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CUBIC SYMMETRIC GRAPHS OF ORDER $10p^3$

Mohsen Ghasemi

Abstract. An automorphism group of a graph is said to be s-regular if it acts regularly on the set of s-arcs in the graph. A graph is s-regular if its full automorphism group is s-regular. In the present paper, all s-regular cubic graphs of order $10p^3$ are classified for each $s \geq 1$ and each prime p.

1. Introduction

Throughout this paper, we consider finite connected graphs without loops or multiple edges. For a graph X, every edge of X gives rise to a pair of opposite arcs. By $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$, we denote the vertex set, the edge set, the arc set and the automorphism group of the graph X, respectively. The *neighborhood* of a vertex $v \in V(X)$, denoted by $N(v)$, is the set of vertices adjacent to v in X. Let a group G act on a set Ω, and let $\alpha \in \Omega$. We denote by G_α the *stabilizer* of α in G, that is, the subgroup of G fixing α. The group G is said to be *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$, and *regular* if G is semiregular and transitive on Ω. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{X}$ is called the *covering graph* and X the *base graph*. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian, then $\widetilde{X}$ is called a *cyclic* or an *elementary abelian covering* of X, respectively. If $\widetilde{X}$ is connected, then K is the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while an element of the covering

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transformation group fixes each fibre setwise. The set of all fibre-preserving automorphisms forms a group called the *fibre-preserving group*.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words, it is a directed walk of length s which never includes a backtracking. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is said to be *edge-transitive* if $\mathrm{Aut}(X)$ is transitive on $E(X)$ and *half-arc-transitive* if X is vertex-transitive and edge-transitive, but not arc-transitive. A subgroup of the automorphism group of a graph X is said to be *s-regular* if it acts regularly on the set of s-arcs of X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X, then X is said to be *s-regular*. Thus, if a graph X is s-regular, then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X. A regular edge- but not vertex-transitive graph will be referred to as a *semisymmetric* graph.

Clearly, a cycle is s-arc-transitive for any $s \geq 0$. Tutte [40, 41] showed that every finite connected cubic symmetric graph is s-regular for some $s \geq 1$ and that this s is at most five. Djoković and Miller [10] constructed an infinite family of cubic 2-regular graphs, and Conder and Praeger [7] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. Several different types of infinite families of tetravalent 1-regular graphs have also been constructed in [29, 33, 38]. The first cubic 1-regular graph was constructed by Frucht [20] and an infinitely many cubic 1-regular graphs of girth 6 were constructed later by Miller [37]. From Cheng and Oxley's classification of symmetric graphs of order $2p$ [5], it can be shown that Miller's construction contains all cubic 1-regular graphs of order $2p$, where $p \geq 13$ is a prime congruent to 1 modulo 3. Marušič and Xu [36] showed a way to construct a cubic 1-regular graph Y from a tetravalent half-arc-transitive graph X with girth 3 by letting the triangles of X be the vertices in Y with two triangles being adjacent whenever they share a common vertex in X. Using Marušič and Xu's result, Miller's construction can be generalized to graphs of order $2n$, where $n \geq 13$ is odd such that 3 divides $\varphi(n)$, the Euler function (see [2, 35]). It may be shown that all cubic 1-regular Cayley graphs on the dihedral groups (see [35]) are exactly those graphs generalized by Miller's construction. Additionally, more cubic 1-regular graphs were constructed by Feng and Kwak [12, 13, 14]. Also, as shown in [35] or in [34], one can see an importance in studying cubic 1-regular graphs in connection with chiral (that is, regular and irreflexible) maps on a surface by means of tetravalent half-arc-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus.

Regular coverings of a graph have received considerable attention. For example, consider the complete graph K_4, the complete bipartite graph $K_{3,3}$, the hypercube Q_3 or the Petersen graph O_3 as graph X. The s-regular cyclic or elementary abelian coverings of X, whose fibre-preserving groups are arc-transitive, have been classified for each $1 \leq s \leq 5$ in refs. [15, 16, 17, 19]. As an application of these classifications, all s-regular cubic graphs of orders $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, and $10p^2$ have been constructed for each $1 \leq s \leq 5$ and each prime p in refs. [15, 16, 17].

Malnič et al. [28] classified the cubic semisymmetric cyclic coverings of the bipartite graph $K_{3,3}$ when the fibre-preserving group contains an edge- but not vertex-transitive subgroup. Using the covering technique, cubic semisymmetric graphs of orders $8p^2$, $6p^2$ and $2p^3$ were classified in [1, 23, 30]. Some general methods of elementary abelian coverings were developed in [11, 26, 27]. Using the covering technique, Malnič and Potočnik [32] classified the vertex-transitive elementary abelian coverings of the Petersen graph when the fibre-preserving group is vertex-transitive. To investigate the s-regular $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$-, $G_1(p) = \langle a, b \mid cma^p = b^p = c^p = 1, c = [a,b], ac = ca, bc = cb\rangle$- or $G_2(p) = \langle a, b \mid a^{p^2} = b^p = 1, [a,b] = a^p\rangle$-coverings of the Petersen graph O_3, we will assume that the fibre-preserving group is arc-transitive. Since the s-regular cyclic or elementary abelian coverings of the Petersen graph O_3 are classified for each $1 \leq s \leq 5$ in [16], we only classify the s-regular $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$-, $G_1(p) = \langle a, b \mid a^p = b^p = c^p = 1, c = [a,b], ac = ca, bc = cb\rangle$- and $G_2(p) = \langle a, b \mid a^{p^2} = b^p = 1, [a,b] = a^p\rangle$-coverings of the Petersen graph O_3 for each $1 \leq s \leq 5$. As an application of these classifications, this paper provides a classification of s-regular cubic graphs of order $10p^3$ for each $1 \leq s \leq 5$ and each prime p.

The following theorem is the main result of this paper.

Theorem 1.1. *A graph X is a cubic connected symmetric graph of order $10p^3$ for some prime p if and only if X is isomorphic to $C80.1$ ($p = 2$, 3-regular), $C1250.1$ ($p = 5$, 2-regular) or $C1250.2$ ($p = 5$, 3-regular).*

2. Preliminaries related to coverings

Let X be a graph and K a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or *K-voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages* and K the *voltage group*. The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$ so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p : X \times_\phi K \to X$, which is called the *natural projection*. By defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering*. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of

X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [21] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ or a projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are, of course, subgroups in $\mathrm{Aut}(\widetilde{X})$ and in $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is, $K=\{\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})\colon\ p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of α, then $K\tilde{\alpha}$ are all the lifts of α.

Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. The problem of whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\bar{\alpha}} = \phi(C^{\alpha}),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^{\alpha})$ are the voltages on C and C^{α}, respectively. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of [24, Theorem 3.5].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then an automorphism α of X lifts if and only if $\bar{\alpha}$ extends to an automorphism of K.*

For more results on the lifts of automorphisms of X, we refer the reader to [3, 4, 9, 25, 31]. Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by X_N the quotient graph corresponding to the orbits of N, that is, the graph having the orbits of N as vertices with two orbits adjacent in X_N whenever there is an edge between these orbits in X. In view of [22, Theorem 9], we have the following.

Proposition 2.2. *Let X be a cubic connected symmetric graph and G an s-regular subgroup of* $\mathrm{Aut}(X)$ *for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular* *subgroup of* $\mathrm{Aut}(X_N)$*, where* X_N *is the quotient graph of* X *corresponding to the orbits of* N*. Furthermore,* X *is a regular covering of* X_N *with the covering transformation group* N*.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2, respectively, are said to be *equivalent* if there exists a graph isomorphism $\tilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{\alpha}p_2 = p_1$. We quote the following proposition.

Proposition 2.3 ([39])**.** *Two connected regular coverings* $X \times_\phi K$ *and* $X \times_\psi K$*, where* ϕ *and* ψ *are* T*-reduced, are equivalent if and only if there exists an automorphism* $\sigma \in \mathrm{Aut}(K)$ *such that* $\phi(u,v)^\sigma = \psi(u,v)$ *for any cotree arc* (u,v) *of* X*.*

3. Regular coverings of O_3 and related classification

As it is well-known, there are exactly five nonisomorphic groups of order p^3, which may be given in the following presentation.

(i) For abelian cases:

$G_1 = \mathbb{Z}_{p^3}$;

$G_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$;

$G_3 = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$;

(ii) For non-abelian cases:

$G_1(p) = \langle a, b \mid a^p = b^p = c^p = 1, c = [a,b], ac = ca, bc = cb\rangle$;

$G_2(p) = \langle a, b \mid a^{p^2} = b^p = 1, [a,b] = a^p\rangle$.

Recall that since the s-regular cyclic or elementary abelian coverings of the Petersen graph O_3 are classified for each $1 \le s \le 5$ in [16], we only classify the s-regular $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$-, $G_1(p)$- or $G_2(p)$-coverings of the Petersen graph O_3 for each $1 \le s \le 5$. As an application of these classifications, we classify s-regular cubic graphs of order $10p^3$ for each $1 \le s \le 5$ and each prime p.

By O_3 we denote the Petersen graph with vertex set $\{a, b, c, d, e, u, v, w, x, y\}$. Let T be a spanning tree of O_3, as shown by dashed lines in Fig. 1. Let ϕ be a such voltage assignment defined by $\phi = 1$ on T and $\phi = z_1, z_2, z_3, z_4$, z_5, and z_6 on the cotree arcs $(\mathbf{u}, \mathbf{v})$, $(\mathbf{a}, \mathbf{c})$, $(\mathbf{a}, \mathbf{d})$, $(\mathbf{b}, \mathbf{e})$, $(\mathbf{b}, \mathbf{v})$, and $(\mathbf{c}, \mathbf{w})$, respectively. Let $\alpha = (\mathbf{abcde})(\mathbf{uvwxy})$, $\beta = (\mathbf{vay})(\mathbf{bcx})(\mathbf{wde})$, and $\gamma = (\mathbf{ex})(\mathbf{bw})(\mathbf{cd})$. Then α, β, and γ are automorphisms of O_3.

Denote by $i_1 i_2 \cdots i_s$ a directed cycle having vertices i_1, i_2, …, i_s in a consecutive order. There are six fundamental cycles **auvwxyu**, **aceyu**, **adxyu**, **auyxdbeyu**, **auyxdbvwxyu**, and **auyecwxyu** in O_3, which are generated by the five cotree arcs $(\mathbf{u}, \mathbf{v})$, $(\mathbf{a}, \mathbf{c})$, $(\mathbf{a}, \mathbf{d})$, $(\mathbf{b}, \mathbf{e})$, $(\mathbf{b}, \mathbf{v})$, and $(\mathbf{c}, \mathbf{w})$, respectively. Each cycle is mapped to a cycle of the same length under the actions of α, β, and γ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of O_3 and $\phi(C)$ denotes the voltage of C. Also note that for abelian cases we use additive symbol.

By Conder [6], there is only one cubic connected symmetric graph of order 80, namely, a 3-regular graph $C80.1$. Also for $p = 3$, there is no cubic symmetric graph of order 10×3^3. Thus we may assume that $p \ge 5$.

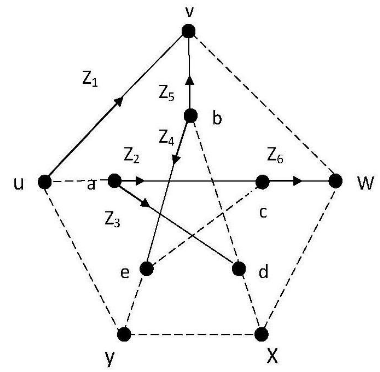

FIGURE 1. The Petersen graph (O_3) with voltage assignment ϕ.

TABLE 1. Fundamental cycles and their images with corresponding voltages

C	$\phi(C)$	C^α	$\phi(C^\alpha)$
$auvwxyu$	z_1	$bvwxyuv$	$z_5z_1z_5^{-1}$
$aceyu$	z_2	$bdauv$	$z_3^{-1}z_1z_5^{-1}$
$adxyu$	z_3	$beyuv$	$z_4z_1z_5^{-1}$
$auyxdbeyu$	z_4	$bvuyecauv$	$z_5z_1^{-1}z_2^{-1}z_1z_5^{-1}$
$auyxdbvwxyu$	z_5	$bvuyecwxyuv$	$z_5z_1^{-1}z_6z_1z_5^{-1}$
$auyecwxyu$	z_6	$bvuadxyuv$	$z_5z_1^{-1}z_3z_1z_5^{-1}$
C^β	$\phi(C^\beta)$	C^γ	$\phi(C^\gamma)$
$yuadbvu$	$z_3z_5z_1^{-1}$	$auvbeyu$	$z_1z_5^{-1}z_4$
$yxwvu$	z_1^{-1}	$adxyu$	z_3
$yebvu$	$z_4^{-1}z_5z_1^{-1}$	$aceyu$	z_2
$yuvbecwvu$	$z_1z_5^{-1}z_4z_6z_1^{-1}$	$auyecwxyu$	z_6
$yuvbecadbvu$	$z_1z_5^{-1}z_4z_2^{-1}z_3z_5z_1^{-1}$	$auyecwvbeyu$	$z_6z_5^{-1}z_4$
$yuvwxdbvu$	$z_1z_5z_1^{-1}$	$auyxdbeyu$	z_4

Lemma 3.1. *There is no connected regular covering of the Petersen graph O_3 whose covering transformation group K is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \cong \langle a\rangle \times \langle b\rangle$ with $o(a) = p^2$ and $o(b) = p$ and whose fibre-preserving group is arc-transitive.*

Proof. Let $\widetilde{X} = O_3 \times_\phi (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ be a covering graph of the graph O_3 satisfying the hypotheses in the theorem, where p is a prime and $\phi = 1$ on the spanning tree T, which is depicted by dashed lines in Fig. 1. We assign voltages z_1, z_2, z_3, z_4, z_5 and z_6 to the cotree arcs as shown in Fig. 1, where $z_i \in K$ (i=1, 2, 3, 4, 5, 6). Note that the vertices of O_3 are labeled by **a**, **b**, **c**, **d**, **e**, **u**, **v**, **w**, **x**, and **y**. By the hypotheses, the fibre-preserving group, say $\widetilde{L}$, of the covering graph $O_3 \times_\phi (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ acts arc-transitively on $O_3 \times_\phi (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. Hence the projection of $\widetilde{L}$, say L, is arc-transitive on the base graph O_3. Clearly, L is also vertex-transitive on O_3. Thus α, $\beta \in L$ and so α and β lift to automorphisms of $O_3 \times_\phi (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. Also, since $O_3 \times_\phi (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is assumed to be connected, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle z_1, z_2, z_3, z_4, z_5, z_6 \rangle$.

Consider the mapping $\bar{\alpha}$ from the set $\{z_1, z_2, z_3, z_4, z_5, z_6\}$ of voltages of the six fundamental cycles of O_3 to the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, which is defined by $(\phi(C))^{\bar{\alpha}} = \phi(C^\alpha)$, where C ranges over the six fundamental cycles. From Table 1, one can see that $z_1^{\bar{\alpha}} = z_1$, $z_2^{\bar{\alpha}} = z_3^{-1} z_1 z_5^{-1}$, $z_3^{\bar{\alpha}} = z_4 z_1 z_5^{-1}$, $z_4^{\bar{\alpha}} = z_2^{-1}$, $z_5^{\bar{\alpha}} = z_6$, and $z_6^{\bar{\alpha}} = z_3$. Similarly, we can define $\bar{\beta}$ and $\bar{\gamma}$. Since $\alpha, \beta \in L$, Proposition 2.1 implies that $\bar{\alpha}$ and $\bar{\beta}$ can be extended to automorphisms of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. We denote by α^* and β^* these extended automorphisms, respectively. By Table 1, $z_4^{\alpha^*} = z_2^{-1}$, $z_5^{\alpha^*} = z_6$, $z_6^{\alpha^*} = z_3$ and $z_2^{\beta^*} = z_1^{-1}$, implying that $o(z_3) = o(z_5) = o(z_6)$ and $o(z_1) = o(z_2) = o(z_4)$, where $o(z)$ denotes the order of $z \in K$. Assume that $K = \langle z_1, z_5 \rangle$. Suppose that $o(z_1) = p^2$ and $o(z_5) = p$. If $\langle z_1 \rangle \cap \langle z_5 \rangle \neq \emptyset$, then $\langle z_5 \rangle$ is a subgroup of $\langle z_1 \rangle$ and $K = \langle z_1 \rangle$, a contradiction. Thus $\langle z_1 \rangle \cap \langle z_5 \rangle = \emptyset$. By Proposition 2.3, one may let $z_1 = a$ and $z_5 = b$ and hence $z_2 = z_1^i z_5^j$, $z_3 = z_1^m z_5^n$, $z_4 = z_1^l z_5^k$, and $z_6 = z_1^x z_5^y$, where $i, m, l, x \in \mathbb{Z}_{p^2}$ and $j, n, k, y \in \mathbb{Z}_p$. By considering the image of $z_2 = z_1^i z_5^j$, and $z_3 = z_1^m z_5^n$ under α^*, we conclude that $z_3^{-1} z_1 z_5^{-1} = z_1^i z_6^j$ and $z_4 z_1 z_5^{-1} = z_1^m z_6^n$. Therefore, we have the following:

(1) $1 - m = i + jx \pmod{p^2}$,
(2) $-n - 1 = jy$,
(3) $1 + l = m + nx \pmod{p^2}$,
(4) $k - 1 = ny$.

As shown in the above equations, in what follows, all equations (unless specified with modulo p^2) are to be taken mod p, but the symbol mod p is omitted. This should cause no confusion. Similarly, by considering the image of $z_4 = z_1^l z_5^k$ and $z_6 = z_1^x z_5^y$ under α^*, we have the following:

(5) $-i = l + kx \pmod{p^2}$,
(6) $-j = ky$,
(7) $m = x + yx \pmod{p^2}$,
(8) $n = y^2$.

By (8), $n = y^2$. Thus by (2) and (4), we have $-y^2 - 1 = jy$ and $k - 1 = y^3$. Now by (6), we have $-y^2 - 1 = -(y^3 + 1)y^2$, so $y = 1$. This implies that $n = 1$, $k = 2$, and $j = -2$. By (7) and (3), $m = 2x$, and $l = 3x - 1$, respectively. So by (1), $i = 1$ and hence by (5), $x = 0$ or $5 = 0 \pmod{p}$. If $x = 0$, then $l = -1$ and $m = 0$. By considering the image of $z_2 = z_1^i z_5^j$ under β^*, we have $-1 = im - i + jl - ji + jm$ and $ni + i + kj - j^2 + nj = 0$. Therefore, $4 = 0$ (by $-1 = im - i + jl - ji + jm$), a contradiction. If $5 = 0 \pmod{p}$, then $-1 = 2x - 1 - 2(3x - 1) + 2 - 4x$ (by $-1 = im - i + jl - ji + jm$). So $x = 1/2 = 3$ and hence $m = 1$ and $l = 2$. Now by $ni + i + kj - j^2 + nj = 0$, $8 = 0$, a contradiction.

For the case when $o(z_1) = p$ and $o(z_5) = p^2$, we have a similar contradiction. Now let $o(z_1) = o(z_5) = p^2$. Then $\langle z_1\rangle \cap \langle z_5\rangle = \langle z_1^p\rangle = \langle z_5^p\rangle$, and hence $z_1^{rp} = z_5^p$ for some $r \in \mathbb{Z}_p^*$. By Proposition 2.3, one may let $z_1 = a$, $z_5 = a^r b$, $z_2 = z_1^{i-jr} z_5^j$, $z_3 = z_1^{m-nr} z_5^n$, $z_4 = z_1^{l-kr} z_5^k$, and $z_6 = z_1^{x-yr} z_5^y$. Considering the image of $z_2 = z_1^{i-jr} z_5^j$ under α^* and β^*, by Table 1, $z_3^{-1} z_1 z_5^{-1} = z_1^{i-jr} z_6^j$ and $z_1^{-1} = z_3^{i-jr} z_5^{i-jr} z_1^{-i+jr} z_4^j z_2^{-j} z_3^j$, which implies the following equations:

(1) $1 - m - r = i - jr + jx \pmod{p^2}$,

(2) $-n - 1 = jy$.

Also, by considering the image $z_3 = z_1^{m-nr} z_5^n$ and $z_4 = z_1^{l-kr} z_5^k$ under α^* and $z_6 = z_1^{x-yr} z_5^y$ under α^* and β^*, we have the following:

(3) $1 + l - r = m - nr + nx \quad \pmod{p^2}$,

(4) $k - 1 = ny$,

(5) $-i = l - kr + kx \quad \pmod{p^2}$,

(6) $-j = ky$,

(7) $m = x - ry + yx \quad \pmod{p^2}$,

(8) $n = y^2$,

(9) $1 = xn + x - ryn - ry + yk - yj + yn$.

By (6), $-jy = ky^2$. Now by (2), $n + 1 = ky^2$. By (7) and (4), $k = y^3 + 1$. Thus $n + 1 = (y^3 + 1)y^2 = y^5 + y^2$. So $y^2 + 1 = y^5 + y^2$. This implies that $y = 1$, and hence $n = 1$, $k = 2$ and $j = -2$. Now by (1), (3), (5), and (9), we have the following equations:

(a) $1 - m - r = i + 2r - 2x$,

(b) $1 + l = m + x$,

(c) $-i = l - 2r + 2x$,

(d) $m = 2x - r$.

By (b), $l = m + x - 1$. So by (d), we have $l = 3x - r - 1$ and hence by (c), $i = 3r - 5x + 1$. Now by (a), one has $x = r$. Now by (8), $4 = 0$, a contradiction.

Now assume that $|\langle z_1, z_5\rangle| = p$. Thus $\langle z_1\rangle = \langle z_5\rangle$. It follows that $\langle z_1\rangle = \langle z_2\rangle = \langle z_3\rangle = \langle z_4\rangle = \langle z_5\rangle = \langle z_6\rangle$. Therefore, K is generated by one of the z_i $(1 \leq i \leq 6)$, a contradiction.

Finally, assume that $|\langle z_1, z_5\rangle| = p^2$. Since α lifts, by Table 1, $|\langle z_1, z_6\rangle| = p^2$. Since $|\langle z_1, z_5\rangle| = p^2$ by Proposition 2.3, we may assume that $z_1 = a$. X is connected and hence one of the z_2, z_3, z_4 or z_6 must be equal to b. If $z_3 = b$ or $z_5 = b$, then $K = \langle z_1, z_5\rangle$ or $K = \langle z_1, z_6\rangle$, a contradiction. Thus $z_2 = b$ or $z_4 = b$. Without loss of generality, we may assume that $z_2 = b$. So $K = \langle z_1, z_2\rangle = \langle z_1, z_4\rangle$. Thus $\langle z_1\rangle \cap \langle z_2\rangle = \langle z_1^p\rangle = \langle z_2^p\rangle$, and $\langle z_1\rangle \cap \langle z_4\rangle = \langle z_1^p\rangle = \langle z_4^p\rangle$, and hence $z_1^{r'p} = z_2^p$ and $z_1^{rp} = z_4^p$ for some $r, r' \in \mathbb{Z}_p^*$. By Proposition 2.3, one may let $z_1 = a$, $z_2 = a^{r'} b$, $z_4 = a^r b$, $z_3 = a^{i-jr} b^j$, $z_5 = a^{m-nr} b^n$, and $z_6 = a^{x-yr} b^y$. By considering the image of $z_5 = a^{m-nr} b^n$ and $z_6 = a^{x-yr} b^y$ under α^*, one has $z_6 = z_1^{m-nr} z_2^{-n}$ and $z_3 = z_1^{x-ry} z_2^{-y}$. Therefore, we have the following equations:

(1) $y = -n$,

(2) $j = -y$.

Similarly, by considering the image of $z_3 = a^{i-jr}b^j$ under α^*, we get $1-n = -j$. So $1 = 0$, a contradiction. □

Lemma 3.2. *There is no connected regular covering of the Petersen graph O_3 whose covering transformation group K is isomorphic to $G_1(p) = \langle a, b \mid a^p = b^p = c^p = 1, c = [a,b], ac = ca, bc = cb\rangle$ and whose fibre-preserving group is arc-transitive.*

Proof. Let $\widetilde{X} = O_3 \times_\phi G_1(p)$ be a covering graph of the graph O_3 satisfying the hypotheses in the theorem, where p is a prime and $\phi = 1$ on the spanning tree T depicted by dashed lines in Fig. 1. Since X is connected, K can be generated by z_1, z_2, z_3, z_4, z_5, and z_6. By the hypotheses, the fibre-preserving subgroup, say $\widetilde{L}$, of the covering graph $O_3 \times_\phi G_1(p)$ acts arc-transitively on $O_3 \times_\phi G_1(p)$. Hence the projection, say L of $\widetilde{L}$, is arc-transitive on the base graph O_3. Thus α, $\beta \in L$. It follows that α and β lift. Since α, $\beta \in L$, Proposition 2.1 implies that $\bar{\alpha}$ and $\bar{\beta}$ can be extended to automorphisms of $\langle a, b \mid a^p = b^p = c^p = 1, c = [a,b], ac = ca, bc = cb\rangle$. We denote by α^* and β^* these extended automorphisms, respectively. By Table 1, $z_6^{\alpha^*} = z_5 z_1^{-1} z_3 z_1 z_5^{-1}$ and $z_5^{\alpha^*} = z_5 z_1^{-1} z_6 z_1 z_5^{-1}$. Also $z_4^{\alpha^*} = z_5 z_1^{-1} z_2^{-1} z_1 z_5^{-1}$ and $z_2^{\beta^*} = z_1^{-1}$. Thus $o(z_3) = o(z_5) = o(z_6)$ and $o(z_1) = o(z_2) = o(z_4)$, where $o(z)$ denotes the order of $z \in K$.

First assume that $K = \langle z_1, z_2\rangle$ and $z_1 = a^{i'} b^{j'} c^{k'}$, $z_2 = a^{l'} b^{m'} c^{n'}$. Since $a^{i'} b^{j'} c^{k'} \mapsto a$, $a^{l'} b^{m'} c^{n'} \mapsto b$ extend to automorphism of K, by Proposition 2.3, one may let $z_1 = a$, $z_2 = b$, $z_3 = a^i b^j c^k$, $z_4 = a^l b^m c^n$, $z_5 = a^f b^g c^r$, and $z_6 = a^x b^y c^z$ $(i, j, k, l, m, n, f, g, r, x, y, z \in \mathbb{Z}_p)$.

Now by Table 1, $(z_1)^{\alpha^*} = (a)^{\alpha^*} = z_5 z_1 z_5^{-1} = a^f b^g c^r a c^{-r} b^{-g} a^{-f}$. By considering $b^j a^i = c^{-ij} a^i b^j$, we have $(a)^{\alpha^*} = ac^{-g}$ and

$$\begin{aligned}
(z_2)^{\alpha^*} = (b)^{\alpha^*} &= z_3^{-1} z_1 z_5^{-1} = c^{-k} b^{-j} a^{-i} a c^{-r} b^{-g} a^{-f} \\
&= c^{-k} a^{1-i} b^{-j} c^{-r} b^{-g} a^{-f} c^{(1-i)j} \\
&= c^{-k} a^{1-i} c^{-r} c^{(1-i)j} a^{-f} b^{-j-g} c^{-f(j+g)} \\
&= a^{1-i-f} b^{-j-g} c^{-k-r+(1-i)j-f(j+g)}.
\end{aligned}$$

Therefore one can get $(c)^{\alpha^*} = [a^{\alpha^*}, b^{\alpha^*}] = c^{-j-g}$. Now since c belongs to $Z(K)$, we have $(a^i)^{\alpha^*} = a^i c^{-gi}$. Also, since $[a^{1-i-f}, b^{-j-g}]$ belongs to $Z(K)$, one has

$$\begin{aligned}
(b^j)^{\alpha^*} &= (a^{1-i-f} b^{-j-g} c^{-k-r+(1-i)j-f(j+g)})^j \\
&= (a^{1-i-f} b^{(-j-g)})^j c^{(-k-r+(1-i)j-f(j+g))j} \\
&= a^{(1-i-f)j} b^{(-j-g)j} [b^{(-j-g)}, a^{1-i-f}]^{\frac{j(j-1)}{2}} c^{(-k-r+(1-i)j-f(j+g))j} \\
&= a^{(1-i-f)j} b^{(-j-g)j} c^{(j+g)(1-i-f)\frac{j(j-1)}{2}} c^{(-k-r+(1-i)j-f(j+g))j}.
\end{aligned}$$

By $(z_3)^{\alpha^*} = (a^i b^j c^k)^{\alpha^*} = z_4 z_1 z_5^{-1}$ and $(z_5)^{\alpha^*} = (a^f b^g c^r)^{\alpha^*} = z_5 z_1^{-1} z_6 z_1 z_5^{-1}$, we have

$$a^i c^{-gi} a^{(1-i-f)j} b^{(-j-g)j} c^{(j+g)(1-i-f)\frac{j(j-1)}{2}} c^{(-k-r+(1-i)j-f(j+g))j} c^{k(-j-g)}$$
$$= a^{l+1-f} b^{m-g} c^{f(m-g)} c^{n-r-m}$$

and

$$a^f c^{-gf} a^{(1-i-f)g} b^{(-j-g)g} c^{(j+g)(1-i-f)\frac{g(g-1)}{2}} c^{(-k-r+(1-i)j-f(j+g))g} c^{(-j-g)r}$$
$$= a^f b^g c^r a^{-1} a^x b^y c^z a c^{-r} b^{-g} a^{-f}$$
$$= a^x b^y c^{-gx-y+yf+z}.$$

Hence by considering the powers of a and b, we have the following:
(1) $i + (1-i-f)j = l + 1 - f$,
(2) $(-j-g)j = m - g$,
(3) $f + (1-i-f)g = x$,
(4) $(-j-g)g = y$.

By similar argument considering the image $z_4 = a^l b^m c^n$ and $z_6 = a^x b^y c^z$ under α^*, and also the image $z_3 = a^i b^j c^k$, $z_4 = a^l b^m c^n$ and $z_5 = a^f b^g c^r$ under β^*, we have the following:
(5) $(-j-g)m = -1$,
(6) $(-j-g)y = j$,
(7) $i(i+f-1) - j = f - l - 1$,
(8) $i(g+j) = g - m$,
(9) $(i+f-1)l - m = x + l - f$,
(10) $(g+j)l = -g + m + y$,
(11) $(g+j)f = j + m - 1$.

By (2) and (8), $(j+g)j = i(g+j)$. So $(g+j)(j-i) = 0$ and hence $g = -j$ or $j = i$. If $j = -g$, then by (5), $1 = 0$, a contradiction. Thus $i = j$. By considering (2), (4), and (10), we have $(g+j)l = -(j+g)j - (g+j)g$. So $(j+g)(l+j+g) = 0$. It follows that $l = -j-g$. Now since $i = j$, by (7) and (9), we have $(i+f-1)(-g) - (j+m) = x - 1$. Now by considering (3), $-f = j + m - 1$ and hence by (11), $-f = (g+j)f$. Thus $f = 0$ or $g + j + 1 = 0$. If $f = 0$, then by (11), $j = 1 - m$. Now by $i = j$ and (1), $l = -m^2$ and so $g = m^2 + m - 1$. By (5), $m = 1$. Therefore, $l = -1$, $j = 0$, $i = 0$, and $g = 1$. So by (3) and (4), $x = 1$ and $y = -1$. Hence by (6), $1 = 0$, a contradiction. Thus $g = -j-1$ and since $l = -j-g$, we have $l = 1$. Also by (4), (5), and (6), we have $y = g$, $m = -1$, and $y = j$. It follows that $j = -1/2$. So $i = -1/2$, $y = -1/2$ and $g = -1/2$. Now by (7), $f = 13/6$ and so by (3), $x = 5/2$. Thus by (11), $-13/6 = -5/2$, a contradiction.

Now assume that $|\langle z_1, z_2\rangle| = p$. Thus $\langle z_1\rangle = \langle z_2\rangle$. Since α lifts, by Table 1, $\langle z_1\rangle = \langle z_4\rangle$. Without loss of generality, we may suppose that $K = \langle z_1, z_5\rangle$ since X is connected. By Proposition 2.3, one may let $z_1 = a$, $z_5 = b$, $z_2 = a^i$, $z_4 = a^j$, $z_3 = a^x b^y c^z$, and $z_6 = a^l b^m c^n$, where $(i,p) = 1$ and $(j,p) = 1$. Now by Table 1, $(z_1)^{\alpha^*} = (a)^{\alpha^*} = z_5 z_1 z_5^{-1} = ac^{-1}$, and $(z_5)^{\alpha^*} =$

$(b)^{\alpha^*} = z_5z_1^{-1}z_6z_1z_5^{-1} = a^lb^mc^nc^{1-l}c^{-1-m}$. So $c^{\alpha^*} = c^m$, $(a^i)^{\alpha^*} = a^ic^{-i}$ and $(b^j)^{\alpha^*} = a^{lj}b^{mj}c^{-lm(\frac{j(j-1)}{2})}c^{nj}c^{(1-l)j}c^{(-1-m)j}$. By $(z_2)^{\alpha^*} = z_3^{-1}z_1z_5^{-1}$ and $(z_4)^{\alpha^*} = z_5z_1^{-1}z_2^{-1}z_1z_5^{-1}$, we have $a^ic^{-i} = a^{1-x}b^{-y-1}c^{-z}c^{(1-x)y}$ and $a^jc^{-j} = a^{-i}c^i$. Therefore, by considering the powers of a and b, we have the following:
(1) $i = 1-x$,
(2) $-y-1 = 0$,
(3) $j = -i$,
(4) $-i = -z + (1-x)y$.
Similarly, by considering the image $z_2 = a^i$ and $z_4 = a^j$ under β^*, we have the following equations:
(5) $(y+1)i = 0$,
(6) $(x-1)j = l + j$,
(7) $(y+1)j = m-1$.
By (2), $j = -1$ and so $i = 1$, by (3). Now by (1), $x = 0$. Also by (5) and (7), $y = -1$ and $m = 1$. Now by (4) and (6), $z = 0$ and $l = 2$. Then

$$z_1 = a, z_2 = a, z_3 = b^{-1}, z_4 = a^{-1}, z_5 = b, z_6 = a^2bc^n.$$

Since α lifts, it follows that $2 = 0$, a contradiction.

Finally, assume that $|\langle z_1, z_2\rangle| = p^2$. Since α lifts, $|\langle z_1, z_4\rangle| = p^2$. By Proposition 2.3, we may assume that $z_1 = a$ and $z_2 = a^ic^j$. Since $\widetilde{X}$ is connected, it implies that $z_3 = b$, $z_5 = b$ or $z_6 = b$. Without loss of generality, we may assume that $z_3 = b$. Thus $K = \langle z_1, z_3\rangle$, and $z_4 = a^{i'}c^{j'}$, $z_5 = a^lb^mc^n$, and $z_6 = a^xb^yc^f$. Now by Table 1, $(z_1)^{\alpha^*} = (a)^{\alpha^*} = z_5z_1z_5^{-1} = ac^{-m}$ and $(z_3)^{\alpha^*} = (b)^{\alpha^*} = z_4z_1z_5^{-1} = a^{i'+1-l}b^{-m}c^{j'-n-ml}$. Therefore, one get $(c)^{\alpha^*} = c^{-m}$. Clearly, $(a^i)^{\alpha^*} = a^ic^{-mi}$ and since $[a^{i'+1-l}, b^{-m}]$ belongs to $Z(K)$, we have $(b^j)^{\alpha^*} = a^{(i'+1-l)j}b^{-mj}c^{m(i'+1-l)}(\frac{j(j-1)}{2})c^{j(j'-n-ml)}$. By $(z_2)^{\alpha^*} = (a^ic^j)^{\alpha^*} = z_3^{-1}z_1z_5^{-1}$, one has $a^ic^{-mi}c^{-mj} = a^{-l+1}b^{-m-1}c^{1-n}c^{-l(1+m)}$ so by considering the power of b in $(z_5)^{\alpha^*} = z_5z_1^{-1}z_6z_1z_5^{-1}$ and $(z_6)^{\alpha^*} = z_5z_1^{-1}z_3z_1z_5^{-1}$, one has $-m^2 = y$ and $-my = 1$. Since $m = -1$ and $y = -1$, it follows that $-1 = 1$, a contradiction. □

Lemma 3.3. *There is no connected regular covering of the Petersen O_3 whose covering transformation group K is isomorphic to $G_2(p) = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p\rangle$ and whose fibre-preserving group is arc-transitive.*

Proof. By the same reasoning as before, α and β lift. Obviously, any automorphism of K is of the form $a \mapsto a^ib^j$, $b \mapsto a^{rp}b$, where $i \in \mathbb{Z}_{p^2}^*$ and $j, r \in \mathbb{Z}_p$. Since $\mathrm{Aut}(K)$ acts transitively on elements of order p^2, one may let $z_1 = a$, $z_2 = a^ib^j$, $z_3 = a^mb^n$, $z_4 = a^kb^l$, $z_5 = a^fb^g$, and $z_6 = a^xb^y$.

Now by Table 1, $(z_1)^{\alpha^*} = (a)^{\alpha^*} = z_5z_1z_5^{-1} = a^fb^gab^{-g}a^{-f}$. By considering $b^ja^i = a^{i-ijp}b^j$, we have $(a)^{\alpha^*} = a^{1-gp}$. Now assume that $(b)^{\alpha^*} = a^{k'p}b$. Then $(b^j)^{\alpha^*} = a^{k'pj}b^j$ since $a^{k'p} \in Z(K)$. By $(z_2)^{\alpha^*} = (a^ib^j)^{\alpha^*} = z_3^{-1}z_1z_5^{-1}$ and

$(z_3)^{\alpha^*} = z_4 z_1 z_5^{-1}$, we have $a^{i-igp}a^{k'pj}b^j = a^{(1-m)+(1-m)np}a^{-f-f(n+g)p}b^{-n-g}$ and $a^{(1-gp)m}a^{k'pn}b^n = a^{k+1-lp}a^{-f+f(l-g)p}b^{l-g}$. Therefore, we have the following:

(1) $i = 1 - m - f$,
(2) $j = -n - g$,
(3) $n = l - g$.

As shown in the above equations, in what follows, all equations are to be taken mod p, but the symbol mod p is to be omitted. This should cause no confusion. By similar argument considering the image $z_5 = a^f b^g$ and $z_6 = a^x b^y$ under α^*, and also the image $z_2 = a^i b^j$ and $z_3 = a^m b^n$ under β^*, we have the following:

(4) $f = x$,
(5) $g = y$,
(6) $x = m$,
(7) $y = n$,
(8) $(n + g)i + j = 0$,
(9) $(n + g)m + n = g - l$.

By (2), $n + g = -j$, so by (8), $-ij + j = 0$. It follows that $j = 0$ or $i = 1$. If $j = 0$, then by (2), $n = -g$. Since $n = g$ by (5) and (7), one has $g = 0$. So $n = 0$ and hence $y = 0$ and $l = 0$. This is a contradiction to the fact that X is connected. Thus $i = 1$, and so by (1), $m = -f$. But by (4) and (6), $f = m$, and hence $m = f = 0$. Now by (3) and (9), $(n + g)m + n = -n$, and so $n = 0$. Therefore, $y = g = l = 0$, a contradiction. □

4. The cubic symmetric graphs of order $10p^3$

Lemma 4.1. *Let p be a prime and let X be a connected cubic symmetric graph of order $10p^3$. If p is one of 7, 11, 13, 17, 19, 23, 29, 31, 47, 59, 79, 239, or 479, then every minimal normal subgroup of A is an elementary abelian group.*

Proof. Since X is at most 5-regular, $|A| \mid 2^5{\cdot}3{\cdot}5{\cdot}p^3$. Let N be a minimal normal subgroup of A. If N is not an elementary abelian group, then $N \cong T_1 \times \cdots \times T_n$, where $T_i \cong T_j$ $(1 \leq i, j \leq n)$. By considering the order of A, one has $n = 1$. Now by checking the simple groups of order less than 10^{25} in [8], $N \cong A_5$, $L_2(7)$, $L_2(11)$, $L_2(16)$ or $L_2(31)$ and $|N| = 2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3 \cdot 7$, $2^2 \cdot 3 \cdot 5 \cdot 11$, $2^4 \cdot 3 \cdot 5 \cdot 17$ or $2^5 \cdot 3 \cdot 5 \cdot 31$. Clearly, N is not transitive on $V(X)$ and hence by Proposition 2.2, N is semiregualr on $V(X)$, a contradiction. So N is an elementary abelian 2-, 3-, 5-, or p-group, respectively. □

Lemma 4.2. *Let X be a connected cubic symmetric graph of order $10 \cdot 7^3$. Also let P be a Sylow 7-subgroup of A. Then $P \lhd A$.*

Proof. Suppose to the contrary that P is not normal in A. Since X is at most 5-regular, $|A| = 2^s{\cdot}3{\cdot}5{\cdot}p^3$ $(1 \leq s \leq 5)$. Now let N be a minimal normal subgroup of A. Then by Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 7-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now assume that N is an elementary abelian 5-group. Then $|X_N| = 2 \cdot 7^3$. By [18, Theorem 3.2], X_N is 1-regular or 2-regular. Also, if X_N is 1-regular, then $X_N \cong \mathrm{Cay}(D_{2\cdot 7^3}, \{a, ab, ab^{-\lambda}\})$, where λ is of the element of order 3 in $\mathbb{Z}_{7^3}^*$, or $X_N \cong \mathrm{Cay}(G_1, \{a, ab, ab^{-k}c\})$, where $G_1 = \langle a, b, c \mid a^2 = b^{7^2} = c^7 = [b, c] = 1, aba = b^{-1}, aca = c^{-1}\rangle$, and k is the element of order 3 in $\mathbb{Z}_7^*$. Moreover, if X_N is 2-regular, then $X_N \cong \mathrm{Cay}(G_2, \{a, ab, ac\})$, where $G_2 = \langle a, b, c, d \mid a^2 = b^7 = c^7 = d^7 = [a, d] = [b, d] = [c, d] = 1, d = [b, c], aba = b^{-1}, aca = c^{-1}\rangle$. If X_N is 1-regular, then $A/N = \mathrm{Aut}(X_N)$. Also, $\mathrm{Aut}(G, S) = \mathrm{Aut}(X_N)_1 \cong \mathbb{Z}_3$, where $G \cong D_{2\cdot 7^3}$ or G_1. Thus X_N is a normal Cayley graph. So $G \lhd A/N$, where $G \cong H/N$. Thus $H \lhd A$ and we have $|H| = 2 \cdot 5 \cdot 7^3$. Now P is characteristic in H and so $P \lhd A$, a contradiction. If X_N is 2-regular, then $|\mathrm{Aut}(X_N) : A/N| \leq 2$. Also, $\mathrm{Aut}(G_2, S) = \mathrm{Aut}(X_N)_1 \cong S_3$ and hence X_N is a normal Cayley graph. If $A/N = \mathrm{Aut}(X_N)$, then $G_2 \lhd A/N$, where $G_2 \cong G/N$. Clearly, a Sylow 7-subgroup of A is normal in A, a contradiction. Now assume that $|\mathrm{Aut}(X_N)| = 2|A/N|$. So $|A/N| = 2 \cdot 3 \cdot 7^3$ and hence a Sylow 7-subgroup of $\mathrm{Aut}(X_N)$ is a Sylow 7-subgroup of A/N. It is easy to see that $P \lhd A$, a contradiction. Now assume that N is an elementary abelian 7-group. Thus $|V(X_N)| = 10$, $10 \cdot 7$ or $10 \cdot 7^2$. If $|V(X_N)| = 10$, then $P \lhd A$, a contradiction. Thus $|V(X_N)| = 10 \cdot 7$ or $10 \cdot 7^2$. By [16, Theorem 5.1], there are no cubic symmetric graphs of orders $10 \cdot 7$ and $10 \cdot 7^2$, a contradiction. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For $p = 2$, by Conder [6], there exists only one connected cubic symmetric graph of order 10×2^3, namely, the cubic 3-regular graph $C80.1$, and for $p = 3$, there is no cubic symmetric graph of order 10×3^3. Also if $p = 5$, then there are two cubic connected symmetric graphs of order 1250, which are the 2-regular graph $C1250.1$ and the 3-regular graph $C1250.2$. Thus we may assume that $p \geq 7$.

Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. Then $|P| = p^3$ and $|A : N_A(P)| = 1 + np$, where $N_A(P)$ is the normalizer of P in A. To prove the theorem, it suffices to show that $P \lhd A$ by [16, Theorem 4.4], Proposition 2.2, and Lemmas 3.1, 3.2, and 3.3. Suppose to the contrary that P is not normal in A. Then $1 + np \geq 8$ since $p \geq 7$. Since X is at most 5-regular, $|A| \mid 2^5 \cdot 3 \cdot 5 \cdot p^3$. It follows that np is one of the following:

7, 11, $14 = 2 \times 7$, 19, 23, 29, 31, $39 = 3 \times 13$, 47, 59, 79, $95 = 5 \times 19$, $119 = 17 \times 7$, $159 = 3 \times 53$, 239, 479. Thus there are five possible cases:

1) $p = 7$, 11, 19, 23, 29, 31, 47, 59, 79, 239, 479 and $n = 1$;
2) $p = 7$ and $n = 2$ or 17;
3) $p = 13$, 53 and $n = 3$;
4) $p = 17$ and $n = 7$;
5) $p = 19$ and $n = 5$.

Case I. $p = 7$, 11, 19, 23, 29, 31, 47, 59, 79, 239, 479 and $n = 1$.

Let $H = N_A(P)$. By considering the right multiplication action of A on the set of right cosets of H in A, $|A/H_A| \mid (p+1)!$, where H_A is the largest normal subgroup of A in H. This forces $p^2 \mid |H_A|$. Let L be a Sylow p-subgroup of H_A. First let $p = 7$ or $p = 11$. So $|A| = 2^s \cdot 3 \cdot 5 \cdot 7^3$ or $2^s \cdot 3 \cdot 5 \cdot 11^3$ $(1 \leq s \leq 5)$. Now let N be a minimal normal subgroup of A. Then by Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, 7, or 11-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now assume that N is an elementary abelian 5-group. Then $|X_N| = 2p^3$. If $p = 7$, then $|X| = 10 \cdot 7^3$. Now by Lemma 4.2, a Sylow 7-subgroup of A is normal in A, a contradiction. Also for $p = 11$, by [18, Theorem 3.2], X_N is 2-regular and $X_N \cong \mathrm{Cay}(G, \{a, ab, ac\})$, where $G = \langle a, b, c, d \mid a^2 = b^{11} = c^{11} = d^{11} = [a, d] = [b, d] = [c, d] = 1, d = [b, c], aba = b^{-1}, aca = c^{-1}\rangle$. Also, $\mathrm{Aut}(G, S) = \mathrm{Aut}(X_N)_1 \cong S_3$. Thus X_N is a normal Cayley graph. Clearly, $|\mathrm{Aut}(X_N) : A/N| \leq 2$. If $A/N = \mathrm{Aut}(X_N)$, then $G \lhd A/N$, where $G \cong G_1/N$. Clearly, a Sylow 11-subgroup of A is normal in A. Now assume that $|\mathrm{Aut}(X_N)| = 2|A/N|$. So $|A/N| = 2 \cdot 3 \cdot 11^3$ and hence a Sylow 11-subgroup of $\mathrm{Aut}(X_N)$ is a Sylow 11-subgroup of A/N. It is easy to see that a Sylow 11-subgroup is normal in A, a contradiction. Now assume that N is an elementary abelian 7- or 11-group. Thus $|V(X_N)| = 10$, $10p$ or $10p^2$, where $p = 7$ or $p = 11$. If $|V(X_N)| = 10$, then a Sylow 7-subgroup or a Sylow 11-subgroup is normal in A, a contradiction. Thus $|V(X_N)| = 10p$ or $10p^2$, where $p = 7$ or $p = 11$. By [16, Theorem 5.1], X_N must be a Coxeter-Frucht graph of order 110, which is 3-regular. Thus $|V(X_N)| = 10 \times 11$ and $|A/N| \mid |\mathrm{Aut}(X_N)| = 2^3 \cdot 3 \cdot 5 \cdot 11$. Now assume that $Y = X_N$ and let T/N be a minimal normal subgroup of A/N. If T/N is not an elementary abelian group, then $T/N \cong T_1 \times T_2 \times \cdots \times T_n$, where $T_i \cong T_j$ $(1 \leq i, j \leq n)$. By considering the order of A, one has $n = 1$. Now by checking the simple groups of order less than 10^{25} in [8], $N \cong A_5$ or $L_2(11)$. Hence T/N is an elementary abelian 2-, 3-, 5-, or 11-group. Clearly, T/N cannot be elementary abelian 2- nor 3-group. If T/N is an elementary abelian 5-group, then $|Y_{T/N}| = 2 \cdot 11$. But by [6], there is no symmetric graph of order 22, a contradiction. Finally, if T/N is an elementary abelian 11-group, then $|T/N| = 11$. So $|T| = 11^3$ and hence a Sylow 11-subgroup is normal in A, a contradiction.

Now let $p > 11$. L is characteristic in H_A and so $L \lhd A$. Also, $L \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. By Proposition 2.2, the quotient graph X_L of X corresponding to the orbits of L is a cubic connected symmetric graph of order $10p$ or $10p^2$. By [16, Theorem 5.1], $p \neq 19$, 23, 29, 31, 47, 59, 79, 119, 239, nor 479.

Case II. $p = 7$ and $n = 2$ or 17.

In this case, $|A : N_A(P)| = 15$ or 120. So $|A| = 3 \cdot 5 \cdot |N_A(P)|$ or $|A| = 2^3 \cdot 3 \cdot 5 \cdot |N_A(P)|$. First assume that $|A| = 3 \cdot 5 \cdot |N_A(P)|$. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 7-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now assume that N is an elementary abelian 5-group. Then $|X_N| = 2 \cdot 7^3$ and so $|X| = 10 \cdot 7^3$. Now by Lemma 4.2, a Sylow 7-subgroup of A is normal in A, a contradiction. Now assume that N is an elementary abelian 7-group. Thus $|V(X_N)| = 10$, $10 \cdot 7$ or $10 \cdot 7^2$. If $|V(X_N)| = 10$, then a Sylow 7-subgroup is normal in A, a contradiction. So $|V(X_N)| = 10 \cdot 7$ or $10 \cdot 7^2$. By [16, Theorem 5.1], there are no symmetric graphs of these orders, a contradiction. Now assume that $|A| = 2^3 \cdot 3 \cdot 5 \cdot |N_A(P)|$. Clearly, X is at least 3-regular. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 7-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Thus N is an elementary abelian 5- or 7-group. Now assume that N is an elementary abelian 5-group. Then $|X_N| = 2 \cdot 7^3$ and so $|X| = 10 \cdot 7^3$. By Lemma 4.2, a Sylow 7-subgroup of A is normal in A, a contradiction. Now assume that N is an elementary abelian 7-group. Then $|V(X_N)| = 10 \cdot 7$, $10 \cdot 7^2$, or 10. Since X is at least 3-regular, by Proposition 2.2, X_N is at least 3-regular. By [16, Theorem 5.1], there are no at least 3-regular graphs of orders $10 \cdot 7$ nor $10 \cdot 7^2$. So $|V(X_N)| = 10$, but in this case, a Sylow 7-subgroup of A is normal in A, a contradiction.

Case III. $p = 13$, 53 and $n = 3$.

In this case, $|A : N_A(P)| = 40$ or 160. So $|A| = 2^3 \cdot 5 \cdot |N_A(P)|$ or $2^5 \cdot 5 \cdot |N_A(P)|$. First assume that $|A| = 2^3 \cdot 5 \cdot |N_A(P)|$. Clearly, X is at least 3-regular. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 13-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now let N be a 5- or 13-group. Then $|V(X_N)| = 2 \cdot 13^3$, $10 \cdot 13$, $10 \cdot 13^2$, or 10. Since X is at least 3-regular, by Proposition 2.2, X_N is at least 3-regular. By [18, Theorem 3.2] and [16, Theorem 5.1], there are no at least 3-regular graphs of orders $2 \cdot 13^3$, $10 \cdot 13$, nor $10 \cdot 13^2$. So $|V(X_N)| = 10$, but in this case, a Sylow 13-subgroup of A is normal in A, a contradiction. Now assume that $|A| = 2^5 \cdot 5 \cdot |N_A(P)|$. Clearly, X is 5-regular. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 53-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now let N be an elementary abelian 5- or 53-group. Then $|V(X_N)| = 2 \cdot 53^3$, $10 \cdot 53$, $10 \cdot 53^2$, or 10. Since X is 5-regular, by Proposition 2.2, X_N is 5-regular. By [18, Theorem 3.2] and [16, Theorem 5.1], there are no 5-regular graphs of orders $2 \cdot 53^3$, $10 \cdot 53$, 10 nor $10 \cdot 53^2$, a contradiction.

Case IV. $p = 17$ and $n = 7$.

In this case, $|A : N_A(P)| = 120$ and $|A| = 2^3 \cdot 3 \cdot 5 \cdot |N_A(P)|$. Clearly, X is at least 3-regular. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 17-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now let N be a 5- or 17-group. Then $|V(X_N)| = 2 \cdot 17^3$, $10 \cdot 17$, $10 \cdot 17^2$, or 10. Since X is at least 3-regular, by Proposition 2.2, X_N is at least 3-regular. Now by [16, Theorem 5.1] and [18, Theorem 3.2], there are no at least 3-regular graphs of orders $2 \cdot 17^3$, $10 \cdot 17$, nor $10 \cdot 17^2$. So $|V(X_N)| = 10$, but in this case, a Sylow 7-subgroup of A is normal in A, a contradiction.

Case V. $p = 19$ and $n = 5$.

In this case, $|A : N_A(P)| = 96$ and $|A| = 2^5 \cdot 3 \cdot |N_A(P)|$. Clearly, X is 5-regular. Now let N be a minimal normal subgroup of A. By Lemma 4.1, N is an elementary abelian 2-, 3-, 5-, or 19-group, respectively. Clearly, N cannot be an elementary abelian 2- nor 3-group. Now let N be an elementary abelian 5- or 19-group. Then $|V(X_N)| = 2 \cdot 19^3$, $10 \cdot 19$, $10 \cdot 19^2$, or 10. Since X is 5-regular, by Proposition 2.2, X_N is 5-regular. Now by [16, Theorem 5.1] and [18, Theorem 3.2], there are no 5-regular graphs of orders $2 \cdot 19^3$, $10 \cdot 19$, $10 \cdot 19^2$ nor 10, a contradiction. □

References

[1] M. Alaeiyan and M. Ghasemi, *Cubic edge-transitive graphs of order* $8p^2$, Bull. Austral. Math. Soc. **77** (2008), no. 2, 315–323.

[2] B. Alspach, D. Marušič, and L. Nowitz, *Constructing graphs which are* 1/2*-transitive*, J. Austral. Math. Soc. Ser. A **56** (1994), no. 3, 391–402.

[3] D. Archdeacon, P. Gvozdnjak, and J. Širan, *Constructing and forbidding automorphisms in lifted maps*, Math. Slovaca **47** (1997), no. 2, 113–129.

[4] D. Archdeacon, R. B. Richter, J. Širan, and M. Škoviera, *Branched coverings of maps and lifts of map homomorphisms*, Australas. J. Combin. **9** (1994), 109–121.

[5] Y. Cheng and J. Oxley, *On weakly symmetric graphs of order twice a prime*, J. Combin. Theory Ser. B **42** (1987), no. 2, 196–211.

[6] M. D. E. Conder, *Trivalent (cubic) symmetric graphs on up to* 2048 *vertices*, http://www.math.auckland.ac.nz/ conder/ (2006).

[7] M. D. E. Conder and C. E. Praeger, *Remarks on path-transitivity in finite graphs*, European J. Combin. **17** (1996), no. 4, 371–378.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of Finite Groups*, Clareandon Press, Oxford, 1985.

[9] D. Ž. Djoković, *Automorphisms of graphs and coverings*, J. Combin. Theory ser. B **16** (1974), 243–247.

[10] D. Ž. Djoković and G. L. Miller, *Regular groups of automorphisms of cubic graphs*, J. Combin. Theory Ser. B **29** (1980), no. 2, 195–230.

[11] S. F. Du, J. H. Kwak, and M. Y. Xu, *Linear criteria for lifting automorphisms of elementary abelian regular coverings*, Linear Algebra Appl. **373** (2003), 101–119.

[12] Y. Q. Feng and J. H. Kwak, *Constructing an infinite family of cubic* 1*-regular graphs*, European J. Combin. **23** (2002), no. 5, 559–565.

[13] ______, *An infinite family of cubic one-regular graphs with unsolvable automorphism groups*, Discrete Math. **269** (2003), no. 1-3, 281–286.

[14] ______, *One-regular cubic graphs of order a small number times a prime or a prime square*, J. Aust. Math. Soc. **76** (2004), no. 3, 345–356.

[15] ______, *Cubic symmetric graphs of order a small number times a prime or a prime square*, J. Combin. Theory Ser. B **97** (2007), no. 4, 627–646.

[16] ______, *Classifying cubic symmetric graphs of order* $10p$ *or* $10p^2$, Sci. China Ser. A **49** (2006), no. 3, 300–319.

[17] Y. Q. Feng, J. H. Kwak, and K. S. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$, European J. Combin. **26** (2005), no. 7, 1033–1052.

[18] Y. Q. Feng, J. H. Kwak, and M. Y. Xu, *Cubic s-regular graphs of order* $2p^3$, J. Graph Theory **52** (2006), no. 4, 341–352.

[19] Y. Q. Feng and K. S. Wang, *s-regular cyclic coverings of the three-dimensional hypercube* Q_3, European J. Combin. **24** (2003), no. 6, 719–731.

[20] R. Frucht, *A one-regular graph of degree three*, Canad. J. Math. **4** (1952), 240–247.

[21] J. L. Gross and T. W. Tucker, *Generating all graph coverings by permutation voltage assignment*, Discrete Math. **18** (1977), no. 3, 273–283.
[22] P. Lorimer, *Vertex-transitive graphs*: *symmetric graphs of prime valency*, J. Graph Theory **8** (1984), no. 1, 55–68.
[23] Z. P. Lu, C. Q. Wang, and M. Y. Xu, *On semisymmetric cubic graphs of order* $6p^2$, Sci. China Ser. A **47** (2004), no. 1, 1–17.
[24] A. Malnič, *Group actions, coverings and lifts of automorphisms*, Discrete Math. **182** (1998), no. 1-3, 203–218.
[25] A. Malnič and D. Marušič, *Imprimitive graphs and graph coverings*, in: D. Jungnickel, S. A. Vanstone (Eds.), Coding Theory, Design Theory, Group Theory: Proc. M. Hall Memorial Conf., J. Wiley and Sons, New York, 1993, pp. 221–229.
[26] A. Malnič, D. Marušič, and P. Potočnik, *On cubic graphs admitting an edge-transitive solvable group*, J. Algebraic Combin. **20** (2004), no. 1, 99–113.
[27] ———, *Elementary abelian covers of graphs*, J. Algebraic Combin. **20** (2004), no. 1, 71–97.
[28] A. Malnič, D. Marušič, P. Potočnik, and C. Q. Wang, *An infinite family of cubic edge-but not vertex-transitive graphs*, Discrete Math. **280** (2004), no. 1-3, 133–148.
[29] A. Malnič, D. Marušič, and N. Seifter, *Constructing infinite one-regular graphs*, European J. Combin. **20** (1999), no. 8, 845–853.
[30] A. Malnič, D. Marušič, and C. Q. Wang, *Cubic edge-transitive graphs of order* $2p^3$, Discrete Math. **274** (2004), no. 1-3, 187–198.
[31] A. Malnič, R. Nedela, and M. Škoviera, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), no. 6, 927–947.
[32] A. Malnič and P. Potočnik, *Invariant subspaces, duality, and covers of the Petersen graph*, European J. Combin. **27** (2006), no. 6, 971–989.
[33] D. Marušič, *A family of one-regular graphs of valency 4*, European J. Combin. **18** (1997), no. 1, 59–64.
[34] D. Marušič and R. Nedela, *Maps and half-transitive graphs of valency 4*, European J. Combin. **19** (1998), no. 3, 345–354.
[35] D. Marušič and T. Pisanski, *Symmetries of hexagonal molecular graphs on the torus*, Croat Chemica Acta **73** (2000), 969–981.
[36] D. Marušič and M. Y. Xu, *A* 1/2*-transitive graph of valency 4 with a nonsolvable group of automorphisms*, J. Graph Theory **25** (1997), 133–138.
[37] R. C. Miller, *The trivalent symmetric graphs of girth at most six*, J. Combin. Theory Ser. B **10** (1971), 163–182.
[38] N. Seifter and V. I. Trofimov, *Automorphism groups of covering graphs*, J. Combin. Theory Ser. B **71** (1997), no. 1, 67–72.
[39] M. Škoviera, *A construction to the theory of voltage groups*, Discrete Math. **61** (1986), 281–292.
[40] W. T. Tutte, *A family of cubical graphs*, Proc. Camb. Phil. Soc. **43** (1947), 459–474.
[41] ———, *On the symmetry of cubic graphs*, Canad. J. Math. **11** (1959), 621–624.

Department of Mathematics
University of Urmia
Urmia 57135, Iran
E-mail address: m.ghasemi@urmia.ac.ir