J. Korean Math. Soc. **50** (2013), No. 2, pp. 331–360
http://dx.doi.org/10.4134/JKMS.2013.50.2.331

CONVOLUTION SUMS ARISING FROM DIVISOR FUNCTIONS

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Abstract. Let $\sigma_s(N)$ denote the sum of the sth powers of the positive divisors of a positive integer N and let $\widetilde{\sigma}_s(N) = \sum_{d|N}(-1)^{d-1}d^s$ with d, N, and s positive integers. Hahn [12] proved that

$$16\sum_{k<N}\widetilde{\sigma}_1(k)\widetilde{\sigma}_3(N-k) = -\widetilde{\sigma}_5(N) + 2(N-1)\widetilde{\sigma}_3(N) + \widetilde{\sigma}_1(N).$$

In this paper, we give a generalization of Hahn's result. Furthermore, we find the formula $\sum_{k=1}^{N-1}\widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^nN-2^nk)$ for m $(0 \le m \le n)$.

1. Introduction

For $N, m, d \in \mathbb{N}$ with $r, s \in \mathbb{N}\cup\{0\}$, we define some necessary divisor functions and infinite products for later use, which also appear in many areas of number theory:

$$\sigma_s(N) = \sum_{d|N} d^s, \qquad \sigma_{s,r}(N;m) = \sum_{\substack{d|N \\ d\equiv r \bmod m}} d^s,$$

$$\widetilde{\sigma}_s(N) = \sum_{d|N}(-1)^{d-1}d^s, \qquad S_1 := \sum_{N \text{ odd}} \sigma_{1,1}(N;2)q^N,$$

$$S_2 := \sum_{N\ge 2 \text{ even}} \sigma_{1,1}(N;2)q^N, \qquad (a;q)_\infty := (a)_\infty := \prod_{N\ge 0}(1-aq^N).$$

We also make use of the following convention:

$$\sigma_s(N) = 0 \text{ if } N \notin \mathbb{Z} \text{ or } N<0, \qquad \sigma(N) := \sigma_1(N) = \sum_{d|N} d.$$

The history of the convolution sums involving the divisor functions $\sigma_s(N)$ goes back to Glaisher [9, 10, 11]. Many recent works on convolution formulas for divisor functions can be found in B. C. Berndt [3]; H. Hahn [12]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [13]; G. Melfi [18]; B. Cho, D. Kim, and J.-K. Koo [4, 5]; and A. Alaca, S. Alaca, and K. S. Williams [1, 2].

Received March 26, 2012; Revised August 20, 2012.
2010 *Mathematics Subject Classification.* 11A67.
Key words and phrases. Weierstrass $\wp(x)$ functions, convolution sums.

In 1997, Melfi [18] considered among others the convolution sums

$$\sum_{k<N/2} \sigma_1(k)\sigma_3(N-2k) \quad \text{and} \quad \sum_{k<N/2} \sigma_3(k)\sigma_1(N-2k),$$

for the case when N is odd, and proved that

$$\sum_{k<N/2} \sigma_1(k)\sigma_3(N-2k) = \frac{1}{48}\sigma_5(N) + \frac{(2-3N)}{48}\sigma_3(N), \quad N \equiv 1 \pmod 2,$$
$$\sum_{k<N/2} \sigma_3(k)\sigma_1(N-2k) = \frac{1}{240}\sigma_5(N) - \frac{1}{240}\sigma_1(N), \quad N \equiv 1 \pmod 2.$$

In 2002, Huard, Ou, Spearman, and Williams [13] extended Melfi's result to

$$\begin{aligned} \sum_{k<N/2} \sigma_1(k)\sigma_3(N-2k) &= \frac{1}{48}\sigma_5(N) + \frac{1}{15}\sigma_5(\frac{N}{2}) + \frac{(2-3N)}{48}\sigma_3(N) \\ &\quad - \frac{1}{240}\sigma_1(\frac{N}{2}), \\ \sum_{k<N/2} \sigma_3(k)\sigma_1(N-2k) &= \frac{1}{240}\sigma_5(N) + \frac{1}{12}\sigma_5(\frac{N}{2}) + \frac{(1-3N)}{24}\sigma_3(\frac{N}{2}) \\ &\quad - \frac{1}{240}\sigma_1(N), \end{aligned} \tag{1}$$

where N is an arbitrary positive integer. In this paper, we give a generalization of (1).

The paper is organized as follows. In Section 2, we derive some basic comments on $\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)]$ from $E_4(\tau) = 1 + 240\sum_{N\geq 1} \sigma_3(N)q^N$. In Section 3, we derive some identities involving $\sum_{k=1}^{N-1} \sigma_{1,1}(N;2)\sigma_{1,1}(N-k;2)$ for certain N. In Section 4, we obtain

$$\begin{aligned} &\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) \\ &= \frac{1}{1680}\{(3 - 2^{n-m+4} - 5\cdot 2^{3n+4} + 15\cdot 2^{4n-m+4})\sigma_5(N) \\ &\quad + 2^{-m+4}(9\cdot 2^m + 2^n + 5\cdot 2^{3n+m} - 15\cdot 16^n)\sigma_5(\frac{N}{2}) \\ &\quad - 10(8^{n+1}-1)(-1+3\cdot 2^{n-m}N)\sigma_3(N) \\ &\quad + 80(8^n-1)(-1+3\cdot 2^{n-m}N)\sigma_3(\frac{N}{2}) - 7(2^{n-m+1}-1)\sigma_1(N) \\ &\quad + 14(2^{n-m}-1)\sigma_1(\frac{N}{2})\} \end{aligned}$$

for any positive integer N and m $(0 \le m \le n)$ with $n \in \mathbb{N} \cup \{0\}$ (Theorem 4.6). This is a generalization of (1). In the last section, we also find

$$\begin{aligned}
&\sum_{k=1}^{N-1} \widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^n(N-k)) \\
= &\frac{1}{112\cdot 2^m}[(16^{n+1} - 2^{3n+m+4} - 2^{n+4} + 9\cdot 2^m)\sigma_5(N) \\
&+ 16(27\cdot 2^m + 2^n + 2^{3n+m} - 16^n)\sigma_5(\frac{N}{2}) \\
&+ \{2^{m+1}(2^{3n+3} - 15) - 2(8\cdot 16^n - 15\cdot 2^n)N\}\sigma_3(N) + 16\{2^m(15 - 2^{3n}) \\
&+ (16^n - 15\cdot 2^n)N\}\sigma_3(\frac{N}{2}) \\
&+ 7(3\cdot 2^m - 2^{n+1})\sigma_1(N) - 14(3\cdot 2^m - 2^n)\sigma_1(\frac{N}{2})]
\end{aligned}$$

for any positive integer N and m $(0 \le m \le n)$ with $n \in \mathbb{N} \cup \{0\}$ (Theorem 5.1). This is a generalization of Hanh's result [12],

$$(2) \qquad 16\sum_{k<N} \widetilde{\sigma}_1(k)\widetilde{\sigma}_3(N-k) = -\widetilde{\sigma}_5(N) + 2(N-1)\widetilde{\sigma}_3(N) + \widetilde{\sigma}_1(N).$$

2. Preliminaries

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathcal{H}$, the complex upper-half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\}.$$

The Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is defined by

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \omega^{-2k}$$

and the normalized Eisenstein series of weight $2k$ with $k > 1$ is given by

$$E_{2k}(\tau) = -\frac{(2k)!B_{2k}}{(2\pi i)^{2k}} G_{2k}(\Lambda_\tau) = 1 - \frac{4k}{B_{2k}} \sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^N, \quad \tau \in \mathcal{H},$$

with B_N the Nth Bernoulli number and $q = e^{2\pi i\tau}$. We use notations $\wp(z)$ and G_{2k} instead of $\wp(z;\Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when lattice Λ_τ has been fixed. Now, the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

Let

$$e_1 = \wp(\frac{\tau}{2}),\ e_2 = \wp(\frac{1}{2}),\ \text{and } e_3 = \wp(\frac{\tau+1}{2}).$$

From [17, p. 251], we get

$$\begin{aligned} e_2 - e_1 &= \pi^2 (q^2;q^2)_\infty^4 \frac{1}{(q;q^2)_\infty^8 (-q^2;q^2)_\infty^8}, \\ e_2 - e_3 &= \pi^2 (q^2;q^2)_\infty^4 (q;q^2)_\infty^8, \end{aligned}$$

and

$$e_3 - e_1 = 2^4 \pi^2 q (q^2;q^2)_\infty^4 \frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^8}$$

with $q = \exp(\pi i \tau)$. Next, we state two identities which appear in [8, pp. 78 and 79], [4]:

$$\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8 (q^4;q^4)_\infty^8} = 1 + 8 \sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\substack{\omega | N \\ \omega \text{ odd}}} \omega, \tag{3}$$

$$\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \sum_{N \text{ odd}} \sigma(N) q^N. \tag{4}$$

Using (3) and (4), we obtain the following identities for $\wp(z)$ (for details, see [14]):

(5)

$$\begin{aligned} \wp(\frac{\tau}{2}) &= -\frac{\pi^2}{3}\left(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8 (q^4;q^4)_\infty^8} + 16 \frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}\right) \\ &= -\frac{\pi^2}{3}\Big(1 + 8 \sum_{N \text{ odd}} q^N \sum_{\substack{\omega | N \\ \omega \text{ odd}}} \omega + 24 \sum_{N \text{ even}} q^N \sum_{\substack{\omega | N \\ \omega \text{ odd}}} \omega + 16 \sum_{N \text{ odd}} \sigma(N) q^N\Big) \\ &= -\frac{\pi^2}{3}\Big(1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N;2) q^N\Big) \\ &= -\frac{\pi^2}{3}(1 + 24 S_1 + 24 S_2). \end{aligned}$$

Similarly, Eqs. (3) and (4) yield the following arithmetic results:

$$\begin{aligned} \wp(\frac{\tau+1}{2}) &= -\frac{\pi^2}{3}\left(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8 (q^4;q^4)_\infty^8} - 32 \frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}\right) \\ &= -\frac{\pi^2}{3}(1 - 24 S_1 + 24 S_2), \end{aligned} \tag{6}$$

$$\begin{aligned}\wp(\frac{1}{2}) &= \frac{2\pi^2}{3}(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}) \\ &= \frac{2\pi^2}{3}(1+24S_2).\end{aligned} \tag{7}$$

Thus we deduce the following result [19, p. 59]:

$$\begin{aligned}E_4(\tau) &= \frac{2^2\cdot 3}{(2\pi)^4}g_2(\tau) \\ &= \frac{2^2\cdot 3}{(2\pi)^4}(-4(e_1e_2+e_2e_3+e_3e_1)) \\ &= \frac{2^2\cdot 3}{(2\pi)^4}\cdot\frac{4\pi^4}{9}[3(1+24S_2)^2+24^2S_1^2].\end{aligned}$$

Note that the right-hand side of the above equation is a power series of q^2. So we change variable q^2 to q, that is, from now on, we always assume $q = \exp(2\pi i\tau)$ unless otherwise specified. Therefore,

$$\begin{aligned}E_4(\tau) = 1 + 240q + \sum_{N=2}^{\infty}[48\sigma_{1,1}(2N;2) &+ 576\sum_{\substack{k=1\\k+l=N}}^{N-1}\sigma_{1,1}(2k;2)\sigma_{1,1}(2l;2) \\ &+ 192\sum_{\substack{k=1\\k+l-1=N}}^{N}\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2l-1;2)]q^N.\end{aligned} \tag{8}$$

From [19, p. 59], we already know that

$$E_4(\tau) = 1 + 240\sum_{N\geq 1}\sigma_3(N)q^N. \tag{9}$$

3. Some convolution sums of $\sigma_{1,1}(k;2)$

In [7, p. 300], Glaisher proved that

$$\sigma(1)\sigma(2N-1)+\sigma(3)\sigma(2N-3)+\cdots+\sigma(2N-1)\sigma(1) = \frac{1}{8}[\sigma_3(2N)-\sigma_3(N)]. \tag{10}$$

Combining (8), (9), and (10), we can obtain

$$\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) = \frac{1}{24}[11\sigma_3(N)-\sigma_3(2N)-2\sigma_{1,1}(N;2)]. \tag{11}$$

In this section, we discuss some new convolution sums derived from the existing ones. By using Eqs. (10) and (11), we can obtain the following lemma.

Lemma 3.1. (a) *If $N \geq 1$ is a positive integer, then*

$$U(N) := \sum_{k=1}^{N}\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2(N-k+1);2) = \frac{1}{24}[\sigma_3(2N+1)-\sigma_1(2N+1)].$$

(b) *If $N \geq 1$ is a positive integer, then*

$$V(N) := \sum_{k=1}^{N} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(2(N-k)+1;2) = \frac{1}{8}[\sigma_3(2N) - \sigma_3(N)].$$

Proof. (a) Note that

$$\sum_{k=1}^{2N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-k;2) = \sum_{k=1}^{N} 2\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2(N-k+1);2).$$

Hence, by (11), we obtain

$$\begin{aligned} &\frac{1}{24}[11\sigma_3(2N+1) - \sigma_3(2(2N+1)) - 2\sigma_{1,1}(2N+1;2)] \\ &= 2\sum_{k=1}^{N} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(2(N-k+1);2). \end{aligned}$$

Since $\sigma_3(2(2N+1)) = 9\sigma_3(2N+1)$, we get the desired result.

(b) It is deduced directly from (10). □

Example 3.2. The first twelve values of $U(N)$ and $V(N)$ are given in Table 1.

Table 1. Examples for $U(N)$ and $V(N)$.

N	2	3	4	5	6	7	8	9	10	11	12	13
$U(N)$	5	14	31	55	91	146	204	285	400	506	655	850
$V(N)$	8	28	64	126	224	344	512	757	1008	1332	1792	2198

Remark 3.3. If $N(\geq 3)$ is an odd integer, then

$$\begin{aligned} U\left(\frac{N-1}{2}\right) &:= \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(N-2k+1;2) \\ &= \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(2k;2)\sigma_{1,1}(N-2k;2) \end{aligned}$$

by Lemma 3.1(a) and (b). Let $S(N) := \sum_{k=1}^{N} k^2$.

In particular, if p is an odd prime integer, then

$$U\left(\frac{p-1}{2}\right) = S\left(\frac{p-1}{2}\right) = \sum_{k=1}^{\frac{p-1}{2}} k^2. \tag{12}$$

Thus we can ask a similar question regarding convolution formulas as follows:

(Question) Can one find $r_1, r_2, s_1, s_2, m, \alpha_1, \beta_1, \beta$ in $\mathbb{Z}$ satisfying

$$\sum_{k<\beta p/\beta_1} \sigma_{r_1,s_1}(\alpha_1 k;m)\sigma_{r_2,s_2}(\beta p - \beta_1 k;m) = \sum_{k=1}^{\frac{p-1}{2}} k^u$$

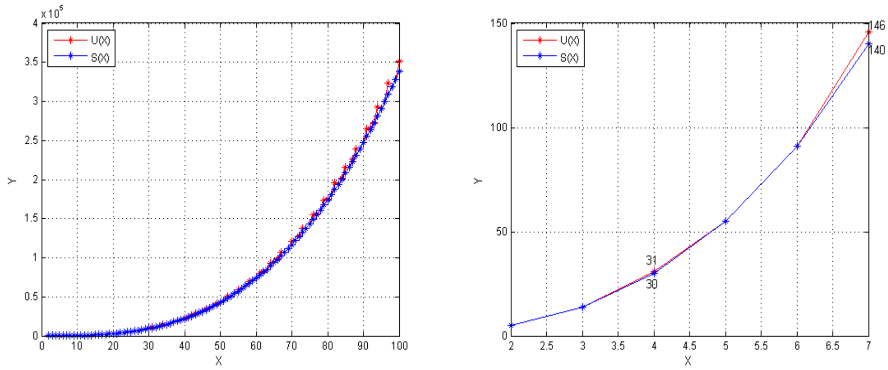

FIGURE 1. $U(X)$ and $S(X)$.

for a fixed u and a fixed odd prime p?

We feel that this sort of problem is generally not easy to solve. Equation (12) is a special case for this question with $u = 2$.

In the following proposition, we state a property of divisor functions, which will be used frequently in our proofs.

Proposition 3.4 ([20, p. 26]). *Let p be a prime. Let $k, N \in \mathbb{N}$. Then*

$$\sigma_k(pN) - (p^k + 1)\sigma_k(N) + p^k \sigma_k(\frac{N}{p}) = 0.$$

Lemma 3.5. *If $N \geq 2$ is a positive integer, then*

$$\sum_{k=1}^{N-1} k\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) = \frac{N}{48}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].$$

Proof. It is obvious from [13, p. 8]. □

Theorem 3.6. *Let N be an odd integer greater than 3. Then*

$$\sum_{L=1}^{N-1}\sum_{l=1}^{L} \sigma_{1,1}(2N-(2L+1);2)\sigma_{1,1}(2l-1;2)\sigma_{1,1}(2(L-l+1);2) \tag{13}$$
$$= \frac{1}{24}[\sigma_5(N) - \sigma_3(N)].$$

Proof. The coefficients of q^{2N} in

$$\sum_{k,l,m=1}^{\infty} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(2l-1;2)\sigma_{1,1}(2m;2)q^{2(k+l+m-1)}$$

with $k+l+m-1=N$ can be expanded as

$$\begin{aligned}
A &:= \sum_{L=1}^{N-1}\sigma_{1,1}(2N-(2L+1);2)\sum_{l=1}^{L}\sigma_{1,1}(2l-1;2)\sigma_{1,1}(2(L-l+1);2)\\
&= \sum_{L=1}^{N-1}\sigma_1(2N-(2L+1))\frac{1}{24}[\sigma_3(2L+1)-\sigma_1(2L+1)]\\
&= \frac{1}{24}\sum_{L=1}^{N-1}\sigma_1(2(N-L-1)+1)\sigma_3(2N-2(N-L-1)-1)\\
&\quad -\frac{1}{24}\sum_{L=1}^{N-1}\sigma_1(2N-(2L+1))\sigma_1(2L+1).
\end{aligned} \tag{14}$$

In [13, p. 24], Eq. (15) is described, for an odd N, as

$$\sum_{k=0}^{N-1}\sigma_1(2k+1)\sigma_3(2N-2k-1)=\sigma_5(N). \tag{15}$$

Substituting (15) and (10) into (14), we get

$$\begin{aligned}
A &= \frac{1}{24}[\sigma_5(N)-\sigma_1(2N-1)]-\frac{1}{24}\{\frac{1}{8}[\sigma_3(2N)-\sigma_3(N)]-\sigma_1(2N-1)\}\\
&= \frac{1}{24}[\sigma_5(N)-\frac{1}{8}\sigma_3(2N)+\frac{1}{8}\sigma_3(N)]\\
&= \frac{1}{24}[\sigma_5(N)-\sigma_3(N)].
\end{aligned}$$

□

Using modular forms, we can give a generalization of Theorem 3.6. Indeed, from pp. 18–19 of [6], we know

$$G_{2,2}(\tau):=-\frac{\pi^2}{3}(1+24\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N)\in M_2(\Gamma_0(2)).$$

Let

$$g(\tau)=\frac{1}{24}+\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N\in M_2(\Gamma_0(2)). \tag{16}$$

From p. 107 of [6], we know

$$d(\Gamma_0(2))=3,\ \epsilon_2(\Gamma_0(2))=1,\ \epsilon_3(\Gamma_0(2))=0,\ \text{and } \epsilon_\infty(\Gamma_0(2))=2.$$

From Theorem 3.1.1 of [6], the genus of modular curve $X(\Gamma_0(2))$ is

$$g=1+\frac{d}{12}-\frac{\epsilon_2}{4}-\frac{\epsilon_3}{3}-\frac{\epsilon_\infty}{2}=0.$$

From Theorem 3.5.1 of [6], we know, for even $k\geq 4$,

$$\dim M_k(\Gamma_0(2))=(k-1)(g-1)+\left\lfloor\frac{k}{4}\right\rfloor\epsilon_2+\left\lfloor\frac{k}{3}\right\rfloor\epsilon_3+\frac{k}{2}\epsilon_\infty=1+\left\lfloor\frac{k}{4}\right\rfloor. \tag{17}$$

Thus $\dim M_6(\Gamma_0(2)) = 2$. Let

$$\widetilde{E_6}(\tau) = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(N) q^N \in M_6(SL_2(\mathbb{Z})) \subset M_6(\Gamma_0(2))$$

(see [16, p. 111]). By Proposition 17 of [16],

$$\widetilde{E_6}(2\tau) = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(\frac{N}{2}) q^N \in M_6(\Gamma_0(2)),$$

where $\sigma_5(\frac{N}{2}) = 0$ if N is odd. Obviously, $\widetilde{E_6}(\tau)$ and $\widetilde{E_6}(2\tau)$ are linearly independent. Hence

$$M_6(\Gamma_0(2)) = \mathbb{C}\widetilde{E_6}(\tau) \oplus \mathbb{C}\widetilde{E_6}(2\tau). \tag{18}$$

Computing the first few Fourier coefficients, we get

$$g(\tau)^3 = \frac{1}{192}\widetilde{E_6}(\tau) - \frac{1}{24}\widetilde{E_6}(2\tau). \tag{19}$$

Comparing the coefficients of the Fourier expansion of (19), we get, for $N \geq 3$,

$$\begin{aligned} &\sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) \\ &\quad + 3\cdot\frac{1}{24}\sum_{\substack{k+l=N \\ k,l>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2) + 3(\frac{1}{24})^2\sigma_{1,1}(N;2) \\ &= \frac{1}{192}\sigma_5(N) - \frac{1}{24}\sigma_5(\frac{N}{2}). \end{aligned}$$

Combining this with Eq. (11), we get

$$\begin{aligned} &\sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) \\ &= \frac{1}{192}\sigma_5(N) - \frac{1}{24}\sigma_5(\frac{N}{2}) - \frac{1}{8}\left[\frac{1}{24}\{11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)\}\right] \\ &\quad - \frac{1}{192}\sigma_{1,1}(N;2) \\ &= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N;2). \end{aligned}$$

The last equality is derived by using Proposition 3.4.

This yields the following theorem.

Theorem 3.7. *For $N \geq 3$,*

$$\sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2)$$

$$= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N;2).$$

Remark 3.8. By Theorem 3.7, for $N \geq 3$,

$$\sum_{\substack{k+l+h=N\\ k,l,h>0}} k\sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2)$$
$$= \frac{1}{3}\sum_{\substack{k+l+h=N\\ k,l,h>0}} (k+l+h)\sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2)$$
$$= \frac{N}{3}\left[-\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N;2)\right].$$

This reproves the formula in [20, p. 133].

Corollary 3.9. *Let $p = 2q+1$ be an odd prime integer.*

(a) $\sum_{k=1}^{p-1}\sigma_{1,1}(k;2)\sigma_{1,1}(p-k;2) = \sum_{k=1}^{q} 2k^2$.
(b) $\sum_{k=1}^{p-1} k\sigma_{1,1}(k;2)\sigma_{1,1}(p-k;2) = p\sum_{k=1}^{q} k^2$.
(c) $\sum_{k+l+h=p}\sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) = (\sum_{k=1}^{q} k)(\sum_{k=1}^{q} k^2)$.
(d) $\sum_{k+l+h=p} k\sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) = (\sum_{k=1}^{q} k^2)^2$.

Proof. From (11), Lemma 3.5, Theorem 3.7 and Remark 3.8, we can deduce the proof. □

Example 3.10. The first thirteen values of $\alpha(X) := \sum_{k=1}^{2X}\sigma_{1,1}(k;2)\sigma_{1,1}(2X+1-k;2)$ and $\beta(X) := \sum_{k=1}^{X} 2k^2$ are given in Table 2.

Table 2. Examples for $\alpha(X)$ and $\beta(X)$.

X	1	2	3	4	5	6	7	8	9	10	11	12	13
$\alpha(X)$	2	10	28	62	110	182	292	408	570	800	1012	1310	1700
$\beta(X)$	2	10	28	60	110	182	280	408	570	770	1012	1300	1638

We can see when $2X+1$ is prime, $\sum_{k=1}^{2X}\sigma_{1,1}(k;2)\sigma_{1,1}(2X+1-k;2)$ coincides with $\sum_{k=1}^{X} 2k^2$ in Figure 2. A similar result for consecutive integers can be found in [15, (2.10)].

Considering Theorem 3.6 from another point of view, we get the following formula.

Proposition 3.11. *For an odd $N \geq 3$, we have*

$$\sum_{k=1}^{N-1}[\sigma_3(2k) - \sigma_3(k)]\sigma_{1,1}(N-k;2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)].$$

Proof. Precisely, the coefficients of

$$\sum_{k,l,m=1}^{\infty}\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2l-1;2)\sigma_{1,1}(2m;2)q^{2(k+l+m-1)}$$

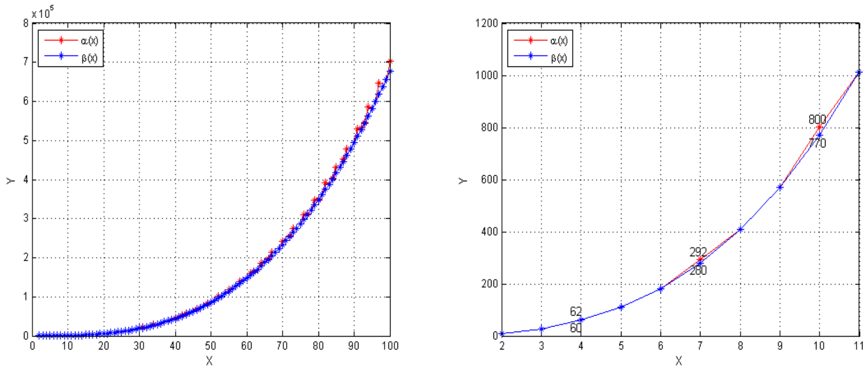

FIGURE 2. $\alpha(X)$ and $\beta(X)$.

can be written as

$$\begin{aligned}(20)\qquad &\sum_{k=1}^{N-1}\sum_{n=1}^{k}\sigma_{1,1}(2n-1;2)\sigma_{1,1}(2k-(2n-1);2)\sigma_{1,1}(2N-2k;2)\\ &=\sum_{k=1}^{N-1}\frac{1}{8}[\sigma_3(2k)-\sigma_3(k)]\sigma_{1,1}(2N-2k;2).\end{aligned}$$

Since Eq. (20) equates with Theorem 13 for an odd N, we have

$$\sum_{k=1}^{N-1}\frac{1}{8}[\sigma_3(2k)-\sigma_3(k)]\sigma_{1,1}(2N-2k;2)=\frac{1}{24}[\sigma_5(N)-\sigma_3(N)],$$

which concludes the proof. □

Theorem 3.12. *If $N(\geq 3)$ is an odd integer, then*

$$\sum_{k=1}^{N-1}\sigma_3(k)\sigma_{1,1}(N-k;2)=\frac{1}{240}\{11\sigma_5(N)-10\sigma_3(N)-\sigma_1(N)\}.$$

Proof. Now let us look at the formula

$$\wp'(z)^2=4\wp(z)^3-g_2(\tau)\wp(z)-g_3(\tau),\ \tau\in\mathcal{H},$$

more closely. Since $\wp'(\frac{\tau}{2})=0$, we get

$$(21)\qquad 4\wp(\frac{\tau}{2})^3=\frac{(2\pi)^4}{2^2\cdot 3}E_4(\tau)\wp(\frac{\tau}{2})+\frac{(2\pi)^6}{2^3\cdot 3^3}E_6(\tau).$$

Substituting

$$\wp(\frac{\tau}{2})=-\frac{\pi^2}{3}(1+24\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N),$$

$$E_4(\tau) = 1 + 240 \sum_{N=1}^{\infty} \sigma_3(N) q^{2N},$$
$$E_6(\tau) = 1 - 504 \sum_{N=1}^{\infty} \sigma_5(N) q^{2N}$$

into Eq. (21), we obtain

$$\begin{aligned} B &:= (1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N;2) q^N)^3 \\ &= 3(1 + 240 \sum_{N=1}^{\infty} \sigma_3(N/2) q^N)(1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N;2) q^N) \\ &\quad - 2(1 - 504 \sum_{N=1}^{\infty} \sigma_5(N/2) q^N). \end{aligned} \tag{22}$$

After applying the usual computation, we will compare the coefficient of q^{2N} and q^{2N-1} on the left-hand side and right-hand side of Eq. (22), separately.

For q^{2N-1}, we can deduce that

$$\begin{aligned} &17280 \sum_{k=1}^{N} \sigma_3(k+1) \sigma_1(2(N-k)+1) \\ &= -72\sigma_1(2N+3) - 17280\sigma_1(2N+1) + 720\sigma_5(2N+3). \end{aligned}$$

For q^{2N}, we have

$$\begin{aligned} &240 \sum_{k=1}^{N-1} \sigma_3(N) \sigma_{1,1}(N-k;2) \\ &= -\sigma_{1,1}(N;2) - 10\sigma_3(N) - 21\sigma_5(N) + 32\sigma_1(2N-1) \\ &\quad + 32 \sum_{k=1}^{N-1} \sigma_1(2(N-k)-1) \sigma_3(2k+1), \end{aligned} \tag{23}$$

where N is any positive integer. From (15),

$$\sum_{k=1}^{N-1} \sigma_1(2(N-k)-1) \sigma_3(2k+1) = \sigma_5(N) - \sigma_1(2N-1).$$

Therefore, (23) is further recalculated and the proof is complete. □

Remark 3.13. Theorem 3.12 can be proved directly by modular forms. Moreover, the proof given below also generalizes Theorem 3.12 to the case of N being even.

From pp. 18–19 of [6], we know that

$$g(\tau) = \frac{1}{24} + \sum_{N=1}^{\infty} \sigma_{1,1}(N;2) q^N \in M_2(\Gamma_0(2)). \tag{24}$$

Let

$$\widetilde{E_4}(\tau) = \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3(N) q^N \in M_4(SL_2(\mathbb{Z})) \subset M_4(\Gamma_0(2)). \tag{25}$$

It is easy to see that

$$g(\tau)\widetilde{E_4}(\tau) \in M_6(\Gamma_0(2)).$$

From (18), $g\widetilde{E_4}$ is a linear combination of $\widetilde{E_6}(\tau)$ and $\widetilde{E_6}(2\tau)$. A direct computation shows that

$$g\widetilde{E_4} = \frac{11}{240}\widetilde{E_6}(\tau) - \frac{2}{15}\widetilde{E_6}(2\tau). \tag{26}$$

Comparing the coefficients of the Fourier expansion in (26), we get

(27)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k) = \frac{11}{240}\sigma_5(N) - \frac{2}{15}\sigma_5(\frac{N}{2}) - \frac{1}{24}\sigma_3(N) - \frac{1}{240}\sigma_{1,1}(N;2)$$

for $N \geq 2$.

4. Special convolution sums

Lemma 4.1. *Let N be any positive integer. Then we have the following*:

(a)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k)$$
$$= \frac{1}{240}(11\sigma_5(N) - 32\sigma_5(\frac{N}{2}) - 10\sigma_3(N) - \sigma_1(N) + 2\sigma_1(\frac{N}{2})).$$

(b)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,0}(N-k;2)$$
$$= \frac{1}{30}\sigma_5(N) - \frac{11}{15}\sigma_5(\frac{N}{2}) - \frac{1}{3}\sigma_3(\frac{N}{2}) - \frac{1}{30}\sigma_1(N) + \frac{1}{15}\sigma_1(\frac{N}{2}).$$

(c)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2)$$
$$= \frac{1}{80}\sigma_5(N) + \frac{3}{5}\sigma_5(\frac{N}{2}) - \frac{1}{24}\sigma_3(N) + \frac{1}{3}\sigma_3(\frac{N}{2}) + \frac{7}{240}\sigma_1(N)$$
$$- \frac{7}{120}\sigma_1(\frac{N}{2}).$$

Proof. (a) We refer to Eq. (27).

(b) We consider the convolution sum $\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{3,0}(N-k;2)$ when N is odd or even. Let N be even. Given the fact that $\sigma_{3,0}(odd;2)=0$ and $\sigma_{3,0}(2N-2k;2)=8\sigma_3(N-k)$, we can obtain

$$\begin{aligned}&\sum_{k=1}^{2N-1}\sigma_{1,1}(k;2)\sigma_{3,0}(2N-k;2)\\&=\sum_{k=1}^{N-1}\sigma_{1,1}(2k;2)\sigma_{3,0}(2N-2k;2)\\&=8\{\sum_{k=1}^{N-1}\sigma_1(k)\sigma_3(N-k)-\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_3(N-k)\}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_3(N-k)&=\sum_{k=1}^{\frac{N}{2}-1}\sigma_{1,0}(2k;2)\sigma_3(N-2k)\\&=2\sum_{k=1}^{\frac{N}{2}-1}\sigma_1(k)\sigma_3(N-2k)\end{aligned}$$

by $\sigma_{1,0}(2k;2)=2\sigma_1(k)$, and therefore, we refer to

$$\begin{aligned}&\sum_{k<\frac{N}{2}}\sigma_1(k)\sigma_3(N-2k)\\&=\frac{1}{240}\{5\sigma_5(N)+(10-15N)\sigma_3(N)+16\sigma_5(\frac{N}{2})-\sigma_1(\frac{N}{2})\}\end{aligned}$$

in [13, Theorem 6]. Lastly, we use $\sigma_5(2N)=33\sigma_5(N)-32\sigma_5(\frac{N}{2})$ and $\sigma_1(2N)=3\sigma_1(N)-2\sigma_1(\frac{N}{2})$. Then

$$\begin{aligned}&\sum_{k=1}^{2N-1}\sigma_{1,1}(k;2)\sigma_{3,0}(2N-k;2)\\&=\frac{1}{30}\sigma_5(2N)-\frac{11}{15}\sigma_5(N)-\frac{1}{3}\sigma_3(N)-\frac{1}{30}\sigma_1(2N)+\frac{1}{15}\sigma_1(N).\end{aligned}\tag{28}$$

To obtain the formula for an odd N, we use Hahn's proof

$$16\sum_{k<N}\widetilde{\sigma}_1(k)\tilde{\sigma}_3(N-k)=-\widetilde{\sigma}_5(N)+2(N-1)\widetilde{\sigma}_3(N)+\widetilde{\sigma}_1(N)\tag{29}$$

in [12, p. 12]. In (29), let us consider the left-hand side

$$\begin{aligned}
&\sum_{k=1}^{N-1} \widetilde{\sigma}_1(k)\widetilde{\sigma}_3(N-k) \\
&= \sum_{k=1}^{N-1} [\sigma_{1,1}(k;2) - \sigma_{1,0}(k;2)][\sigma_{3,1}(N-k;2) - \sigma_{3,0}(N-k;2)] \\
&= \sum_{k=1}^{N-1} [\sigma_{1,1}(k;2) - \sigma_{1,0}(k;2)][-\sigma_3(N-k) + 2\sigma_{3,1}(N-k;2)] \qquad (30)\\
&= -\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k) + 2\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2) \\
&\quad + \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_3(N-k) - 2\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{3,1}(N-k;2).
\end{aligned}$$

Then

$$\begin{aligned}
& -\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k) \\
&= -\sum_{k=1}^{N-1} [\sigma_1(k) - \sigma_{1,0}(k;2)]\sigma_3(N-k) \\
&= -\sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(N-k) + \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k;2)\sigma_3(N-2k) \\
&= -\frac{1}{240}[21\sigma_5(N) + (10-30N)\sigma_3(N) - \sigma_1(N)] \\
&\quad + \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k;2)\sigma_3(N-2k),
\end{aligned}$$

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_3(N-k) = \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k;2)\sigma_3(N-2k),$$

and

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{3,1}(N-k;2) = \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k;2)\sigma_{3,1}(N-2k;2).$$

Hence we can rewrite (30) as

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_1(k)\widetilde{\sigma}_3(N-k) = -\frac{1}{240}[21\sigma_5(N) + (10-30M)\sigma_3(N) - \sigma_1(N)]$$

$$+ 2 \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2).$$

From (29), we can get the exact value of (30); therefore,

$$\begin{aligned}
&\frac{1}{16}[-\sigma_5(N) + 2(N-1)\sigma_3(N) + \sigma_1(N)] \\
= &-\frac{1}{240}[21\sigma_5(N) + (10-30N)\sigma_3(N) - \sigma_1(N)] \\
&+ 2 \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2).
\end{aligned}$$

Thus

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2) = \frac{1}{80}\sigma_5(N) - \frac{1}{24}\sigma_3(N) + \frac{7}{240}\sigma_1(N). \tag{31}$$

Comparing Eq. (27) with Eq. (31), we obtain the formula

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,0}(N-k;2).$$

So the proof is complete.

(c) This follows readily from (a) and (b) that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,1}(N-k;2) \\
= &\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k) - \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{3,0}(N-k;2) \\
= &\{\frac{1}{240}(11\sigma_5(N) - 32\sigma_5(\frac{N}{2}) - 10\sigma_3(N) - \sigma_1(N) + 2\sigma_1(\frac{N}{2}))\} \\
&- \{\frac{1}{30}\sigma_5(N) - \frac{11}{15}\sigma_5(\frac{N}{2}) - \frac{1}{3}\sigma_3(\frac{N}{2}) - \frac{1}{30}\sigma_1(N) + \frac{1}{15}\sigma_1(\frac{N}{2})\}.
\end{aligned}$$

□

Corollary 4.2. *Let $N \geq 3$ be odd. Then*

(a) $\sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{3,0}(N-k;2) = \frac{1}{30}[\sigma_5(N) - \sigma_1(N)]$.
(b) $\sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(k;2)\sigma_{3,0}(N-2k;2) = 0$.
(c) $\sum_{k=1}^{N-1} \sigma_{3,0}(2k;2)\sigma_{1,1}(N-k;2) = \frac{11}{30}\sigma_5(N) - \frac{1}{3}\sigma_3(N) - \frac{1}{30}\sigma_1(N)$.
(d) $\sum_{k=1}^{N-1} \sigma_{1,1}(N-k;2)\sigma_3(2k) = \frac{91}{240}\sigma_5(N) - \frac{3}{8}\sigma_3(N) - \frac{1}{240}\sigma_1(N)$.

Proof. (a) Using $\sigma_{1,1}(2k;2) = \sigma_{1,1}(k;2)$ in Lemma 4.1(a), (a) is obtained.

(b) This follows from $\sigma_{3,0}(odd;2) = 0$.

(c) By $\sigma_{3,0}(2k;2) = 8\sigma_3(k)$, we can get

$$\sum_{k=1}^{N-1} \sigma_{3,0}(2k;2)\sigma_{1,1}(N-k;2) = 8\sum_{k=1}^{N-1} \sigma_3(k)\sigma_{1,1}(N-k;2)$$
$$= 8\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k).$$

Now, by Lemma 4.1(b), we can obtain

$$\sum_{k=1}^{N-1} \sigma_{3,0}(2k;2)\sigma_{1,1}(N-k;2) = \frac{11}{30}\sigma_5(N) - \frac{1}{3}\sigma_3(N) - \frac{1}{30}\sigma_1(N).$$

(d) In Proposition 3.11, for an odd $N \geq 3$, we found that

$$\sum_{k=1}^{N-1} [\sigma_3(2k) - \sigma_3(k)]\sigma_{1,1}(N-k;2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)].$$

From this equation, we can deduce

$$\sum_{k=1}^{N-1} \sigma_3(2k)\sigma_{1,1}(N-k;2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)] + \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_3(N-k).$$

The last term is obtained by applying Lemma 4.1(b). □

Lemma 4.3. *Let N be any positive integer. Then we have the following*:

(a)

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_3(N-k)$$
$$= \frac{1}{120}[5\sigma_5(N) + (10-15N)\sigma_3(N) + 16\sigma_5(\frac{N}{2}) - \sigma_1(\frac{N}{2})].$$

(b)

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{3,0}(N-k;2)$$
$$= \frac{1}{15}[21\sigma_5(\frac{N}{2}) + (10-15N)\sigma_3(\frac{N}{2}) - \sigma_1(\frac{N}{2})].$$

(c)

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{3,1}(N-k;2)$$
$$= \frac{1}{24}\sigma_5(N) - \frac{19}{15}\sigma_5(\frac{N}{2}) + \frac{2-3N}{24}\sigma_3(N) + \frac{3N-2}{3}\sigma_3(\frac{N}{2})$$
$$+ \frac{7}{120}\sigma_1(\frac{N}{2}).$$

Proof. (a) We know that

$$\begin{aligned}\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_3(N-k) &= \sum_{k<N/2}\sigma_{1,0}(2k;2)\sigma_3(N-2k)\\ &= 2\sum_{k<N/2}\sigma_1(k)\sigma_3(N-2k).\end{aligned}$$

From (1), we obtain the desired result.

(b) If N is an odd integer, then

$$\sigma_{1,0}(k;2)\sigma_{3,0}(N-k;2) = \sigma_{1,0}(2k;2)\sigma_{3,0}(N-2k;2)$$

is zero since $\sigma_{3,0}(odd;2) = 0$. And if $N = 2L$ is an even integer, then

$$\begin{aligned}\sum_{k=1}^{2L-1}\sigma_{1,0}(k;2)\sigma_{3,0}(2L-k;2) &= \sum_{k=1}^{L-1}\sigma_{1,0}(2k;2)\sigma_{3,0}(2L-2k;2)\\ &= 16\sum_{k=1}^{L-1}\sigma_1(k)\sigma_3(L-k)\end{aligned}$$

since $\sigma_{1,0}(2k;2) = 2\sigma_1(k)$ and $\sigma_{3,0}(2k;2) = 8\sigma_3(k)$. Next we refer to

$$\sum_{k=1}^{N-1}\sigma_1(k)\sigma_3(N-k) = \frac{1}{240}[21\sigma_5(N) + (10-30N)\sigma_3(N) - \sigma_1(N)] \tag{32}$$

in [13] and obtain

$$\sum_{k=1}^{2L-1}\sigma_{1,0}(k;2)\sigma_{3,0}(2L-k;2) = \frac{1}{15}[21\sigma_5(L) + (10-30L)\sigma_3(L) - \sigma_1(L)].$$

(c) This is directly derived from (a) and (b). □

Corollary 4.4. *Let $N \geq 3$ be odd. Then*

(a) $\sum_{k=1}^{N-1} k\sigma_{1,0}(2k;2)\sigma_3(N-k) = \frac{1}{120}[7N\sigma_5(N) - 6N^2\sigma_3(N) - N\sigma_1(N)]$.

(b) $\sum_{k=1}^{N-1} k^2\sigma_{1,0}(2k;2)\sigma_3(N-k) = \frac{1}{40}N^2\sigma_5(N) - \frac{1}{60}N^3\sigma_3(N) - \frac{1}{120}N^2\sigma_1(N)$.

Proof. (a) Using $\sigma_{1,0}(2k;2) = 2\sigma_1(k)$, we deduce that

$$\sum_{k=1}^{N-1} k\sigma_{1,0}(2k;2)\sigma_3(N-k) = 2\sum_{k=1}^{N-1} k\sigma_1(k)\sigma_3(N-k).$$

Then we can use

$$\sum_{k=1}^{N-1} k\sigma_1(k)\sigma_3(N-k) = \frac{1}{240}[7N\sigma_5(N) - 6N^2\sigma_3(N) - N\sigma_1(N)]$$

in [13, (3.13)].

(b) The proof is similar to that of (a), except that we refer to

$$\sum_{k=1}^{N-1} k^2\sigma_1(k)\sigma_3(N-k) = \frac{1}{80}N^2\sigma_5(N) - \frac{1}{120}N^3\sigma_3(N) - \frac{1}{240}N^2\sigma_1(N),$$

in [20, p. 156]. □

We give the following proposition in order to grasp Theorem 4.6.

Proposition 4.5. *Let N be any positive integer. Then we have the following*:

(a)

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(\frac{N-k}{2}) = \frac{1}{240}\{\sigma_5(N) + 20\sigma_5(\frac{N}{2}) + (10-30N)\sigma_3(\frac{N}{2}) - \sigma_1(N)\}.$$

(b)

$$\sum_{k=1}^{N-1} \sigma_1(\frac{k}{2})\sigma_3(N-k) = \frac{1}{48}\sigma_5(N) + \frac{1}{15}\sigma_5(\frac{N}{2}) + \frac{(2-3N)}{48}\sigma_3(N) - \frac{1}{240}\sigma_1(\frac{N}{2}).$$

(c)

$$\sum_{k=1}^{N-1} \sigma_1(\frac{k}{2})\sigma_3(\frac{N-k}{2}) = \frac{1}{240}[21\sigma_5(\frac{N}{2}) + (10-15N)\sigma_3(\frac{N}{2}) - \sigma_1(\frac{N}{2})].$$

Proof. (a) From [13, p. 25], we see that an odd N satisfies

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(\frac{N-k}{2}) = \frac{1}{240}\sigma_5(N) - \frac{1}{240}\sigma_1(N) \tag{33}$$

and that an even N, i.e., $N = 2L$, yields

(34)

$$\sum_{k=1}^{2L-1} \sigma_1(k)\sigma_3(\frac{2L-k}{2}) = \sum_{k=1}^{L-1} \sigma_{1,1}(k;2)\sigma_3(L-k) + 2\sum_{k=1}^{L-1} \sigma_1(k)\sigma_3(L-k)$$
$$= \frac{1}{240}\{20\sigma_5(L) + \sigma_5(2L) + (10-60L)\sigma_3(L) - \sigma_1(2L)\}$$

by Lemma 4.1(b) and (32). Thus we combine (33) and (34) to complete the proof.

(b) This is written as

$$\sum_{k=1}^{N-1}\sigma_1(\frac{k}{2})\sigma_3(N-k)=\sum_{k<N/2}\sigma_1(k)\sigma_3(N-2k),$$

so refer to (1).

(c) We can obtain the desired result using (32). □

Theorem 4.6. *Let N be any positive integer. If m $(0\le m\le n)$ is any nonnegative integer with $n\in\mathbb{N}\cup\{0\}$, then the following assertions hold*:

(a)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_1(2^{n-m}k)\sigma_3(2^n(N-k))\\ &=\frac{1}{1680}\{(3-2^{n-m+4}-5\cdot 2^{3n+4}+15\cdot 2^{4n-m+4})\sigma_5(N)\\ &+2^{-m+4}(9\cdot 2^m+2^n+5\cdot 2^{3n+m}-15\cdot 16^n)\sigma_5(\frac{N}{2})\\ &-10(8^{n+1}-1)(-1+3\cdot 2^{n-m}N)\sigma_3(N)\\ &+80(8^n-1)(-1+3\cdot 2^{n-m}N)\sigma_3(\frac{N}{2})\\ &-7(2^{n-m+1}-1)\sigma_1(N)+14(2^{n-m}-1)\sigma_1(\frac{N}{2})\}.\end{aligned}$$

(b)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_3(2^{n-m}k)\sigma_1(2^n(N-k))\\ &=\frac{1}{1680}\{(15\cdot 2^{4n-3m+4}-5\cdot 2^{3n-3m+4}-2^{n+4}+3)\sigma_5(N)\\ &+16(5\cdot 2^{3(n-m)}-15\cdot 2^{4n-3m}+2^n+9)\sigma_5(\frac{N}{2})\\ &+10(2^{3(n-m+1)}-1)(1-3\cdot 2^nN)\sigma_3(N)\\ &+80(2^{3(n-m)}-1)(-1+3\cdot 2^nN)\sigma_3(\frac{N}{2})\\ &-7(2^{n+1}-1)\sigma_1(N)+14(2^n-1)\sigma_1(\frac{N}{2})\}.\end{aligned}$$

Proof. (a) We use induction on m to obtain the general formula for $\sigma_1(2^mk)$. If $m=1$, then

$$\sigma_1(2k)=3\sigma_1(k)-2\sigma_1(\frac{k}{2}).$$

If $m = 2$, then

$$\sigma_1(2^2k) = (3^2 - 2)\sigma_1(k) - 3 \cdot 2\sigma_1(\frac{k}{2}).$$

Continuing this process, we get

$$\sigma_1(2^m k) = (2^{m+1} - 1)\sigma_1(k) + (2 - 2^{m+1})\sigma_1(\frac{k}{2}).$$

Similarly, we obtain

$$\sigma_3(2^n(N-k)) = \frac{8^{n+1}-1}{7}\sigma_3(N-k) + \frac{8-8^{n+1}}{7}\sigma_3(\frac{N-k}{2}).$$

Therefore, we know that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_1(2^m k)\sigma_3(2^n(N-k)) \\
&= \sum_{k=1}^{N-1} \{(2^{m+1}-1)\sigma_1(k) + (2-2^{m+1})\sigma_1(\frac{k}{2})\} \\
&\quad \{\frac{8^{n+1}-1}{7}\sigma_3(N-k) + \frac{8-8^{n+1}}{7}\sigma_3(\frac{N-k}{2})\} \\
&= (2^{m+1}-1) \cdot \frac{8^{n+1}-1}{7} \sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(N-k) \\
&\quad + (2^{m+1}-1) \cdot \frac{8-8^{n+1}}{7} \sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(\frac{N-k}{2}) \\
&\quad + (2-2^{m+1}) \cdot \frac{8^{n+1}-1}{7} \sum_{k=1}^{N-1} \sigma_1(\frac{k}{2})\sigma_3(N-k) \\
&\quad + (2-2^{m+1}) \cdot \frac{8-8^{n+1}}{7} \sum_{k=1}^{N-1} \sigma_1(\frac{k}{2})\sigma_3(\frac{N-k}{2}).
\end{aligned} \tag{35}$$

Then we use Proposition 4.5 and change $m \to n - m$.

(b) This is similar to Theorem 4.6(a), but we refer to

$$\begin{aligned}
\sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(N-k) &= \sum_{t=1}^{N-1} \sigma_3(N-t)\sigma_1(t), \\
\sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(\frac{N-k}{2}) &= \sum_{t=1}^{N-1} \sigma_3(N-2t)\sigma_1(t), \\
\sum_{k=1}^{N-1} \sigma_3(\frac{k}{2})\sigma_1(N-k) &= \sum_{t=1}^{N-1} \sigma_3(t)\sigma_1(N-2t),
\end{aligned}$$

$$\sum_{k=1}^{N-1}\sigma_3(\frac{k}{2})\sigma_1(\frac{N-k}{2})=\sum_{t=1}^{N-1}\sigma_3(\frac{N}{2}-t)\sigma_1(t).$$
□

Corollary 4.7. *Let N be any positive integer. Then*

$$\sum_{k=1}^{N}\sigma_1(2k-1)\sigma_3(2N-2k+1)=\frac{1}{32}(\sigma_5(2N)-\sigma_5(N)).$$

In particular, if N is odd, then

$$\sum_{k=1}^{N}\sigma_1(2k-1)\sigma_3(2N-2k+1)=\sigma_5(N),$$

which is also the result of Huard, Ou, Spearman, and Williams [13, Corollary 3].

Proof. Consider

$$\begin{aligned}&\sum_{k=1}^{2N-1}\sigma_1(k)\sigma_3(2N-k)\\ &=\sum_{k=1}^{N}\sigma_1(2k-1)\sigma_3(2N-2k+1)+\sum_{k=1}^{N-1}\sigma_1(2k)\sigma_3(2N-2k).\end{aligned}$$

Then by replacing N with $2N$ in [13, (3.12)] and $n=1, m=0$ in

$$\sum_{k=1}^{N-1}\sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)),$$

we get

$$\begin{aligned}&\sum_{k=1}^{N}\sigma_1(2k-1)\sigma_3(2N-2k+1)\\ &=\frac{1}{240}\{21\sigma_5(2N)+(10-60N)\sigma_3(2N)-\sigma_1(2N)\}\\ &\quad-\{\frac{151}{80}\sigma_5(N)-\frac{9}{5}\sigma_5(\frac{N}{2})+\frac{3}{8}\sigma_3(N)-\frac{9N}{4}\sigma_3(N)-\frac{1}{3}\sigma_3(\frac{N}{2})+2N\sigma_3(\frac{N}{2})\\ &\quad-\frac{1}{80}\sigma_1(N)+\frac{1}{120}\sigma_1(\frac{N}{2})\}.\end{aligned}$$

Then we use $\sigma_5(2N)=33\sigma_5(N)-32\sigma_5(\frac{N}{2})$, $\sigma_3(2N)=9\sigma_3(N)-8\sigma_3(\frac{N}{2})$, and $\sigma_1(2N)=3\sigma_1(N)-2\sigma_1(\frac{N}{2})$. □

Corollary 4.8. *Let N be any positive integer with $n\in\mathbb{N}\cup\{0\}$. Then we have the following*:

(a)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_{1,0}(2^{n-m}k;2)\sigma_3(2^n(N-k))\\&=\frac{1}{840\cdot 2^m}[(3\cdot 2^m-2^{n+3}-5\cdot 2^{3n+m+4}+15\cdot 2^{4n+3})\sigma_5(N)\\&\quad+8(9\cdot 2^{m+1}+2^n+5\cdot 2^{3n+m+1}-15\cdot 2^{4n})\sigma_5(\frac{N}{2})\\&\quad+5\{2^m(2^{3n+4}-2)+3\cdot 2^nN(1-2^{3n+3})\}\sigma_3(N)\\&\quad+40(2^{3n}-1)(3\cdot 2^nN-2^{m+1})\sigma_3(\frac{N}{2})\\&\quad-7(2^n-2^m)\sigma_1(N)+7(2^n-2^{m+1})\sigma_1(\frac{N}{2})].\end{aligned}$$

(b)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_{1,1}(2^{n-m}k;2)\sigma_3(2^n(N-k))\\&=\frac{1}{1680}\{(5\cdot 2^{3n+4}-3)\sigma_5(N)-16(5\cdot 2^{3n}+9)\sigma_5(\frac{N}{2})\\&\quad+10(1-2^{3n+3})\sigma_3(N)+80(2^{3n}-1)\sigma_3(\frac{N}{2})-7\sigma_1(N)+14\sigma_1(\frac{N}{2})\}.\end{aligned}$$

(c)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_3(2^{n-m}k)\sigma_{1,0}(2^n(N-k);2)\\&=\frac{1}{840}[(3-2^{n+3}-5\cdot 2^{3n-3m+4}+15\cdot 2^{4n-3m+3})\sigma_5(N)\\&\quad+8(18+2^n+5\cdot 2^{3n-3m+1}-15\cdot 2^{4n-3m})\sigma_5(\frac{N}{2})\\&\quad+5\{-2+2^{3n-3m+4}-3\cdot 2^nN(2^{3(n-m+1)}-1)\}\sigma_3(N)\\&\quad+40\{2-2^{3n-3m+1}+3\cdot 2^nN(2^{3(n-m)}-1)\}\sigma_3(\frac{N}{2})\\&\quad-7(2^n-1)\sigma_1(N)+7(2^n-2)\sigma_1(\frac{N}{2})].\end{aligned}$$

(d)

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_3(2^{n-m}k)\sigma_{1,1}(2^n(N-k);2)\\&=\frac{1}{1680}\{(5\cdot 2^{3n-3m+4}-3)\sigma_5(N)-16(5\cdot 2^{3(n-m)}+9)\sigma_5(\frac{N}{2})\end{aligned}$$

$$- 10(2^{3(n-m+1)} - 1)\sigma_3(N) + 80(2^{3(n-m)} - 1)\sigma_3(\frac{N}{2})$$
$$- 7\sigma_1(N) + 14\sigma_1(\frac{N}{2})\}.$$

(e)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(2^{n-m}k; 2)\sigma_{3,0}(2^n(N-k); 2)$$
$$= \frac{1}{210}\{(5 \cdot 2^{3n+1} - 3)\sigma_5(N) - 2(5 \cdot 8^n + 72)\sigma_5(\frac{N}{2}) - 10(8^n - 1)\sigma_3(N)$$
$$+ 10(8^n - 8)\sigma_3(\frac{N}{2}) - 7\sigma_1(N) + 14\sigma_1(\frac{N}{2})\}.$$

Proof. (a) From the fact that

$$\sum_{k=1}^{N-1} \sigma_{1,0}(2^{n-m}k; 2)\sigma_3(2^n(N-k)) = \sum_{k=1}^{N-1} 2\sigma_1(2^{n-m-1}k)\sigma_3(2^n(N-k))$$
$$= 2\sum_{k=1}^{N-1} \sigma_1(2^{n-(m+1)}k)\sigma_3(2^n(N-k)),$$

we put $m \to m+1$ in Theorem 4.6 when $m < n$. If $m = n$, then we use $2\sigma_1(\frac{k}{2}) = 3\sigma_1(k) - \sigma_1(2k)$.

(b)-(e) This is similar to Corollary 4.8(a). □

Remark 4.9. Corollary 4.8(b) can be proved directly via Hecke operators. Let

$$g(\tau) = \frac{1}{24} + \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) \in M_2(\Gamma_0(2)),$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H}$. From (17), we know that, for even $k \geq 4$,

$$\dim M_k(\Gamma_0(2)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor.$$

In particular,

$$\dim M_4(\Gamma_0(2)) = \dim M_6(\Gamma_0(2)) = 2.$$

Let

$$\widetilde{E_4} = \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3(N)q^N \in M_4(SL_2(\mathbb{Z})) \subset M_4(\Gamma_0(2)),$$
$$\widetilde{E_6} = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(N)q^N \in M_6(SL_2(\mathbb{Z})) \subset M_6(\Gamma_0(2))$$

(see [16, p. 111]). By Proposition 17 of [16],

$$\widetilde{E_4}(2\tau) = \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3(\frac{N}{2})q^N \in M_4(\Gamma_0(2)),$$

$$\widetilde{E_6}(2\tau) = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(\frac{N}{2})q^N \in M_6(\Gamma_0(2)),$$

where $\sigma_3(\frac{N}{2}) = \sigma_5(\frac{N}{2}) = 0$ if $2 \nmid N$. Therefore,

$$M_4(\Gamma_0(2)) = \mathbb{C}\widetilde{E_4}(\tau) \oplus \mathbb{C}\widetilde{E_4}(2\tau),$$

$$M_6(\Gamma_0(2)) = \mathbb{C}\widetilde{E_6}(\tau) \oplus \mathbb{C}\widetilde{E_6}(2\tau).$$

The Hecke operator,

$$T_2 : M_k(\Gamma_0(2)) \to M_k(\Gamma_0(2)),$$

is a linear map such that, for any $\sum a_N q^N \in M_k(\Gamma_0(2))$,

$$T_2(\sum a_N q^N) = \sum a_{2N} q^N$$

(see Proposition 37 of Chapter 3 of [16]). It is easy to see that $\widetilde{E_4}(\tau)$ and $T_2\widetilde{E_4}(\tau)$ are linearly independent in $M_4(\Gamma_0(2))$. Taking them as a basis, we get

$$T_2(\widetilde{E_4}, T_2\widetilde{E_4}) = (\widetilde{E_4}, T_2\widetilde{E_4}) \begin{pmatrix} 0 & -8 \\ 1 & 9 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & -8 \\ 1 & 9 \end{pmatrix}^n = \frac{1}{7} \begin{pmatrix} 8 - 8^n & 8 - 8^{n+1} \\ -1 + 8^n & -1 + 8^{n+1} \end{pmatrix},$$

we get

$$T_2^n \widetilde{E_4} = \frac{1}{7}(8 - 8^n)\widetilde{E_4} + \frac{1}{7}(8^n - 1)T_2\widetilde{E_4}. \tag{36}$$

Since $g\widetilde{E_4}, gT_2\widetilde{E_4} \in M_6(\Gamma_0(2))$, they are linear combinations of $\widetilde{E_6}(\tau)$ and $\widetilde{E_6}(2\tau)$. A direct computation shows that

$$g\widetilde{E_4}(\tau) = \frac{11}{240}\widetilde{E_6}(\tau) - \frac{2}{15}\widetilde{E_6}(2\tau), \tag{37}$$

$$gT_2\widetilde{E_4}(\tau) = \frac{91}{240}\widetilde{E_6}(\tau) - \frac{7}{15}\widetilde{E_6}(2\tau). \tag{38}$$

Combining (36), (37), and (38), we obtain

(39)

$$\begin{aligned} & gT_2^n\widetilde{E_4} \\ &= \frac{1}{7}(8 - 8^n)(\frac{11}{240}\widetilde{E_6}(\tau) - \frac{2}{15}\widetilde{E_6}(2\tau)) + \frac{1}{7}(8^n - 1)T_2(\frac{91}{240}\widetilde{E_6}(\tau) - \frac{7}{15}\widetilde{E_6}(2\tau)) \\ &= (-\frac{1}{560} + \frac{8^n}{21})\widetilde{E_6}(\tau) + (-\frac{9}{105} - \frac{8^n}{21})\widetilde{E_6}(2\tau). \end{aligned}$$

Comparing the coefficients of q^N $(N \geq 2)$ on both sides of (39), we get

$$\sum_{\substack{k+l=N \\ k,l>0}} \sigma_{1,1}(k;2)\sigma_3(2^n l) + \frac{1}{24}\sigma_3(2^n N) + \frac{1}{240}\sigma_{1,1}(N;2)$$
$$= (-\frac{1}{560} + \frac{8^n}{21})\sigma_5(N) + (-\frac{3}{35} - \frac{8^n}{21})\sigma_5(\frac{N}{2}).$$

This yields Corollary 4.8(b).

5. Formulation of $\sum_{k=1}^{M-1} \tilde{\sigma_1}(2^{n-m}k)\tilde{\sigma_3}(2^n(M-k))$

In Hahn's paper, we see that

$$16\sum_{k<N} \widetilde{\sigma}_1(k)\widetilde{\sigma}_3(N-k) = -\widetilde{\sigma}_5(N) + 2(N-1)\widetilde{\sigma}_3(N) + \widetilde{\sigma}_1(N) \tag{40}$$

by [12, Theorem 4.3]. Now we generalize (40) so that we can use Theorem 4.6 and find the convolution formula for the summation

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^n(N-k))$$

with N being odd.

Theorem 5.1. *Let N be any positive integer. If m $(0 \leq m \leq n)$ is any positive integer with $n \in \mathbb{N} \cup \{0\}$, then*

(a)

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^n(N-k))$$
$$= \frac{1}{112 \cdot 2^m}\{(16^{n+1} - 2^{3n+m+4} - 2^{n+4} + 9 \cdot 2^m)\sigma_5(N)$$
$$+ 16(27 \cdot 2^m + 2^n + 2^{3n+m} - 16^n)\sigma_5(\frac{N}{2})$$
$$+ 2(2^{3(n+1)} - 15)(2^m - 2^n N)\sigma_3(N) + 16(2^{3n} - 15)(-2^m + 2^n N)\sigma_3(\frac{N}{2})$$
$$+ 7(3 \cdot 2^m - 2^{n+1})\sigma_1(N) - 14(3 \cdot 2^m - 2^n)\sigma_1(\frac{N}{2})\}.$$

(b)

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_3(2^{n-m}k)\widetilde{\sigma}_1(2^n(N-k))$$
$$= \frac{1}{112 \cdot 2^{3m}}\{(16^{n+1} - 2^{3n+4} - 2^{n+3m+4} + 9 \cdot 2^{3m})\sigma_5(N)$$
$$+ 16(27 \cdot 2^{3m} + 2^{3n} + 2^{3m+n} - 16^n)\sigma_5(\frac{N}{2})$$
$$+ 2(2^{3(n+1)} - 15 \cdot 2^{3m})(1 - 2^n N)\sigma_3(N)$$

$$+ 16(15 \cdot 2^{3m} - 2^{3n})(1 - 2^n N)\sigma_3(\frac{N}{2})$$
$$- 7 \cdot 8^m (2^{n+1} - 3)\sigma_1(N) + 7 \cdot 2^{3m+1}(2^n - 3)\sigma_1(\frac{N}{2})\}.$$

Proof. (a) If $n = 0$ in Theorem 5.1(a), then Theorem 5.1(a) is equivalent to Eq. (40). So we assume $n \geq 1$. We can expand the convolution sum as

(41)
$$\sum_{k=1}^{N-1} \widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^n(N-k))$$
$$= \sum_{k=1}^{N-1} \{\sigma_1(2^{n-m}k) - 4\sigma_1(2^{n-m-1}k)\}\{\sigma_3(2^n(N-k)) - 16\sigma_3(2^{n-1}(N-k))\}$$

by using $\widetilde{\sigma}_s(N) = \sigma_s(N) - 2^{s+1}\sigma_s(N/2)$, where $N \in \mathbb{N}$ (see [12, (1.12)]). Then the right-hand side of (41) can be written as

(42)
$$\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^{n-1}(N-k))$$
$$- 4\sum_{k=1}^{N-1} \sigma_1(2^{n-m-1}k)\sigma_3(2^n(N-k)) + 64\sum_{k=1}^{N-1} \sigma_1(2^{n-m-1}k)\sigma_3(2^{n-1}(N-k))$$
$$= \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^{n-1}(N-k))$$
$$+ \sum_{k=1}^{N-1} \{2\sigma_1(2^{n-m+1}k) - 6\sigma_1(2^{n-m}k)\}\sigma_3(2^n(N-k))$$
$$+ \sum_{k=1}^{N-1} \{-32\sigma_1(2^{n-m+1}k) + 96\sigma_1(2^{n-m}k)\}\sigma_3(2^{n-1}(N-k)),$$

where an elementary formula $\sigma_1(2N) = 3\sigma_1(N) - 2\sigma_1(N/2)$ is used with $N \in \mathbb{N}$. Now, Eq. (42) becomes

(43)
$$\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16\sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)}k)\sigma_3(2^{n-1}(N-k))$$
$$+ 2\sum_{k=1}^{N-1} \sigma_1(2^{n-(m-1)}k)\sigma_3(2^n(N-k)) - 6\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k))$$
$$- 32\sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-2)}k)\sigma_3(2^{n-1}(N-k))$$

$$+ 96 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)}k)\sigma_3(2^{n-1}(N-k))$$

$$= -5 \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k))$$

$$+ 80 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)}k)\sigma_3(2^{n-1}(N-k))$$

$$+ 2 \sum_{k=1}^{N-1} \sigma_1(2^{n-(m-1)}k)\sigma_3(2^n(N-k))$$

$$- 32 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-2)}k)\sigma_3(2^{n-1}(N-k)).$$

Finally, we apply Theorem 4.6(a) to get the result.

(b) This is similar to Theorem 5.1(a). □

Remark 5.2. Let $\sigma_s^*(N) := \sum_{\substack{d|N \\ \frac{N}{d} \text{ odd}}} d^s$. For instance, the function $\sigma_s^*(N)$ has the formula [20, p. 27]

$$\sigma_s^*(N) = \sigma_s(N) - \sigma_s(\frac{N}{2}). \tag{44}$$

Using (44), we can rewrite Theorem 5.1 as

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_1(2^{n-m}k)\widetilde{\sigma}_3(2^n(N-k))$$

$$= \frac{1}{16}\{-\widetilde{\sigma}_5(N) + 2(N-1)\widetilde{\sigma}_3(N) + \widetilde{\sigma}_1(N)\} + \frac{1}{56}(2^{n-m}-1)\{8(8^n-1)\sigma_5^*(N)$$

$$+ 15N\sigma_{3,1}(N;2) - 8^{n+1}N\sigma_3^*(N) - 7\sigma_1^*(N)\} - \frac{1}{7}(8^n-1)(N-1)\sigma_3^*(N),$$

$$\sum_{k=1}^{N-1} \widetilde{\sigma}_3(2^{n-m}k)\widetilde{\sigma}_1(2^n(N-k))$$

$$= \frac{1}{16}\{-\widetilde{\sigma}_5(N) + 2(N-1)\widetilde{\sigma}_3(N) + \widetilde{\sigma}_1(N)\}$$

$$+ \frac{1}{7}(8^{n-m}-1)\{(2^n-1)\sigma_5^*(N) - (2^nN-1)\sigma_3^*(N)\}$$

$$+ \frac{1}{8}(2^n-1)\{N\widetilde{\sigma}_3(N) - \sigma_1^*(N)\}.$$

Acknowledgements. The authors are enormously grateful to the anonymous referee whose comments and suggestions lead to a large improvement of the paper.

References

[1] A. Alaca, S. Alaca, and K. S. Williams, *The convolution sum* $\sum_{m<n/16}\sigma(m)\sigma(n-16m)$, Canad. Math. Bull. **51** (2008), no. 1, 3–14.
[2] ———, *The convolution sums* $\sum_{l+24m=n}\sigma(l)\sigma(m)$ *and* $\sum_{3l+8m=n}\sigma(l)\sigma(m)$, Math. J. Okayama Univ. **49** (2007), 93–111.
[3] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989.
[4] B. Cho, D. Kim, and J.-K. Koo, *Divisor functions arising from q-series*, Publ. Math. Debrecen **76** (2010), no. 3-4, 495–508.
[5] ———, *Modular forms arising from divisor functions*, J. Math. Anal. Appl. **356** (2009), no. 2, 537–547.
[6] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Springer-Verlag, 2005.
[7] L. E. Dickson, *History of the Theory of Numbers. Vol. I*: *Divisibility and Primality*, Chelsea Publishing Co., New York 1966.
[8] N. J. Fine, *Basic Hypergeometric Series and Applications*, American Mathematical Society, Providence, RI, 1988.
[9] J. W. L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 156–163.
[10] ———, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1–20.
[11] ———, *Expressions for the five powers of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **15** (1885), 33–36.
[12] H. Hahn, *Convolution sums of some functions on divisors*, Rocky Mountain J. Math. **37** (2007), no. 5, 1593–1622.
[13] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, Number theory for the millennium, II (Urbana, IL, 2000), 229–274, A K Peters, Natick, MA, 2002.
[14] D. Kim and M. Kim, *Divisor functions and Weierstrass functions arising from q series*, Bull. Korean Math. Soc. **49** (2012), no. 4, 693–704.
[15] M.-S. Kim, *A p-adic view of mutiple sums of powers*, Int. J. Number Theory **7** (2011), no. 8, 2273–2288.
[16] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1993.
[17] S. Lang, *Elliptic Functions*, Addison-Wesly, 1973.
[18] G. Melfi, *On Some Modular Identities*, Number theory (Eger, 1996), 371–382, de Gruyter, Berlin, 1998.
[19] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.
[20] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, 2011.

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