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ASCREEN LIGHTLIKE HYPERSURFACES OF AN INDEFINITE SASAKIAN MANIFOLD

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Abstract. In this paper, we study lightlike hypersurfaces of an indefinite Sasakian manifold $\bar{M}$. First, we construct a type of lightlike hypersurface according to the form of the structure vector field of $\bar{M}$, named by ascreen lightlike hypersurface. Next, we characterize the geometry of such ascreen lightlike hypersurfaces.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the electromagnetic field theory. The study of such notion was initiated by Duggal and Bejancu [1] and later studied by many authors (see up-to date results in two books [2, 5]). Recently several authors studied the geometry of lightlike submanifolds of indefinite Sasakian manifolds [3, 4, 6, 9]. Most of authors that wrote on lightlike submanifolds M of indefinite contact manifolds $\bar{M}$ fail to treat with the case the structure vector field ζ of $\bar{M}$ is not tangent to M, but studied only to the case ζ is tangent to M (which is called *tangential lightlike submanifold* [3, 4, 9] of $\bar{M}$). There are few papers on non-tangential lightlike geometry studied by Jin [7, 8].

The objective of this paper is to study lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$. There are many different types of non-tangential lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$ according to the form of the structure vector field of $\bar{M}$. We study a type of them here, named by *ascreen lightlike hypersurfaces.* In Section 3, we construct ascreen lightlike hypersurface of an indefinite Sasakian manifold and characterize the geometry of such ascreen lightlike hypersurfaces. In Section 4, we study totally geodesic ascreen lightlike hypersurfaces and prove several new results.

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2. Lightlike Hypersurfaces

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite contact manifold* if there exists a set $(J,\ \theta,\ \zeta,\ \bar{g})$, where J is a $(1,1)$-type tensor field, ζ a vector field which is called the structure vector field and θ a 1-form satisfying

$$\left\{\begin{array}{l} J^2X = -X + \theta(X)\zeta,\ J\zeta = 0,\ \theta \circ J = 0,\ \theta(\zeta) = 1, \\ \bar{g}(\zeta,\zeta) = \epsilon, \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \epsilon\,\theta(X)\theta(Y), \\ \theta(X) = \epsilon\bar{g}(\zeta, X), \quad d\theta(X,Y) = \bar{g}(JX,Y),\ \epsilon = \pm 1, \end{array}\right. \tag{2.1}$$

for any vector fields X, Y of $\bar{M}$. Then the set $(J,\ \theta,\ \zeta,\ \bar{g})$ is called an indefinite contact structure on $\bar{M}$. We say that $\bar{M}$ has a normal indefinite contact structure if $N_J + d\theta \otimes \zeta = 0$, where N_J is the Nijenhuis tensor field of J. A normal indefinite contact manifold is called an *indefinite Sasakian manifold* for which we have

$$\bar{\nabla}_X\zeta = JX, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = \epsilon\,\theta(Y)X - \bar{g}(X,Y)\zeta, \tag{2.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$ [3, 4, 6, 9]. Due to [6], it is known that the structure vector field ζ is a unit spacelike vector field on $\bar{M}$.

A hypersurface M of $\bar{M}$ is called a *lightlike hypersurface* if the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1, and coincides with the radical distribution $Rad(TM)$. Therefore there exists a non-degenerate complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM, which is called a *screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(\bar{M})$ the algebra of smooth functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of smooth sections of any vector bundle E over $\bar{M}$. It is well-known [1] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$, of rank 1, in $S(TM)^\perp$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N,N) = \bar{g}(N,X) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$ respectively.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas for M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.7}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are the linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are linear operators on TM which are called the *shape operators* on TM and $S(TM)$ respectively and τ is a 1-form on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$, we show that B is independent of the choice of $S(TM)$ and

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.8}$$

The induced connection ∇ of M is not a metric one and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{2.9}$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM).$$

But the connection ∇^* on $S(TM)$ is a metric one. The above two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.10}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.11}$$

From (2.10), we show that A^*_ξ is $S(TM)$-valued self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.12}$$

M is called *totally geodesic* [1] if the local second fundamental form B, or equivalently the shape operator A^*_ξ, vanishes identically, i.e., $B = A^*_\xi = 0$.

A lightlike hypersurface M equipped with a degenerate metric g and a linear connection ∇ is said to be of *constant curvature* c if there exists a constant c such that the curvature tensor R of the connection ∇ satisfies

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{2.13}$$

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.14}$$

In general, $R^{(0,2)}$ is not symmetric [1, 2]. If $R^{(0,2)}$ is symmetric, then it is called the *induced Ricci tensor* of M and denote it by *Ric*. It is well known that $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$ on $\mathcal{U} \subset M$ [1].

For any $X \in \Gamma(TM)$, let $\nabla^{\perp}_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $tr(TM)$. Then $\nabla^{\perp}$ is a linear connection on $tr(TM)$. We say that $\nabla^{\perp}$ is the *transversal connection* of M. We define the curvature tensor $R^{\perp}$ of $tr(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N, \quad \forall\, X,\, Y \in \Gamma(TM).$$

If $R^{\perp}$ vanishes identically, then the transversal connection is said to be *flat* [6].

From (2.5) and the definition of the transversal connection $\nabla^{\perp}$, we have

$$\nabla^{\perp}_X N = \tau(X)N, \quad R^{\perp}(X,Y)N = 2d\tau(X,Y)N, \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this result we deduce the following theorem [6]:

Theorem 2.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the following assertions are equivalent:*

(1) *The transversal connection of M is flat, i.e., $R^{\perp} = 0$.*
(2) *The 1-form τ is closed, i.e., $d\tau = 0$, on M.*
(3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

3. Ascreen Lightlike Hypersurfaces

Let M be a lightlike hypersurface of an indefinite contact manifold $\bar{M}$. In general, the functions a and b defined by $a = \theta(N)$ and $b = \theta(\xi)$ are non-vanishing smooth functions on $\bar{M}$. We introduce the following result considered by Jin [6, 7]:

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite contact manifold $\bar{M}$. Then $J(Rad(TM))$ and $J(tr(TM))$ are vector subbundles of $S(TM)$ of rank* 1.

Definition 1. A lightlike hypersurface M of an indefinite contact manifold $\bar{M}$ is said to be an *ascreen lightlike hypersurface* [7, 8] if ζ belongs to the orthogonal complement $S(TM)^{\perp} = Rad(TM) \oplus tr(TM)$ of the screen distribution $S(TM)$.

For an ascreen lightlike hypersurface M, ζ is decomposed as

$$\zeta = a\xi + bN. \tag{3.1}$$

As $\bar{g}(\zeta, \zeta) = 1$, we have $2ab = 1$. Thus we show that $a \neq 0$ and $b \neq 0$.

Theorem 3.2. *Let M be a lightlike hypersurface of an indefinite contact manifold $\bar{M}$. Then M is an ascreen lightlike hypersurface of $\bar{M}$ if and only if*

$$J(Rad(TM)) = J(tr(TM)).$$

Proof. ($\Rightarrow$) Applying J to (3.1) and using the fact $J\zeta = 0$, we get $J\xi = kJN$, where $k = -\frac{b}{a} \neq 0$. This implies that $J(Rad(TM)) \cap J(tr(TM)) \neq \{0\}$. As $J(Rad(TM))$ and $J(tr(TM))$ have rank 1, it follow that $J(Rad(TM)) = J(tr(TM))$.

($\Leftarrow$) As $J(Rad(TM)) = J(tr(TM))$, there exists a non-vanishing smooth function k such that $J\xi = kJN$. Taking the scalar product with $J\xi$ and JN to this by turns, we get $b^2 = k(ab-1)$ and $ka^2 = ab - 1$ respectively. From these equations we have $a \neq 0$; $b \neq 0$ and $b^2 = (ka)^2$. The last equation implies $b = ka$ or $b = -ka$. If $b = ka$, then we have $ab = ka^2 = ab - 1$. It is a contradiction. Thus we have $b = -ka$. In this case we get $2ab = 1$. Since $k = -\frac{b}{a}$, $a \neq 0$ and $J\xi = kJN$, we see that $aJ\xi + bJN = 0$. Applying J to this equation and using $(2.1)_1$ and $2ab = 1$, we show that $\zeta = a\xi + bN$. Thus M is an ascreen lightlike hypersurface of $\bar{M}$. □

Remark 1. Now $\bar{M} = (R_q^{2m+1}, J, \zeta, \theta, \bar{g})$ will denote the semi-Euclidean manifold R_q^{2m+1} equipped with its usual indefinite Sasakian structure given by

$$\theta = \tfrac{1}{2}(dz - \textstyle\sum_{i=1}^m y_i dx_i), \qquad \zeta = 2\partial z,$$
$$\bar{g} = \theta \otimes \theta - \tfrac{1}{4}\textstyle\sum_{i=1}^{q/2}(dx_i \otimes dx_i + dy_i \otimes dy_i) + \tfrac{1}{4}\textstyle\sum_{i=q+1}^m (dx_i \otimes dx_i + dy_i \otimes dy_i),$$
$$J(\textstyle\sum_{i=1}^m (X_i \partial x_i + Y_i \partial y_i) + Z\partial z) = \textstyle\sum_{i=1}^m (Y_i \partial x_i - X_i \partial y_i) + \textstyle\sum_{i=1}^m Y_i y_i \partial z,$$

where (x_i, y_i, z) are the Cartesian coordinates and $\bar{g}$ is a semi-Euclidean metric of signature $(-, +, \cdots, +; -, +, \cdots, +; +)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \cdots, \partial x_m; \partial y_1, \partial y_2, \cdots, \partial y_m; \partial z\}.$$

This construction will help in understanding how the indefinite Sasakian structure is recovered in the following example.

Example 1. Consider a submanifold M of $\bar{M} = (R_2^3, J, \zeta, \theta, \bar{g})$ given by

$$X(x, y) = \left(x, y, \frac{1}{\sqrt{2}}(x+y)\right).$$

By direct calculations we easily check that

$$TM = \text{Span}\{\xi = \partial x + \partial y + \sqrt{2}\partial z,\ U = \partial x - \partial y\},$$
$$Rad(TM) = \text{Span}\{\xi\}, \qquad S(TM) = \text{Span}\{U\},$$
$$tr(TM) = \text{Span}\{N = \frac{1}{4}(-\partial x - \partial y + \sqrt{2}\partial z)\}.$$

From these equations, we show that $J\xi = U$, $Rad(TM) \cap J(Rad(TM)) = \{0\}$, $JN = -\frac{1}{4}U$, $JN = -\frac{1}{4}J\xi$, $J(Rad(TM)) = J(tr(TM))$ and $J\zeta = 0$. As $\zeta = \frac{1}{2\sqrt{2}}\xi + \sqrt{2}N$, M is an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$.

From now and in the sequel, let M be an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. From Theorem 3.2, TM is decomposed as

$$TM = Rad(TM) \oplus_{orth} \{J(Rad(TM)) \oplus_{orth} \mathcal{D}\}, \tag{3.2}$$

where $\mathcal{D}$ is a non-degenerate and almost complex distribution with respect to J.

Consider a local timelike vector field V on $S(TM)$ and its 1-form v defined by

$$V = -b^{-1}J\xi = a^{-1}JN, \quad v(X) = -g(X, V), \ \forall\, X \in \Gamma(TM). \tag{3.3}$$

Applying J to the first equation of (3.3)[denote $(3.3)_1$] and using $(2.1)_1$, we get

$$JV = a\xi - bN, \tag{3.4}$$

due to $2ab = 1$. Denote by S the projection morphism of TM on $\mathcal{D}$ with respect to the decomposition (3.2). Any vector field X on M is expressed as follows

$$X = SX + \eta(X)\xi + v(X)V.$$

Applying J to this and using (3.3), (3.4) and the fact $\theta(X) = b\eta(X)$, we have

$$JX = FX + av(X)\xi - \theta(X)V - bv(X)N, \tag{3.5}$$

where F is a tensor field of type $(1,1)$ globally defined on TM by

$$FX = JSX, \quad \forall\, X \in \Gamma(TM).$$

Applying J to (3.5) and using (2.1), (3.3), (3.4) and $2a\theta(X) = \eta(X)$, we have

$$F^2X = -X + \eta(X)\xi + v(X)V, \quad \text{i.e.,} \quad F^2X = -SX. \tag{3.6}$$

Substituting (3.5) into $\bar{g}(JX, JY) = g(X,Y) - \theta(X)\theta(Y)$, we have

$$g(FX, FY) = g(X,Y) + v(X)v(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.7}$$

Theorem 3.3. *Let M be an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then $(M_{\mathcal{D}}, g, F)$ is an almost complex manifold, where $M_{\mathcal{D}}$ is a leaf of the almost complex distribution $\mathcal{D}$.*

Proof. Take $X,\, Y \in \Gamma(\mathcal{D})$. From (3.6) and (3.7), we show that

$$F^2X = -X, \quad g(FX, FY) = g(X,Y).$$

From these equations we deduce our result. □

Definition 2. (1) A screen distribution $S(TM)$ is called *totally umbilical* [1] in M if there exist a smooth function γ on a neighborhood $\mathcal{U}$ in M such that

$$C(X, PY) = \gamma\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

In particular, if $\gamma = 0$ on $\mathcal{U}$, then we say that $S(TM)$ is *totally geodesic* in M.

(2) A lightlike hypersurface M of an indefinite contact manifold $\bar{M}$ is *nearly screen conformal* if the shape operators A_N and A^*_ξ of M and $S(TM)$ respectively are related by $A_N = \varphi\, A^*_\xi + \psi\, \eta \otimes V$, or equivalently, the second fundamental forms B and C of M and $S(TM)$ respectively satisfy

$$C(X, PY) = \varphi B(X, Y) - \psi\, \eta(X) v(Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{3.8}$$

where φ is a non-vanishing smooth function on a coordinate neighborhood $\mathcal{U}$ in M. In particular, if $\psi = 0$ ($\psi \neq 0$) on $\mathcal{U}$, then M is called *screen conformal* (*proper nearly screen conformal*).

For the rest of this paper, by saying that M is nearly screen conformal we shall mean M is either proper nearly screen conformal or screen conformal.

Theorem 3.4. (1) *There exist no ascreen lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M.*

(2) *There exist no nearly screen conformal ascreen lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$.*

Proof. Applying $\bar{\nabla}_X$ to (3.1) and using (2.2), (2.4), (2.5), (2.7), (2.8), we get

$$JX = -aA^*_\xi X - bA_N X + \{X[a] - a\tau(X)\}\xi + \{X[b] + b\tau(X)\}N.$$

Comparing this equation with (3.5), for all $X \in \Gamma(TM)$, we have

$$X[a] - a\tau(X) = av(X), \quad X[b] + b\tau(X) = -bv(X), \tag{3.9}$$

$$aA^*_\xi X + bA_N X = \theta(X)V - FX. \tag{3.10}$$

Replacing X by ξ to (3.10) and using (2.12) and the fact $F\xi = 0$, we have

$$A_N \xi = V, \quad \text{i.e.,} \quad C(\xi, V) = -1. \tag{3.11}$$

(1) If $S(TM)$ is totally umbilical in M, then, using $(3.11)_2$, we have

$$-1 = C(\xi, V) = \gamma g(\xi, V) = \gamma\, 0 = 0.$$

It is a contradiction. Thus there exist no ascreen lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M.

(2) If M is screen conformal, then, using (2.8) and $(3.11)_2$, we show that

$$-1 = C(\xi, V) = \varphi B(\xi, V) = \varphi\, 0 = 0.$$

It is a contradiction. Thus there exist no screen conformal ascreen lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$.

If M is proper nearly screen conformal, then, replacing X by ξ and Y by V to (3.8) and using $(3.11)_2$, we show that $\psi = 1$. Taking the scalar product with Y to (3.10) and using (3.8) and the fact $\psi = 1$, we have

$$(a + b\varphi)B(X, Y) = -g(FX, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.12}$$

Using (2.1), we show that the equation $\bar{g}(X, JY) + \bar{g}(JX, Y) = 0$ is equivalent to the equation $\bar{g}(JX, JY) = g(X, Y) - \theta(X)\theta(Y)$. From this result and (3.5), we have $g(FX, Y) + g(X, FY) = 0$ for all $X,\, Y \in \Gamma(S(TM))$. This result implies that $\phi(X, Y) = g(FX, Y)$ is skew-symmetric. As the left term of (3.12) is symmetric and the right term is skew-symmetric, we get $(a + b\varphi)B(X, Y) = 0$ and

$$g(FX, Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

From the fact $S(TM)$ is non-degenerate, we have $FX = 0$ for all $X \in \Gamma(TM)$. Using this result and (3.7), we have $g(X, Y) + v(X)v(Y) = 0$ for all $X,\, Y \in \Gamma(TM)$. Taking $X,\, Y \in \Gamma(\mathcal{D})$, we have $g(X, Y) = 0$. This implies $\mathcal{D}$ is degenerate. As $\mathcal{D}$ is non-degenerate, it is a contradiction. Thus there exist no nearly screen conformal ascreen lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$. □

Theorem 3.5. *Let M be an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then $S(TM)$ is not parallel distribution. Moreover if $\dim \bar{M} > 3$, then $S(TM)$ is not integrable distribution.*

Proof. For $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$, we have $C(X, Y) = g(\nabla_X Y, N)$ by (2.6). If $S(TM)$ is parallel distribution, then we have $\nabla_X Y = 0$. This results imply that $C = 0$, i.e., $S(TM)$ is totally geodesic in M. It is a contradiction to (1) of Theorem 3.4. Thus $S(TM)$ is not parallel distribution.

For all $X,\, Y \in \Gamma(S(TM))$, by (2.6) we have $C(X, Y) = g(\nabla_X Y, N)$. Thus

$$C(X, Y) - C(Y, X) = g(\nabla_X Y - \nabla_Y X, N) = \eta([X, Y]).$$

If $S(TM)$ is integrable, then $\eta([X, Y]) = 0$ for all $X,\, Y \in \Gamma(S(TM))$. This implies that C is symmetric on $S(TM)$. For any $Y \in \Gamma(S(TM))$, we show that $\bar{g}(\zeta, Y) = 0$. Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, Y) = 0$ and using (2.2), (2.4) and (2.6), we have

$$\bar{g}(JX, Y) + bC(X, Y) + aB(X, Y) = 0, \quad \forall\, X \in \Gamma(TM).$$

Using this equation and the fact that B and C are symmetric, we get

$$\bar{g}(JX,Y)-\bar{g}(X,JY)=0,\quad \forall\, X,\,Y\in\Gamma(S(TM)).$$

On the other hand, since $\bar{g}(X,JY)+\bar{g}(JX,Y)=0$, we have $\bar{g}(JX,Y)=0$ for all $X,\,Y\in\Gamma(S(TM))$. From this, (3.1) and (3.5), we have

$$g(FX,\,Y)=0,\quad \forall\, X,\,Y\in\Gamma(S(TM)).$$

As $S(TM)$ is non-degenerate, we have $FX=0$ for all $X\in\Gamma(S(TM))$. From this and (3.6), we have $X=0$ for all $X\in\Gamma(S(TM))$, i.e., $S(TM)=\{0\}$. It is a contradiction as $\dim\bar{M}>3$. Thus $S(TM)$ is not integrable. □

Theorem 3.6. *Let M be an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. $\mathcal{D}$ is an integrable distribution on M if and only if we have*

$$(\nabla_X F)Y=(\nabla_Y F)X,\quad \forall\, X,\,Y\in\Gamma(\mathcal{D}).$$

Proof. Taking $Y\in\Gamma(\mathcal{D})$, we show that $FY=JY\in\Gamma(\mathcal{D})$ and $\theta(Y)=v(Y)=0$. Applying $\bar{\nabla}_X$ to $JY=FY$ and using (2.3), (2.4), (3.3) and (3.5), we have

$$B(X,FY)=b\{g(\nabla_X Y,V)-g(X,Y)\},\tag{3.13}$$

$$(\nabla_X F)Y=-a\{g(\nabla_X Y,V)+g(X,Y)\}\xi+\{aB(X,Y)-\theta(\nabla_X Y)\}V.\tag{3.14}$$

From (3.14), for all $X,\,Y\in\Gamma(TM)$, we show that

$$(\nabla_X F)Y-(\nabla_Y F)X=-ag([X,Y],\,V)\xi-bg([X,Y],\,N)V.\tag{3.15}$$

If $\mathcal{D}$ is an integrable distribution, then $[X,Y]\in\Gamma(\mathcal{D})$ for any $X,\,Y\in\Gamma(\mathcal{D})$. This implies $g([X,Y],V)=0$ and $g([X,Y],\,N)=0$. Thus we get $(\nabla_X F)Y=(\nabla_Y F)X$ for all $X,\,Y\in\Gamma(\mathcal{D})$. Conversely if $(\nabla_X F)Y=(\nabla_Y F)X$ for all $X,\,Y\in\Gamma(\mathcal{D})$, then, taking the scalar product with N and V to (3.15) by turns, we have $g([X,Y],V)=0$ and $g([X,Y],\,N)=0$. This imply $[X,Y]\in\Gamma(\mathcal{D})$ for all $X,\,Y\in\Gamma(\mathcal{D})$. Thus $\mathcal{D}$ is an integrable distribution of M. □

4. Totally Geodesic Ascreen Lightlike Hypersurfaces

Theorem 4.1. *Let M be an ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. If M is totally geodesic, then the Ricci type tensor $R^{(0,\,2)}$ is an induced symmetric Ricci tensor of M and the transversal connection is flat.*

Proof. Applying ∇_X to $Y[a]=a\tau(Y)+av(Y)$ and using $(3.9)_1$, we have

$$\begin{aligned}XY[a]&=aX(\tau(Y))+aX(v(Y))+a\tau(X)\tau(Y)+av(X)\tau(Y)\\&\quad+a\tau(X)v(Y)+av(X)v(Y),\quad \forall\, X,\,Y\in\Gamma(TM).\end{aligned}$$

From this, $(3.9)_1$ and the facts $[X, Y] = XY - YX$ and $a \neq 0$, we have

$$d\tau(X, Y) + dv(X, Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Denote by $\bar{R}$ and R the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ and the induced connection ∇ of M respectively. Using the local Gauss-Weingarten formulas for M, for all $X,\, Y,\, Z \in \Gamma(TM)$, we obtain the Gauss equation for M:

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + B(X, Z)A_{N}Y - B(Y, Z)A_{N}X \\ & + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N.\end{aligned} \tag{4.1}$$

Assume that M is totally geodesic. Applying $\bar{\nabla}_X$ to $bV = -J\xi$ and using (2.3), (2.4), (2.7), (3.3), (3.5), $(3.9)_2$ and the fact $A^*_{\xi} = B = 0$, we get

$$\nabla_X V = -X + v(X)V, \quad \forall\, X \in \Gamma(TM). \tag{4.2}$$

Using (4.2) and the fact that ∇ is a torsion free connection, we have

$$R(X, Y)V = 2dv(X, Y)V + v(X)Y - v(Y)X, \ \forall\, X,\, Y \in \Gamma(TM).$$

Replacing Z by V to (4.1) and using the fact $B = 0$, we get

$$\bar{R}(X, Y)V = 2dv(X, Y)V + v(X)Y - v(Y)X, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking the scalar product with V to this equation and using $(3.3)_2$, we have $dv = 0$. Thus we have $d\tau = 0$. By Theorem 2.1, we show that $R^{(0,\,2)}$ is an induced symmetric Ricci tensor of M and the transversal connection $\nabla^{\perp}$ of M is flat. □

Definition 4. We say that M is *semi-symmetric* [6] if its curvature tensor R satisfies $R(X, Y)R = 0$ for any $X,\, Y \in \Gamma(TM)$.

Theorem 4.2. *Let M be a totally geodesic ascreen lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. If M is semi-symmetric, then M is of constant positive curvature* 1.

Proof. Assume that M is totally geodesic. As $dv = 0$, we obtain

$$R(X, Y)V = v(X)Y - v(Y)X, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.3}$$

Applying $\bar{\nabla}_X$ to $v(Y) = -g(Y, V)$ and using (2.4) and (4.2), we have

$$(\nabla_X v)(Y) = g(X, Y) + v(X)v(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.4}$$

Applying ∇_Z to (4.3) and using (4.2), (4.3) and (4.4), we have

$$(\nabla_Z R)(X, Y)V - R(X, Y)Z = g(X, Z)Y - g(Y, Z)X, \tag{4.5}$$

for all X, Y, $Z \in \Gamma(TM)$. Replacing Z by V to (4.5) and using (4.3) and (4.5), we have $(\nabla_V R)(X,Y)V = 0$. Substituting (4.5) into $(R(U,Z)R)(X,Y)V = 0$ and using (4.2), (4.3) and (4.5), we have

$$v(Z)(\nabla_U R)(X,Y)V - v(U)(\nabla_Z R)(X,Y)V = 0, \tag{4.6}$$

for all X, Y, Z, $U \in \Gamma(TM)$. Replacing U by V to (4.6) and using $(\nabla_V R)(X,Y)V = 0$, we have $(\nabla_Z R)(X,Y)V = 0$. From this and (4.5), we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

for all X, Y, $Z \in \Gamma(TM)$. Thus M is a space of constant positive curvature 1. □

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