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ORTHOGONAL ALMOST COMPLEX STRUCTURES ON THE RIEMANNIAN PRODUCTS OF EVEN-DIMENSIONAL ROUND SPHERES

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Abstract. We discuss the integrability of orthogonal almost complex structures on Riemannian products of even-dimensional round spheres and give a partial answer to the question raised by E. Calabi concerning the existence of complex structures on a product manifold of a round 2-sphere and of a round 4-sphere.

1. Introduction

It is well-known that a $2n$-dimensional sphere S^{2n} admits an almost complex structure if and only if $n = 1$ or 3 and that any almost complex structure on S^2 is integrable. Also, the complex structure on S^2 is unique with respect to the conformal structure on it. A 2-dimensional sphere S^2 equipped with this complex structure is biholomorphic to a complex projective line $\mathbb{C}P_1$. On the contrary, it is a long-standing open problem whether S^6 admits an integrable almost complex structure (namely, a complex structure) or not. Lebrun [4] gave a partial answer to this problem, that is, proved that any orthogonal almost complex structure on a round 6-sphere is never integrable (see also [6], Corollary 5.2). On one hand, Sutherland proved that a connected product of even-dimensional spheres admits an almost complex structure if and only if it is a product of copies of S^2, S^6 and $S^2 \times S^4$ under a more general setting ([7], Theorem 3.1). In [1], Calabi raised the problem of whether the product manifold $V^2 \times S^4$ (V^2 is any closed orientable surface) can admit an integrable almost complex structure or not. In the present note, we discuss the integrability of orthogonal almost complex structures on a Riemannian product of round 2-spheres, 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere, and prove the following.

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Theorem A. *An orthogonal almost complex structure on a Riemannian product of round 2-spheres, round 6-spheres, and Riemannian product manifolds of a round 2-sphere and a round 4-sphere is integrable if and only if it is the product of the canonical complex structures on round 2-spheres.*

Remark 1. Let M be any Riemannian product of round 2-spheres. Then the product of the canonical complex structures of round 2-spheres is necessarily an orthogonal complex structure on M.

From Theorem A, we have the following partial answer to the above mentioned problem by Calabi.

Corollary B. *Any orthogonal almost complex structure on a Riemannain product of a round 2-sphere and a round 4-sphere is never integrable.*

Remark 2. An explicit example of an orthogonal almost Hermitian structure on a Riemannian product of a round 2-sphere and a round 4-sphere was introduced and its geometric property was discussed in [3].

We denote by $S^m(\kappa)$ an m-dimensional round sphere of positive constant sectional curvature κ. Throughout the present paper, we shall mean by *a round m-sphere* an oriented m-dimensional sphere with constant sectional curvature.

2. Preliminaries

Let $M = (M, J, \langle , \rangle)$ be a $2n$-dimensional almost Hermitian manifold. We denote by ∇ the Levi-Civita connection and R the curvature tensor of M defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}Z \tag{2.1}$$

for X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. We denote the Ricci $*$-tensor of M by ρ^* which is defined by

$$\begin{aligned} \rho^*(X,Y) &= \mathrm{tr}\big(Z \longmapsto R(X,JZ)JY\big) \\ &= \frac{1}{2}\mathrm{tr}\big(Z \longmapsto R(X,JY)JZ\big) \end{aligned} \tag{2.2}$$

for X, Y, $Z \in \mathfrak{X}(M)$. Here we note that the Ricci $*$-tensor ρ^* satisfies the following equality

$$\rho^*(X,Y) = \rho^*(JY,JX) \tag{2.3}$$

for X, $Y \in \mathfrak{X}(M)$. Thus from (2.3), we see that ρ^* is symmetric if and only if ρ^* is J-invariant. We also denote by N the Nijenhuis tensor of the almost complex structure J defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{2.4}$$

for X, $Y \in \mathfrak{X}(M)$. It follows from the celebrated theorem by Newlander and Nirenberg [5] that the almost complex structure J is integrable if and only if $N = 0$ holds everywhere on M. An almost Hermitian manifold with an integrable almost complex structure is called a Hermitian manifold.

Now we set

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle \tag{2.5}$$

for X, Y, Z, $W \in \mathfrak{X}(M)$. Gray [2] proved the following result which plays an important role in our forthcoming arguments of the present paper.

Theorem 2.1. *The curvature tensor R of a Hermitian manifold $M = (M, J, \langle , \rangle)$ satisfies the following identity*:

$$\begin{aligned}R_{WXYZ} &+ R_{JWJXJYJZ} - R_{JWJXYZ} - R_{JWXJYZ}\\ &- R_{JWXYJZ} - R_{WJXJYZ} - R_{WJXYJZ} - R_{WXJYJZ} = 0\end{aligned}$$

for any W, X, Y, $Z \in \mathfrak{X}(M)$.

3. Lemmas

We shall prove several lemmas prior to the proof of Theorem A. First of all, we note that orthogonal almost complex structures on the Riemannian products of even-dimensional round spheres do not depend on the order of the factors. We now consider the Riemannian product $M = S^2(\alpha) \times M'$, where M' is a Riemannian product of round 2-spheres, round 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere.

Lemma 3.1. *Let J be an orthogonal complex structure on M. Then J induces a canonical complex structure on $S^2(\alpha)$ and an orthogonal almost complex structure on $\{p_1\} \times M'$ for each point $p_1 \in S^2(\alpha)$.*

Proof. We denote by π_1 and π_2 the canonical projections defined by $\pi_1 : M \to S^2(\alpha)$ and $\pi_2 : M \to M'$, respectively. We set

$$x_1 = d\pi_1(x), \quad x_2 = d\pi_2(x) \tag{3.1}$$

for any $x \in T_pM$, $p = (p_1, p_2) \in S^2(\alpha) \times M'$. The tangent space T_pM is identified with the orthogonal direct sum of $T_{p_1}S^2(\alpha)$ and $T_{p_2}M'$ in the natural way. Let x, $y \in T_{p_1}S^2(\alpha)$ with $x \perp y$, $|x| = |y| = 1$. Then we get

$$R(x, y, x, y) = -\alpha. \tag{3.2}$$

Here, since $\dim S^2(\alpha) = 2$, we may set

$$(Jx)_1 = \langle Jx, y\rangle y, \qquad (Jy)_1 = \langle Jy, x\rangle x. \tag{3.3}$$

Now, taking account of (3.3), we further have

$$\begin{aligned}&R(Jx, Jy, Jx, Jy)\\ &= R((Jx)_1 + (Jx)_2, (Jy)_1 + (Jy)_2, (Jy)_1 + (Jy)_2, (Jx)_1 + (Jx)_2)\\ &= R((Jx)_1, (Jy)_1, (Jx)_1, (Jy)_1) + R((Jx)_2, (Jy)_2, (Jx)_2, (Jy)_2)\\ &= -\alpha\big(|(Jx)_1|^2|(Jy)_1|^2 - \langle (Jx)_1, (Jy)_1\rangle^2\big)\\ &\quad + R((Jx)_2, (Jy)_2, (Jx)_2, (Jy)_2)\end{aligned} \tag{3.4}$$

$$= \ - \alpha|(Jx)_1|^2|(Jy)_1|^2 + R_2((Jx)_2,(Jy)_2,(Jx)_2,(Jy)_2),$$

where R_2 is the curvature tensor of M'.

$$\begin{aligned} R(Jx,Jy,x,y) &= R((Jx)_1,(Jy)_1,x,y) \\ &= \langle Jx,y\rangle\langle x,Jy\rangle R(y,x,x,y) \\ &= \alpha\langle Jx,y\rangle\langle x,Jy\rangle \\ &= -\alpha\langle x,Jy\rangle^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} R(Jx,y,Jx,y) &= R((Jx)_1,y,(Jx)_1,y) \\ &= \langle Jx,y\rangle^2 R(y,y,y,y) \\ &= 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} R(Jx,y,x,Jy) &= R((Jx)_1,y,x,(Jy)_1) \\ &= -\langle Jx,y\rangle^2 R(y,y,x,x) \\ &= 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} R(x,Jy,x,Jy) &= R(x,(Jy)_1,x,(Jy)_1) \\ &= \langle Jy,x\rangle^2 R(x,x,x,x) \\ &= 0. \end{aligned} \tag{3.8}$$

Thus, from Theorem 2.1 and (3.2)∼(3.8), we have

$$\begin{aligned} 0 &= R(x,y,x,y) + R(Jx,Jy,Jx,Jy) - 2R(Jx,Jy,x,y) \\ &\quad - R(Jx,yJx,y) - 2R(Jx,y,x,Jy) - R(x,Jy,x,Jy) \\ &= \ - \alpha\{1 - |(Jx)_1|^2|(Jy)_1|^2\} + R_2((Jx)_2,(Jy)_2,(Jx)_2,(Jy)_2). \end{aligned} \tag{3.9}$$

Since M' is non-negatively curved, we see that

$$R_2((Jx)_2,(Jy)_2,(Jx)_2,(Jy)_2) \leq 0 \tag{3.10}$$

for all x, $y \in T_{p_1}S^2(\alpha)$. Thus, from (3.9) and (3.1), we see that

$$|(Jx)_1| = 1 \quad \text{and} \quad |(Jy)_1| = 1 \tag{3.11}$$

and hence $Jx \in T_{p_1}S^2(\alpha)$ and $Jy \in T_{p_1}S^2(\alpha)$ for any orthogonal pair $\{x,y\}$ in $T_{p_1}S^2(\alpha)$. Since $d\pi_1$ is a linear map from T_pM onto $T_{p_1}S^2(\alpha)$, from (3.11), we may easily see that $Jx \in T_{p_1}S^2(\alpha)$ for all $x \in T_{p_1}S^2(\alpha)$, and hence $J(T_{p_1}S^2(\alpha)) = T_{p_1}S^2(\alpha)$. Therefore we see also that $J(T_{p_2}M') = T_{p_2}M'$. □

Now, for each $p_1 \in S^2(\alpha)$, we denote by $J' = J'(p_1)$ the induced almost complex structure on $\{p_1\} \times M'$ as in Lemma 3.1. Then we have the following.

Lemma 3.2. *The almost complex structure J' is integrable (and hence defines a complex structure on $\{p_1\} \times M'$).*

Proof. Let N' be the Nijenhuis tensor of the almost complex structure J'. Taking account of Lemma 3.1, we have

$$\begin{aligned} N'(X',Y') &= [J'X',J'Y'] - [X',Y'] - J'[J'X',Y'] - J'[X',J'Y'] \\ &= [JX',JY'] - [X',Y'] - J'[JX',Y'] - J'[X',JY'] \\ &= [JX',JY'] - [X',Y'] - J[JX',Y'] - J[X',JY'] \\ &= N(X',Y') \\ &= 0 \end{aligned} \tag{3.12}$$

for all X', $Y' \in \mathfrak{X}(M')$. Therefore, from (3.12), we see that the induced almost complex structure J' on $\{p_1\} \times M'$ is integrable for each $p_1 \in S^2(\alpha)$. □

From Lemmas 3.1 and 3.2, if M' has a round 2-sphere as a factor, by a suitable reordering of the factors, we may assume that M is expressed in the form $M' = S^2(\alpha)\times M''$, where M'' is defined similarly as M'. Applying Lemma 3.2 to M', it follows that the orthogonal complex structure J' induces a complex structure on M''. By repeating similar operations, we may assume that M is expressed in the form $M = M_1 \times M_2$, where $M_1 = S^2_1(\alpha_1) \times \cdots \times S^2_s(\alpha_s)$ $(0 \leq \alpha_1 \leq \cdots \leq \alpha_s)$ and M_2 does not involve a round 2-sphere, and further that the orthogonal almost complex structure J on M induces a canonical orthogonal complex structure on $M_1 \times \{p_2\}$ for each point $p_2 \in M_2$ and an orthogonal almost complex structure on $\{p_1\} \times M_2$ for each point $p_1 \in M_1$, respectively. Thus, taking account of the result due to Sutherland ([7], Theorem 3.1), we have the following.

Lemma 3.3. *Let M be a Riemannian product of round 2-spheres, round 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere, and J be an orthogonal complex structure on M. Then M takes of the form $M = M' \times M''$ (after suitable reordering of the factors), where M' (resp., M'') is a Riemannian product of round 2-spheres (resp., a Riemannian product of round 6-spheres), and further, J induces a canonical orthogonal complex structure on $M' \times \{p''\}$ for each point $p'' \in M''$ and an orthogonal complex structure on $\{p'\} \times M''$ for each point $p' \in M'$, respectively.*

Now, we shall show the following.

Lemma 3.4. *Let $M = (M,\langle,\rangle)$ be the Riemannian product of round 6-spheres $S^6_a(\beta_a)=(S^6,\langle,\rangle_a)$ $(0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_t, a=1,2,\ldots,t)$, and J be an orthogonal almost complex structure on M. Then for each point $(p_1,\ldots,p_{a-1},p_{a+1},\ldots,p_t) \in S^6_1(\beta_1)\times\cdots\times S^6_{a-1}(\beta_{a-1})\times S^6_{a+1}(\beta_{a+1})\times\cdots\times S^6_t(\beta_t)$, J induces an orthogonal almost complex structure on $\{(p_1,\ldots,p_{a-1},p_{a+1},\ldots,p_t)\}\times S^6_a(\beta_a)$.*

Proof. Let $p = (p_1,p_2,\ldots,p_t) \in M$ $(p_a \in S^6_a(\beta_a)$, $a = 1,2,\ldots,t)$ be any point of M and $\{e(a)_i\}$ $(i = 1,2,\ldots,6)$ be any orthonormal basis of $T_{p_a}S^6_a(\beta_a)$. We denote by $R_{(a)}$ the curvature tensor of $S^6_a(\beta_a)$. Then we have

$$R(x,y)z = R_{(a)}(x,y)z \tag{3.13}$$

and

$$R_{(a)}(x,y)z = \beta_a\big(\langle y,z\rangle_a x - \langle x,z\rangle_a y\big) \tag{3.14}$$

for x, y, $z \in T_{p_a}S_a^6(\beta_a)$. Now, we set

$$Je(a)_i = \sum_{c=1}^{t}\big(\sum_{j=1}^{6} J(a,c)_{ij}e(c)_j\big) \tag{3.15}$$

for $1 \le i \le 6$ and $1 \le a \le t$. Then since $\langle Je(a)_i, e(b)_j\rangle = -\langle e(a)_i, Je(b)_j\rangle$, from (3.15), we have

$$\begin{aligned}\langle Je(a)_i, e(b)_j\rangle &= \langle \sum_c\sum_k J(a,c)_{ik}e(c)_k, e(b)_j\rangle\\ &= \sum_c\sum_k J(a,c)_{ik}\delta_{cb}\delta_{kj}\\ &= J(a,b)_{ij}\end{aligned}$$

and

$$\begin{aligned}\langle e(a)_i, Je(b)_j\rangle &= \langle e(a)_i, \sum_c\sum_k J(b,c)_{jk}e(c)_k\rangle\\ &= \sum_c\sum_k J(b,c)_{jk}\delta_{ac}\delta_{ik}\\ &= J(b,a)_{ji}.\end{aligned}$$

Hence we have

$$J(a,b)_{ij} = -J(b,a)_{ji} \tag{3.16}$$

for $1 \le a,b \le t$ and $1 \le i,j \le 6$. On one hand, since $J^2 = -id$, from (3.15), we have

$$\begin{aligned}-e(a)_i &= J(Je(a)_i)\\ &= J\big(\sum_c\sum_j J(a,c)_{ij}e(c)_j\big)\\ &= \sum_c\sum_d\sum_{j,k} J(a,c)_{ij}J(c,d)_{jk}e(d)_k\end{aligned}$$

and hence,

$$\sum_c\sum_j J(a,c)_{ij}J(c,d)_{jk} = -\delta_{ik}\delta_{ad} \tag{3.17}$$

for $1 \le i,k \le 6$ and $1 \le a,d \le t$. Here, we shall calculate the components of the Ricci $*$-tensor ρ^*. From (3.13), (3.15), (3.16) and (3.17), we have

$$\begin{aligned}&\rho^*(e(a)_i, e(a)_j) \qquad (3.18)\\ &= -\frac{1}{2}\sum_c\sum_k R(e(a)_i, Je(a)_j, e(c)_k, Je(c)_k)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_k R(e(a)_i, Je(a)_j, e(a)_k, Je(a)_k)\\
&= -\frac{1}{2}\sum_k R_{(a)}\big(e(a)_i, \sum_l J(a,a)_{jl}e(a)_l, e(a)_k, \sum_u J(a,a)_{ku}e(a)_u\big)\\
&= -\frac{1}{2}\sum_{k,l,u} J(a,a)_{jl}J(a,a)_{ku}R_{(a)}(e(a)_i, e(a)_l, e(a)_k, e(a)_u)\\
&= -\frac{\beta_a}{2}\sum_{k,l,u} J(a,a)_{jl}J(a,a)_{ku}\{\delta_{lk}\delta_{iu} - \delta_{ik}\delta_{lu}\}\\
&= -\frac{\beta_a}{2}\{-\delta_{ji} - \delta_{ji}\}\\
&= \beta_a\delta_{ij},
\end{aligned}
$$

$$
\begin{aligned}
(3.19)\qquad &\rho^*(e(a)_i, e(b)_j)\\
&= -\frac{1}{2}\sum_c\sum_k R(e(a)_i, Je(b)_j, e(c)_k, Je(c)_k)\\
&= -\frac{1}{2}\sum_k R(e(a)_i, \sum_l J(b,a)_{jl}e(a)_l, e(a)_k, \sum_u J(a,a)_{ku}e(a)_u)\\
&= -\frac{1}{2}\sum_{k,l,u} J(b,a)_{jl}J(a,a)_{ku}R_{(a)}(e(a)_i, e(a)_l, e(a)_k, e(a)_u)\\
&= -\frac{\beta_a}{2}\sum_{k,l,u} J(b,a)_{jl}J(a,a)_{ku}\{\delta_{lk}\delta_{iu} - \delta_{ik}\delta_{lu}\}\\
&= -\frac{\beta_a}{2}\{J(b,a)_{jk}J(a,a)_{ki} - \sum_l J(b,a)_{jl}J(a,a)_{il}\}\\
&= -\frac{\beta_a}{2}\{-\delta_{ji}\delta_{ba} - \delta_{ji}\delta_{ba}\}\\
&= \beta_a\delta_{ij}\delta_{ab},
\end{aligned}
$$

$$
\begin{aligned}
(3.20)\qquad &\rho^*(e(a)_i, Je(a)_j)\\
&= \frac{1}{2}\sum_c\sum_k R(e(a)_i, e(a)_j, e(c)_k, Je(c)_k)\\
&= \frac{1}{2}\sum_k R(e(a)_i, e(a)_j, e(a)_k, Je(a)_k)\\
&= \frac{1}{2}\sum_{k,l} J(a,a)_{kl}R_{(a)}(e(a)_i, e(a)_j, e(a)_k, e(a)_l)\\
&= \frac{\beta_a}{2}\sum_{k,l} J(a,a)_{kl}\{\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}\}
\end{aligned}
$$

$$= \frac{\beta_a}{2}\{J(a,a)_{ji} - J(a,a)_{ij}\}$$
$$= \beta_a J(a,a)_{ji},$$

(3.21)
$$\rho^*(e(a)_i, Je(b)_j)$$
$$= \frac{1}{2}\sum_c \sum_k R(e(a)_i, e(b)_j, e(c)_k, Je(c)_k)$$
$$= \frac{1}{2}\sum_{c,d} \sum_{k,l} J(c,d)_{kl} R(e(a)_i, e(b)_j, e(c)_k, e(d)_l)$$
$$= -\beta_a \delta_{ab} J(a,b)_{ij},$$

(3.22)
$$\rho^*(Je(a)_i, e(a)_j)$$
$$= -\frac{1}{2}\sum_c \sum_k R(Je(a)_i, Je(a)_j, e(c)_k, Je(c)_k)$$
$$= -\frac{1}{2}\sum_c \sum_{k,l,u,v} J(a,c)_{il} J(a,c)_{ju} J(c,c)_{kv} R_{(c)}(e(c)_l, e(c)_u, e(c)_k, e(c)_v)$$
$$= -\frac{1}{2}\sum_c \beta_c \sum_{k,l,u,v} J(a,c)_{il} J(a,c)_{ju} J(c,c)_{kv}\{\delta_{uk}\delta_{lv} - \delta_{lk}\delta_{uv}\}$$
$$= -\frac{1}{2}\sum_c \beta_c \Big\{\sum_{k,l} J(a,c)_{il} J(a,c)_{jk} J(c,c)_{kl}$$
$$- \sum_{k,u} J(a,c)_{ik} J(a,c)_{ju} J(c,c)_{ku}\Big\}$$
$$= -\frac{1}{2}\sum_c \beta_c \Big\{ -\sum_l J(a,c)_{il}\delta_{jl}\delta_{ac} + \sum_u J(a,c)_{ju}\delta_{iu}\delta_{ac}\Big\}$$
$$= \frac{1}{2}\beta_a J(a,a)_{ij} - \frac{1}{2}\beta_a J(a,a)_{ji}$$
$$= \beta_a J(a,a)_{ij},$$

(3.23)
$$\rho^*(Je(b)_i, e(a)_j)$$
$$= -\frac{1}{2}\sum_c \sum_k R(Je(b)_i, Je(a)_j, e(c)_k, Je(c)_k)$$
$$= -\frac{1}{2}\sum_c \sum_{k,l,u,v} J(b,c)_{il} J(a,c)_{ju} J(c,c)_{kv} R_{(c)}(e(c)_l, e(c)_u, e(c)_k, e(c)_v)$$
$$= -\frac{1}{2}\sum_c \beta_c \sum_{k,l,u,v} J(b,c)_{il} J(a,c)_{ju} J(c,c)_{kv}\{\delta_{uk}\delta_{lv} - \delta_{lk}\delta_{uv}\}$$
$$= -\frac{1}{2}\sum_c \beta_c \Big\{\sum_{k,l} J(b,c)_{il} J(a,c)_{jk} J(c,c)_{kl}$$

$$
\begin{aligned}
&- \sum_{k,u} J(b,c)_{ik} J(a,c)_{ju} J(c,c)_{ku} \bigr\} \\
= &- \frac{1}{2} \sum_c \beta_c \bigl\{ - \sum_l \delta_{jl}\delta_{ac} J(b,c)_{il} + \sum_u \delta_{iu}\delta_{bc} J(a,c)_{ju} \bigr\} \\
= &\frac{1}{2}\beta_a J(b,a)_{ij} - \frac{1}{2}\beta_a J(a,b)_{ji} \\
= &- \beta_a J(a,b)_{ij}.
\end{aligned}
$$

Thus, from (3.18) and (3.19), we see that ρ^* is symmetric (and hence J-invariant). Further, from (3.21), (3.23) and taking account of the symmetry of ρ^*, we have $J(a,b)_{ij} = 0$ for $a \neq b$. Hence

$$
J(T_{p_a} S^6_a(\beta_a)) = T_{p_a} S^6_a(\beta_a), \quad a = 1, 2, \ldots, t. \tag{3.24}
$$

Therefore, from (3.24), we see that J induces an almost complex structure on $\{(p_1, \ldots, p_{a-1},\ p_{a+1}, \ldots, p_t)\} \times S^6_a(\beta_a)$ for each $(p_1, \ldots, p_{a-1},\ p_{a+1}, \ldots, p_t) \in S^6_1(\beta_1) \times \cdots \times S^6_{a-1}(\beta_{a-1}) \times\ S^6_{a+1}(\beta_{a+1}) \times \cdots \times S^6_t(\beta_t)$. □

Lemma 3.5. *Any orthogonal almost complex structure on a Riemannian product of round 6-spheres is never integrable.*

Proof. Let $M = (M, \langle,\rangle)$ be a Riemannian product of round 6-spheres $S^6_a(\beta_a)$ $(a = 1, 2, \ldots, t)$ and assume that M admits an orthogonal complex structure denoted by J. Then taking account of the results in [4], it suffices to consider the case when $t \geq 2$. From Lemma 3.4, for each point $(p_1, \ldots, p_{t-1}) \in S^6_1(\beta_1) \times \cdots \times S^6_{t-1}(\beta_{t-1})$, J induces an orthogonal almost complex structure on $\{(p_1, \ldots, p_{t-1})\} \times S^6_t(\beta_t)$. Then we may show that the induced orthogonal almost complex structure is integrable by slightly modifying the proofs of Lemmas 3.1 and 3.2. But this is a contradiction. □

4. Proof of Theorem A

In this section, we prove Theorem A based on the arguments in §3. Let $M = (M, \langle,\rangle)$ be a Riemannian product of round 2-spheres, round 6-spheres, and Riemannian product manifolds of a round 2-sphere and a round 4-sphere. Assume that M admits an orthogonal complex structure and denote it by J. Then from Lemma 3.3, we see that M is of the form $M = M' \times M''$, where M' is of the form $M' = S^2(\alpha_1) \times \cdots \times S^2(\alpha_s)$ and M'' is of the form $M'' = S^6(\beta_1) \times \cdots \times S^6(\beta_t)$, respectively. Further, J induces an orthogonal complex structure on $\{p'\} \times M''$ for each point $p' \in M'$. Therefore, from Lemmas 3.1 and 3.2 and by the uniqueness of the canonical complex structure on a round 2-sphere, J is an orthogonal complex structure on M. Therefore, taking account of Lemma 3.1, we see that J is a product of the canonical complex structures on these round 2-spheres. The converse is evident by Remark 1. This completes the proof of Theorem A.

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