$-LCM-stable.*
(4) *$R[X] \subseteq T[X]_{N_v(T)}$ is LCM-stable.*
(5) *$N_v(R) \subseteq N_v(T)$ and $R[X]_{N_v(R)} \subseteq T[X]_{N_v(T)}$ are LCM-stable.*

Proof. (1) ⇔ (2) This follows easily from the fact that $(a) \cap (b) = ((a) : (b))(b)$.

(1) ⇒ (3) Assume that $R \subseteq T$ is $w(e)$-LCM-stable. Then $R \subseteq T$ is $\mathcal{R}_2$-stable, and since R is a Krull domain, T is t-linked over R by Corollary 2.8, and hence $T[X]$ is t-linked over $R[X]$ [20, Theorem 3.5]. Thus for any $0 \neq f, g \in R[X]$, we have $f :_{T[X]} g = ((f :_{R[X]} g)T[X])_v$ [21, Proposition 8]. Note

that $f :_{R[X]} g = (R :_{qf(R)} I)fR[X]$, where $I = c_R(f) + c_R(g)$, [20, Lemma 3.9] and $((R :_{qf(R)} I)T[X])_w = ((R :_{qf(R)} I)T[X])_v$ by Lemmas 2.9 and 2.10. Thus

$$((f :_{R[X]} g)T[X])_w = ((f :_{R[X]} g)T[X])_v = f :_{T[X]} g,$$

which implies that $R[X] \subseteq T[X]$ is $w(e)$-LCM-stable.

(3) ⇒ (4) This follows from Corollary 2.3 and Example 2.6 because each maximal ideal of $T[X]_{N_v(T)}$ is extended from a maximal t-ideal of $T[X]$ [15, Propositions 2.1 and 2.2].

(4) ⇒ (1) Let $0 \neq a, b \in R$. Since $R[X] \subseteq T[X]_{N_v(T)}$ is LCM-stable, we have

$$\begin{aligned}(aR[X] \cap bR[X])T[X]_{N_v(T)} &= aT[X]_{N_v(T)} \cap bT[X]_{N_v(T)} \\ &= (aT \cap bT)T[X]_{N_v(T)},\end{aligned}$$

and thus by Lemma 1.3, we obtain

$$\begin{aligned}((aR \cap bR)T)_{w_T} &= (aR[X] \cap bR[X])T[X]_{N_v(T)} \cap qf(T) \\ &= (aT \cap bT)T[X]_{N_v(T)} \cap qf(T) \\ &= (aT \cap bT)_{w_T} \\ &= aT \cap bT.\end{aligned}$$

(4) ⇒ (5) Note that R is of finite t-character. Also, $R \subseteq T$ is $w(e)$-LCM-stable by (4) ⇒ (1) above. So T is t-linked over R by Corollary 2.8 and hence $N_v(R) \subseteq N_v(T)$. Thus $R[X]_{N_v(R)} \subseteq T[X]_{N_v(T)}$ is LCM-stable [20, Corollary 1.5].

(5) ⇒ (1) This can be proved in the same way as the proof of (4) ⇒ (1). □

Corollary 2.12. *Let T be an overring of R. If R is a Krull domain, then $R \subseteq T$ is $w(e)$-LCM-stable if and only if T is t-linked over R.*

Proof. Assume that T is t-linked over R. Then $T = \bigcap_{P\in\Lambda} R_P$, where Λ is a set of height-one prime ideals of R [15, Theorem 3.8]. Hence for all $0 \neq a, b \in R$, we have

$$\begin{aligned}((aR \cap bR)T)_{w_T} &= \bigcap_{P\in\Lambda} ((aR \cap bR)T)R_P \\ &= \bigcap_{P\in\Lambda} (aR_P \cap bR_P) \\ &= \left(\bigcap_{P\in\Lambda} aR_P\right) \cap \left(\bigcap_{P\in\Lambda} bR_P\right) \\ &= \left(\bigcap_{P\in\Lambda} (aT)R_P\right) \cap \left(\bigcap_{P\in\Lambda} (bT)R_P\right) \\ &= (aT)_{w_T} \cap (bT)_{w_T} \\ &= aT \cap bT.\end{aligned}$$

Thus $R \subseteq T$ is $w(e)$-LCM-stable. The converse follows from Theorem 2.11. $\square$

3. *w*-LCM stableness

Let $R \subseteq T$ be an extension of integral domains, X be an indeterminate over T, and $T[X]$ be the polynomial ring over T. Let $R[X]$ be the polynomial ring and $N_v(R) = \{f \in R[X] \mid c_R(f)_v = R\}$.

Recall that $R \subseteq T$ is *w-LCM-stable* if $((aR \cap bR)T)_{w_R} = (aT \cap bT)_{w_R}$ for all $0 \neq a, b \in R$. By Lemma 1.5, LCM-stable extensions are w-LCM-stable, but w-LCM-stable extensions need not be LCM-stable (see Example 3.9).

Lemma 3.1. *If M is a torsionfree R-module, then* $M_{w_R} = \bigcap_{P \in t\text{-Max}(R)} M_P$. *Hence $(M_{w_R})_P = M_P$ for all nonzero prime ideals P of R with $P_t \subsetneq R$.*

Proof. This appears in [22, Proposition 3.4] and [23, Theorem 3.9]. $\square$

Our first result of this section is a characterization of w-LCM-stable extensions via LCM-stable extensions.

Theorem 3.2. *The following statements are equivalent.*

(1) *$R \subseteq T$ is w-LCM-stable.*
(2) *If $D = T[X]_{N_v(R)} \cap qf(T)$, then $R \subseteq D$ is w-LCM-stable.*
(3) *$R_P \subseteq T_P$ is LCM-stable for all nonzero prime ideals P of R with $P_t \subsetneq R$.*
(4) *$R_P \subseteq T_P$ is LCM-stable for all maximal t-ideals P of R.*

Proof. (1) $\Rightarrow$ (2) Clearly, $R \subseteq D$. Let $0 \neq a, b \in R$. Then $((aR \cap bR)T)_{w_R} = (aT \cap bT)_{w_R}$, and hence by Lemma 1.3, we have

$$\begin{aligned}
((aR \cap bR)D)_{w_R} &= (aR \cap bR)D[X]_{N_v(R)} \cap qf(T) \\
&= (aR \cap bR)T[X]_{N_v(R)} \cap qf(T) \\
&= ((aR \cap bR)T)_{w_R}[X]_{N_v(R)} \cap qf(T) \\
&= (aT \cap bT)_{w_R}[X]_{N_v(R)} \cap qf(T) \\
&= (aT \cap bT)[X]_{N_v(R)} \cap qf(T) \\
&= (aT[X]_{N_v(R)} \cap bT[X]_{N_v(R)}) \cap qf(T) \\
&= (aD[X]_{N_v(R)} \cap bD[X]_{N_v(R)}) \cap qf(T) \\
&= (aD \cap bD)[X]_{N_v(R)} \cap qf(T) \\
&= (aD \cap bD)_{w_R}.
\end{aligned}$$

(2) $\Rightarrow$ (3) Let P be a nonzero prime ideals P of R with $P_t \subsetneq R$. For $0 \neq x, y \in R_P$, there is an $s \in R \setminus P$ such that $sx, sy \in R$. So $((sxR \cap syR)D)_{w_R} = (sxD \cap syD)_{w_R}$ by assumption. Thus by (2) and Lemma 3.1, we have

$$\begin{aligned}
(xR_P \cap yR_P)T_P &= (xR_P \cap yR_P)D_P \\
&= (sxR \cap syR)D_P \\
&= (((sxR \cap syR)D)_{w_R})D_P
\end{aligned}$$

$$\begin{aligned} &= ((xD \cap yD)_{w_R})D_P \\ &= (xD \cap yD)D_P \\ &= xD_P \cap yD_P \\ &= xT_P \cap yT_P. \end{aligned}$$

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let $0 \neq a, b \in R$. For each $P \in t\text{-Max}(R)$, since $R_P \subseteq T_P$ is LCM-stable, $(aR \cap bR)T_P = (aR_P \cap bR_P)T_P = aT_P \cap bT_P = (aT \cap bT)_P$. Hence

$$\begin{aligned} ((aR \cap bR)T)_{w_R} &= \bigcap_{P \in t\text{-Max}(R)} (aR \cap bR)T_P \\ &= \bigcap_{P \in t\text{-Max}(R)} (aT \cap bT)_P \\ &= (aT \cap bT)_{w_R} \end{aligned}$$

by Lemma 3.1. □

Remark 3.3. Let $R \subseteq T$ be an extension of integral domains.

(1) It is easy to show that $T_{w_R} = T[X]_{N_v(R)} \cap qf(T)$ is t-linked over R and that if D is an overring of T such that D is t-linked over R, then $T_{w_R} \subseteq D$ (*cf.* [5, Remark 3.3]).

(2) By Theorem 2.2, when we study w-LCM-stable extensions, it suffices to consider the case when T is t-linked over R.

Corollary 3.4. *If $R \subseteq T$ is w-LCM-stable, then $R_{S_1} \subseteq T_{S_2}$ is w-LCM-stable for any multiplicative subsets S_1 and S_2 of R and T, respectively, with $S_1 \subseteq S_2$.*

Proof. Let Q be a maximal t-ideal of R_{S_1}. Then $Q = PR_{S_1}$ for some prime t-ideal P of R (cf. [15, Lemma 3.17]) and hence $(R_{S_1})_Q = R_P$ and $(T_{S_2})_Q = (T_{S_2})_{R \setminus P} = (T_{R \setminus P})_{S_2}$. By Theorem 3.2, $R_P \subseteq T_{R \setminus P}$ is LCM-stable, and thus $(R_{S_1})_Q = R_P \subseteq (T_{R \setminus P})_{S_2} = (T_{S_2})_Q$ is LCM-stable [20, Corollary 1.5]. Thus again by Theorem 3.2, $R_{S_1} \subseteq R_{S_2}$ is w-LCM-stable. □

We note in Example 2.6(1) that if each maximal ideal of T is a t-ideal, then the extension $R \subseteq T$ being $w(e)$-LCM-stable implies that $R \subseteq T$ is LCM-stable. The next result is the w-LCM-stable extension analog.

Corollary 3.5. *If each maximal ideal of R is a t-ideal, then $R \subseteq T$ is w-LCM-stable if and only if $R \subseteq T$ is LCM-stable.*

Proof. Assume that $R \subseteq T$ is w-LCM-stable. Let M be a maximal ideal of T. If $M \cap R = (0)$, then $R_{M \cap R}$ is a field, and hence $R_{M \cap R} \subseteq T_M$ is LCM-stable. Next, if $M \cap R \neq (0)$, then $(M \cap R)_t \subsetneq R$ by assumption and hence $R_{M \cap R} \subseteq T_M$ is LCM-stable by Theorem 3.2 and [20, Corollary 1.5]. Thus $R \subseteq T$ is LCM-stable [20, Proposition 1.6]. The converse follows from Lemma 1.5. □

Let M be an R-module. We say that M is a *w-locally flat R-module* if M_P is a flat R_P-module for all maximal t-ideals P of R. Although the notions of w-locally flat and t-flat are generalizations of flatness, they are different as shown in [4]. We next give the w-locally flat analog of Corollary 2.4(1).

Corollary 3.6. *If $R \subseteq T$ is w-locally flat, then $R \subseteq T$ is w-LCM-stable.*

Proof. This follows from Theorem 3.2 and [20, Proposition 1.1]. □

It is clear that $(a) \cap (b) = ((a) :_R (b))(b)$ for all $0 \neq a, b \in R$. Thus $R \subseteq T$ is w-LCM-stable if and only if $((a :_R b)T)_{w_R} = (a :_T b)_{w_R}$ for all $0 \neq a, b \in R$. In particular, if T is t-linked over R, then $R \subseteq T$ is w-LCM-stable if and only if $((a :_R b)T)_{w_R} = a :_T b$ for all $0 \neq a, b \in R$ (see Proposition 3.7(2)).

Proposition 3.7. *Assume that T is t-linked over R.*

(1) *$(aT \cap bT)_{w_R} = aT \cap bT$ for all $0 \neq a, b \in R$.*
(2) *$R \subseteq T$ is w-LCM-stable if and only if $((aR \cap bR)T)_{w_R} = aT \cap bT$ for all $0 \neq a, b \in R$.*
(3) *If $R \subseteq T$ is w-LCM-stable, then $R \subseteq T$ is $w(e)$-LCM-stable.*

Proof. (1) Note that $N_v(R) \subseteq N_v(T)$ because T is t-linked over R. Hence if I is a nonzero fractional ideal of T, then

$$I_{w_R} = IT[X]_{N_v(R)} \cap qf(T) \subseteq IT[X]_{N_v(T)} \cap qf(T) = I_{w_T}$$

by Lemma 1.3. So $I_{w_R} \subseteq (I_{w_R})_{w_T} = I_{w_T}$. Hence we have

$$(aT \cap bT)_{w_R} \subseteq (aT \cap bT)_{w_T} = aT \cap bT \subseteq (aT \cap bT)_{w_R}.$$

Thus $(aT \cap bT)_{w_R} = aT \cap bT$.

(2) If $R \subseteq T$ is w-LCM-stable, then $((aR \cap bR)T)_{w_R} = (aT \cap bT)_{w_R}$. Thus $((aR \cap bR)T)_{w_R} = aT \cap bT$ by (1). Conversely, if $((aR \cap bR)T)_{w_R} = aT \cap bT$ for all $0 \neq a, b \in R$, then we have

$$((aR \cap bR)T)_{w_R} = (((aR \cap bR)T)_{w_R})_{w_R} = (aT \cap bT)_{w_R}$$

by Lemma 1.3. Thus $R \subseteq T$ is w-LCM-stable.

(3) This follows from (2) and the fact that $(I_{w_R})_{w_T} = I_{w_T}$ for all nonzero fractional ideals I of T (see the proof of (1) above). □

Corollary 3.8. *The following statements are equivalent.*

(1) *R is a PvMD.*
(2) *$R \subseteq T$ is $w(e)$-LCM-stable for any t-linked overring T of R.*
(3) *$R \subseteq T$ is w-LCM-stable for any t-linked overring T of R.*
(4) *$R \subseteq T$ is w-LCM-stable for any overring T of R.*

Proof. (1) ⇔ (2) Corollary 2.5.

(3) ⇔ (4) Theorem 3.2.

(3) ⇒ (2) Proposition 3.7.

(1) ⇒ (4) Let P be a maximal t-ideal of R. Then R_P is a valuation domain and $T_{R\setminus P}$ is an overring of R_P. Hence $T_{R\setminus P}$ is a quotient ring of R_P [11,

Theorem 17.6], and thus $T_{R\setminus P}$ is flat over R_P. Thus $R \subseteq T$ is w-LCM-stable by Theorem 3.2. □

We next give an example of w-LCM-stable extensions that are neither $w(e)$-LCM-stable nor LCM-stable.

Example 3.9. Let R be a GCD-domain that is not a Prüfer domain (for example, let R be the polynomial ring over $\mathbb{Z}$). Then there exists an $\alpha \in qf(R)$ such that $R \subseteq R[\alpha]$ is not LCM-stable [20, Corollary 1.8]. Moreover, since R is a GCD-domain, $R \subseteq R[\alpha]$ is not $w(e)$-LCM-stable by Example 2.6. But note that $R \subseteq R[\alpha]$ is w-LCM-stable by Corollary 3.8 because GCD-domains are PvMDs.

It is known that if R is a GCD-domain, then $R \subseteq T$ is LCM-stable if and only if $R[X] \subseteq T[X]$ is LCM-stable. Clearly, if R is a field, then R is a GCD-domain and $R \subseteq T$ is LCM-stable. Thus we have:

Lemma 3.10. *If R is a field, then $R[X] \subseteq T[X]$ is LCM-stable.*

We next give a w-LCM-stable extension analog of Theorem 2.11 and [21, Theorem 11] that if R is a Krull domain, then $R \subseteq T$ is LCM-stable (resp., $w(e)$-LCM-stable) if and only if $R[X] \subseteq T[X]$ is also LCM-stable (resp., $w(e)$-LCM-stable).

Theorem 3.11. *The following statements are equivalent for a PvMD R.*

(1) $R \subseteq T$ *is w-LCM-stable.*
(2) $R[X] \subseteq T[X]$ *is w-LCM-stable.*
(3) $R[X]_{N_v(R)} \subseteq T[X]_{N_v(R)}$ *is LCM-stable.*

Proof. (1) ⇒ (2) Let Q be a maximal t-ideal of $R[X]$, and let $P = Q \cap R$. By Theorem 3.2, it suffices to show that $R[X]_Q \subseteq T[X]_{R[X]\setminus Q}$ is LCM-stable.

Case 1. $P = (0)$. Then $K = R_{R\setminus\{0\}} \subseteq T_{R\setminus\{0\}}$, where $K = qf(R)$. Since K is a field, $K[X] \subseteq T_{R\setminus\{0\}}[X]$ is LCM-stable by Lemma 3.10. Thus $R[X]_Q = K[X]_{QK[X]} \subseteq T_{R\setminus\{0\}}[X]_{R[X]\setminus Q} = T[X]_{R[X]\setminus Q}$ is LCM-stable [20, Corollary 1.5].

Case 2. $P \neq (0)$. Then $Q = P[X]$, where P is a maximal t-ideal of R [13, Proposition 1.1] and $R[X]_{P[X]} = R_P[X]_{PR_P[X]}$. Note that $R_P \subseteq T_{R\setminus P}$ is LCM-stable by Theorem 3.2 and R_P is a valuation domain; so $R_P[X] \subseteq T_{R\setminus P}[X]$ is LCM-stable [20, Corollary 3.7]. Note also that

$$R[X]_{P[X]} = R_P[X]_{PR_P[X]} \text{ and } T[X]_{R[X]\setminus P[X]} = T_{R\setminus P}[X]_{R_P[X]\setminus PR_P[X]}.$$

Thus $R[X]_{P[X]} \subseteq T[X]_{R[X]\setminus P[X]}$ is LCM-stable.

(2) ⇒ (3) We first note that $R[X]_{N_v(R)} \subseteq T[X]_{N_v(R)}$ is w-LCM-stable by Corollary 3.4. Note also that each maximal ideal of $R[X]_{N_v(R)}$ is a t-ideal [15, Propositions 2.1 and 2.2]. Thus the result follows from Corollary 3.5.

(3) ⇒ (1) Let $0 \neq a, b \in R$ and $N_v = N_v(R)$. Then we have

$$((aR \cap bR)T)_{w_R} = ((aR \cap bR)T)T[X]_{N_v} \cap qf(T)$$

$$\begin{aligned}
&= ((aR \cap bR)R[X]_{N_v})T[X]_{N_v} \cap qf(T) \\
&= (aR[X]_{N_v} \cap bR[X]_{N_v})T[X]_{N_v} \cap qf(T) \\
&= (aT[X]_{N_v} \cap bT[X]_{N_v}) \cap qf(T) \\
&= (aT \cap bT)T[X]_{N_v} \cap qf(T) \\
&= (aT \cap bT)_{w_R},
\end{aligned}$$

where the first and the sixth equalities follow from Lemma 1.3. Thus $R \subseteq T$ is w-LCM-stable. □

It is well-known that R is a Prüfer domain if and only if R is a PvMD whose maximal ideals are t-ideals. So by Corollary 3.5 and Theorem 3.11, we have:

Corollary 3.12. *The following assertions are equivalent for a Prüfer domain R.*

(1) $R \subseteq T$ *is LCM-stable.*
(2) $R[X] \subseteq T[X]$ *is w-LCM-stable.*
(3) $R[X]_N \subseteq T[X]_N$ *is LCM-stable, where* $N = \{f \in R[X] \mid c_R(f) = R\}$.

The proofs (2) ⇒ (3) ⇒ (1) in Theorem 3.11 also show the following result.

Corollary 3.13. *If $R[X] \subseteq T[X]$ is w-LCM-stable, then $R \subseteq T$ is w-LCM-stable.*

We give a new characterization of PvMDs. This is a w-LCM-stable extension analog of the fact that R is a Prüfer domain if and only if $R[X] \subseteq T[X]$ is LCM-stable for each domain T containing R as a subring [6].

Corollary 3.14. *R is a PvMD if and only if $R[X] \subseteq T[X]$ is w-LCM-stable for each overring T of R.*

Proof. (⇒) This follows from Corollary 3.8 and Theorem 3.11.
(⇐) This follows from Corollaries 3.13 and 3.8. □

As mentioned in the Introduction, it was shown by Condo that R is a Dedekind domain if and only if $R[\![X]\!] \subseteq T[\![X]\!]$ is LCM-stable for any domain T containing R as a subring. Thus the following question arises naturally.

Question 3.15. Is it true that R is a Krull domain if and only if $R[\![X]\!] \subseteq T[\![X]\!]$ is w-LCM-stable (or $w(e)$-LCM-stable) for any domain T containing R as a subring such that $R \subseteq T$ is t-linked?

Appendix

In this appendix, we give a diagram in order to help the readers better understand the correlation among some properties including well-known facts related to w-LCM-stableness and $w(e)$-LCM-stableness.

Recall that D is a *DW-domain* (or *t-linkative*) if each nonzero ideal of D is a w-ideal. It was shown that D is a DW-domain if and only if each maximal

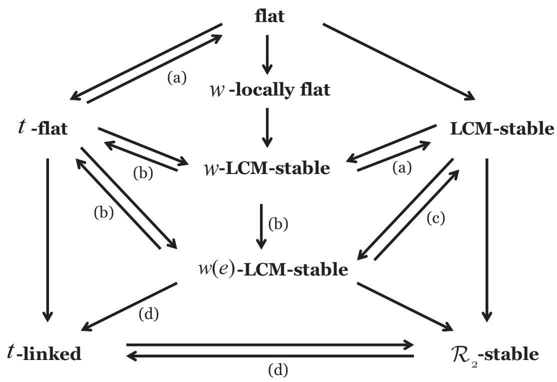

FIGURE 1. Correlations among some properties related to w-LCM-stableness and $w(e)$-LCM-stableness

ideal of D is a w-ideal, if and only if each prime ideal of D is a w-ideal [17, Proposition 2.2].

Remark 3.16. Let $R \subseteq T$ be an extension of integral domains in Figure 1. Then we have the following assertions.

(1) The arrows without indices always hold.
(2) If R is a DW-domain, then the implications with the index (a) hold.
(3) If T is t-linked over R, then the implications with the index (b) hold.
(4) The implication with the index (c) holds in each of the following cases:
  (i) Each maximal ideal of T is a t-ideal.
  (ii) R is a GCD-domain.
  (iii) R is a UFD.
  (iv) T is a Prüfer domain.
  (v) T is an integral domain of (Krull) dimension one.
(5) If R is of finite t-character, then the implication with the index (d) holds.

Acknowledgements. The first author's work was supported by the University of Incheon Research Fund in 2012 (2012-0047), while the second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0011996). Also, the third author was supported by the Brain Korea 21 Project Team to Nurture the Next Generation of First-class Mathematical Scientists by the Korean Government.

References

[1] T. Akiba, *LCM-stableness, Q-stableness and flatness*, Kobe J. Math. **2** (1985), no. 1, 67–70.

[2] D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461–2475.
[3] D. D. Anderson, E. G. Houston, and M. Zafrullah, *t-linked extensions, the t-class group, and Nagata's theorem*, J. Pure Appl. Algebra **86** (1993), no. 2, 109–124.
[4] D. F. Anderson and G. W. Chang, *Overrings as intersections of localizations of an integral domain*, preprint.
[5] G. W. Chang, *$*$-Noetherian domains and the ring $D[X]_{N_*}$*, J. Algebra **297** (2006), no. 1, 216–233.
[6] J. T. Condo, *LCM-stability of power series extensions characterizes Dedekind domains*, Proc. Amer. Math. Soc. **123** (1995), no. 8, 2333–2341.
[7] D. E. Dobbs, *On the criteria of D. D. Anderson for invertible and flat ideals*, Canad. Math. Bull. **29** (1986), no. 1, 25–32.
[8] D. E. Dobbs, E. G. Houston, T. G. Lucas, and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains*, Comm. Algebra **17** (1989), no. 11, 2835–2852.
[9] R. Gilmer, *An embedding theorem for HCF-rings*, Proc. Cambridge Philos. Soc. **68** (1970), 583–587.
[10] ______, *Finite element factorization in group rings*, Ring theory, 47–61, Lecture Notes in Pure and Appl. Math., Vol. 7, Dekker, New York, 1974.
[11] ______, *Multiplicative Ideal Theory*, Queen's Papers in Pure Appl. Math. **90**, Queen's University, Kingston, Ontario, 1992.
[12] J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.
[13] E. G. Houston and M. Zafrullah, *On t-invertibility. II*, Comm. Algebra **17** (1989), no. 8, 1955–1969.
[14] B. G. Kang, *$*$-operations on integral domains*, Ph.D. Dissertation, Univ. Iowa 1987.
[15] ______, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), no. 1, 151–170.
[16] D. J. Kwak and Y. S. Park, *On t-flat overrings*, Chinese J. Math. **23** (1995), no. 1, 17–24.
[17] A. Mimouni, *Integral domains in which each ideal is a w-ideal*, Comm. Algebra **33** (2005), no. 5, 1345–1355.
[18] S. Oda and K. Yoshida, *Remarks on LCM-stableness and reflexiveness*, Math. J. Toyama Univ. **17** (1994), 93–114.
[19] J. Sato and K. Yoshida, *The LCM-stability on polynomial extensions*, Math. Rep. Toyama Univ. **10** (1987), 75–84.
[20] H. Uda, *LCM-stableness in ring extensions*, Hiroshima Math. J. **13** (1983), no. 2, 357–377.
[21] ______, *G_2-stableness and LCM-stableness*, Hiroshima Math. J. **18** (1988), no. 1, 47–52.
[22] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.
[23] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222.
[24] M. Zafrullah, *Putting t-invertibility to use*, Non-Noetherian commutative ring theory, 429–457, Math. Appl., **520**, Kluwer Acad. Publ., Dordrecht, 2000.

Gyu Whan Chang
Department of Mathematics
University of Incheon
Incheon 406-772, Korea
E-mail address: whan@incheon.ac.kr

Hwankoo Kim
Department of Information Security
Hoseo University
Asan 336-795, Korea
E-mail address: hkkim@hoseo.edu

Jung Wook Lim
Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: lovemath@postech.ac.kr, jwlim@sogang.ac.kr