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Monodromy Groups on Knot Surgery 4-manifolds

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Abstract. In the article we show that nondiffeomorphic symplectic 4-manifolds which admit marked Lefschetz fibrations can share the same monodromy group. Explicitly we prove that, for each integer $g > 0$, every knot surgery 4-manifold in a family $\{E(2)_K \mid K$ is a fibered 2-bridge knot of genus g in $S^3\}$ admits a marked Lefschetz fibration structure which has the same monodromy group.

1. Introduction

Seiberg-Witten invariants are one of the most powerful invariants in the classification of smooth 4-manifolds and Fintushel-Stern's knot surgery method is one of the most effective methods to modify smooth structures on a given 4-manifold. But Seiberg-Witten invariants are not complete invariants and there are known examples of nondiffeomorphic symplectic 4-manifolds which share the same Seiberg-Witten invariants [3, 15].

R. Fintushel and R. Stern showed that Seiberg-Witten invariants of knot surgery 4-manifold

$$E(2)_K = E(2)\sharp_{F=m_K\times S^1}(M_K \times S^1)$$

can be computed by using the Alexander polynomial of the related knot K [2]. If we restrict our attention to a fibered knot K, then $E(2)_K$ naturally has a symplectic structure by a result of R. Gompf [6]. Since there are infinitely many fibered knots of genus $g \geq 2$ which share the same Alexander polynomial, we could have an infinite family of symplectic 4-manifolds which share the same Seiberg-Witten invariants. But most of these manifolds cannot be distinguished in smooth category even though they are expected to be nondiffeomorphic each other.

On the one hand, by a result of S. Donaldson and R. Gompf [7], a symplectic 4-manifold is characterized by its Lefschetz pencil or Lefschetz fibration structure. It is also well known that a Lefschetz fibration over S^2 with fiber genus bigger than one is characterized by its monodromy factorization [12]. Moreover, R. Fintushel

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and R. Stern [4] figured out a Lefschetz fibration structure on $E(2)_K$ and its explicit monodromy factorization was known [17]. So it is very natural to ask whether one can define an invariant coming from monodromy factorization. A conjugacy class of a monodromy group, which is a subgroup of the mapping class group generated by each single letter in monodromy factorization, is a well-defined invariant of a Lefschetz fibration up to Lefschetz fibration isomorphism. By using this fact, we could give a family of simply connected symplectic 4-manifolds which have more than one inequivalent Lefschetz fibration structures [13, 14].

In this article, we show that this conjugacy class of a monodromy group is a very rough invariant and there is a family of knot surgery 4-manifolds which share the same conjugacy class of monodromy group even though they are not diffeomorphic each other.

Theorem 1.1. *For each positive integer $g > 0$, every knot surgery 4-manifold lying in $\{E(2)_K \mid K \text{ is a fibered 2-bridge knot of genus } g \text{ in } S^3\}$ admits a marked Lefschetz fibration structure which shares the same monodromy group.*

In [15], we constructed a family of smooth 4-manifolds which share the same Seiberg-Witten invariants even though they are not diffeomorphic each other. Such examples come from a family of fibered 2-bridge knots $K(n,i)$ with the same Alexander polynomial (Definition 3.4.) and they are distinguished by using a covering method. So it may also be possible to distinguish

$$\{E(2)_{K(n,i)} | n \in \mathbb{Z}_+, \quad i = 0, 1, \cdots, 2^n - 1\}$$

in smooth category even though we don't know the answer yet.

Corollary 1.2. *For each integer $n \geq 1$ and $i = 0, 1, 2, \cdots, 2^n - 1$, every knot surgery 4-manifold $E(2)_{K(n,i)}$ admits a marked monodromy factorization which shares the same monodromy group.*

One the other hand, R. Fintushel and R. Stern [4] constructed various families of 4-manifolds which share the same Seiberg-Witten invariants. One of them is

$$Y(2 : K, K') = E(2)_K \sharp_{\Sigma_{2g+1}} E(2)_{K'}$$

where K and K' are fibered knots of genus g. In [14], we proved that smooth structure of $Y(2 : K, K')$ does not determine the knot type of K and K'. Such examples are constructed by using a pair of Kanenobu's knots. In this article, we also show that $Y(2 : K, K')$ are diffeomorphic each other for any given pair of fibered 2-bridge knots K and K' of same genus.

Corollary 1.3. *For any fibered 2-bridge knots K and K' of the same genus $g > 0$,*

$$Y(2; K, K') = E(2)_K \sharp_{\Sigma_{2g+1}} E(2)_{K'}$$

are all diffeomorphic to each other.

2. Lefschetz Fibration and Its Monodromy Factorization

In the section we will briefly review some well-known facts about symplectic Lefschetz fibration and its monodromy factorization.

Definition 2.1. Let X be a compact smooth oriented 4-manifold and B be a compact oriented smooth two manifold. A smooth map $f : M \to B$ is a *Lefschetz fibration* of genus g if

(1) $f^{-1}(\partial B) = \partial X$

(2) f has finitely many and nonempty set of critical values

$$\{b_1, b_2, \cdots, b_n\} \subset \mathrm{Int}(B)$$

and f is a smooth genus g fiber bundle over $B - \{b_1, b_2, \cdots, b_n\}$

(3) there is unique critical point p_i in $f^{-1}(b_i)$ and f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$

(4) we also assume that there is no -1 sphere on each $f^{-1}(b_i)$.

Definition 2.2. For a given Lefschetz fibration $f : X \to S^2$ with n singular values, we can consider X as

$$(F \times D^2) \cup (h_1^2 \cup h_2^2 \cup \cdots \cup h_n^2) \cup (F \times D^2)$$

where F is a closed oriented Riemann surface and h_i^2 is a 4-dimensional two handle $D^2 \times D^2$ whose attaching sphere is a simple closed curve c_i on F so that

$$t_{c_n} t_{c_{n-1}} \cdots t_{c_2} t_{c_1}$$

is the identity element in the surface mapping class group $\mathrm{Mod}(F)$. This ordered sequence of right handed Dehn twists is called *monodromy factorization* of the Lefschetz fibration and we denote it by $t_{c_n} \cdot t_{c_{n-1}} \cdots t_{c_2} \cdot t_{c_1}$.

It is known that a Lefschetz fibration is characterized by its monodromy factorization, an ordered sequence of right handed Dehn twists [9, 12]. A right handed Dehn twist along a simple closed curve c is denoted by t_c. In the article we use usual function notation, *i.e.* $t_c t_d$ means that we apply t_d first and then apply t_c.

For any element $f \in \mathrm{Mod}(F)$ and a simple closed curve c on F,

$$f(t_c) = f t_c f^{-1} = t_{f(c)}.$$

A monodromy factorization is well defined up to Hurwitz equivalences which come from a choice of Hurwitz system and simultaneous conjugation equivalences which come from a choice of generic fiber of the Lefschetz fibration.

Definition 2.3. Two monodromy factorizations W_1 and W_2 are *Hurwitz equivalence*, denoted by $W_1 \sim W_2$, if W_1 can be changed to W_2 in finitely many steps by using the following two operations:

(1) *Hurwitz move:* $t_{c_n} \cdot ... \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot ... \cdot t_{c_1} \to t_{c_n} \cdot ... \cdot t_{c_{i+1}}(t_{c_i}) \cdot t_{c_{i+1}} \cdot ... \cdot t_{c_1}$

(2) *inverse Hurwitz move:* $t_{c_n} \cdot ... \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot ... \cdot t_{c_1} \to t_{c_n} \cdot ... \cdot t_{c_i} \cdot t_{c_i}^{-1}(t_{c_{i+1}}) \cdot ... \cdot t_{c_1}$.

The *simultaneous conjugation equivalence* of two monodromy factorizations is given by

$$t_{c_n} \cdot t_{c_{n-1}} \cdot ... \cdot t_{c_2} \cdot t_{c_1} \equiv f(t_{c_n}) \cdot f(t_{c_{n-1}}) \cdot ... \cdot f(t_{c_2}) \cdot f(t_{c_1})$$

for some $f \in \mathrm{Mod}(F)$. We will consider $f(w_k \cdot ... \cdot w_2 \cdot w_1)$ as $f(w_k) \cdot ... \cdot f(w_2) \cdot f(w_1)$.

Definition 2.4. [12] Two Lefschetz fibrations $f : M \to B$, $f' : M' \to B'$ are *isomorphic* if there are orientation preserving diffeomorphisms $H : M \to M'$ and $h : B \to B'$ such that

$$\begin{array}{ccc} M & \xrightarrow{H} & M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{h} & B' \end{array}$$

commutes *i.e* $f' \circ H = h \circ f$.

Theorem 2.5. [9, 12] *Let $X_i \to \mathbb{CP}^1$, $i = 1, 2$, be Lefschetz fibrations of genus g with monodromy factorization W_i corresponding to a fixed generic fiber F_i. Then the two Lefschetz fibrations are isomorphic if and only if W_1 can be changed to W_2 by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences.*

From the above theorem, we can define a map from the set of all isomorphic class of genus g Lefschetz fibration over S^2 to the set of all conjugacy classes of subgroups of mapping class group $\mathrm{Mod}(\Sigma_g)$ of oriented closed surface of genus g.

Definition 2.6. For a given Lefschetz fibration $f : X \to S^2$ with n singular values and generic fiber F, let us fix an identification of generic fiber F with oriented closed Riemann surface Σ_g (which is called *marked Lefschetz fibration*), then its monodromy factorization is given by $t_{c_n} \cdot t_{c_{n-1}} \cdots t_{c_2} \cdot t_{c_1}$ by using isotopy class of simple closed curves $\{c_1, c_2, \cdots, c_n\}$ on Σ_g. The subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by $\{t_{c_1}, t_{c_2}, \cdots, t_{c_n}\}$ is called the *monodromy group* of the marked Lefschetz fibration and it is denoted by

$$G_F(t_{c_n} \cdot t_{c_{n-1}} \cdots t_{c_2} \cdot t_{c_1}).$$

Remark 2.7. Hurwitz move and inverse Hurwitz move does not change monodromy group. Two marked Lefschetz fibrations are isomorphic if one monodromy factorization can be changed to the other monodromy factorization by a finite sequence of Hurwitz equivalences. So in the case two corresponding monodromy groups are the same.

Now we will briefly explain how to consider knot surgery 4-manifold $E(n)_K$ with fibered knot K can be considered as a Lefschetz fibration.

Definition 2.8. Let $M(n,g)$ be the desingularization of the two fold covering of $\Sigma_g \times S^2$ with branch set

$$(\Sigma_g \times \{2 \text{ pts}\}) \cup (\{2n \text{ pts}\} \times S^2).$$

Then this manifold is diffeomorphic to $(\Sigma_g \times S^2)\sharp 4n\overline{\mathbb{CP}}^2$.

If we consider it as a singular genus $(2g+n-1)$ fibration over S^2 with two singular fibers, then after a local perturbation we get a Lefschetz fibration with $2(2g+4n-2)$ singular values by computing the Euler number and signature of $(\Sigma_g \times S^2)\sharp 4n\overline{\mathbb{CP}}^2$. It is known to various authors [12, 11, 8, 16] that the corresponding involution can be written as a product of right handed Dehn twists.

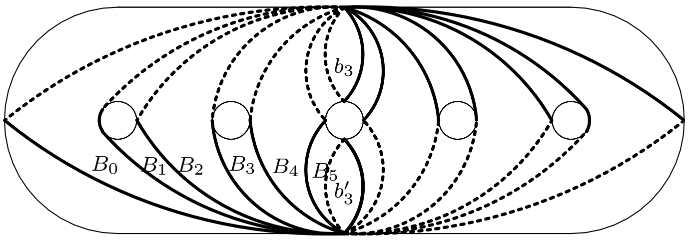

Figure 1: Simple closed curves for Korkmaz word: $g=2$ and $n=2$ case

Theorem 2.9. [11] *Monodromy of $M(2,g)$ is given by η_g^2 where*

$$\eta_g = t_{B_0} \cdot t_{B_1} \cdots t_{B_{2g+1}} \cdot t^2_{b_{g+1}} \cdot t^2_{b'_{g+1}}.$$

Theorem 2.10. [4, 17] *Let K be a fibered knot of genus g such that*

$$S^3 \setminus \nu(K) = ([0,1] \times \Sigma_g^1)/_{(1,x)\sim(0,\phi_K(x))}$$

and let

$$\Phi_K = \phi_K \sharp id \sharp id : \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g \to \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g,$$

then monodromy factorization of $E(2)_K$ is given by $\Phi_K(\eta_g^2) \cdot \eta_g^2$.

3. Monodromy Group

Assume that p and q are relatively prime integers with p odd. Let us consider a 2-bridge knot $b(p,q)$ which is defined as follows:

Definition 3.1.[1] A 2-*bridge knot* $b(p,q)$ is of the form

$$C(n_1, -n_2, n_3, -n_4, \cdots, (-1)^{k-1}n_k)$$

as in Figure 2, where

$$\frac{q}{p} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots \cfrac{1}{n_{k-1} + \frac{1}{n_k}}}}} = [n_1, n_2, \cdots, n_k].$$

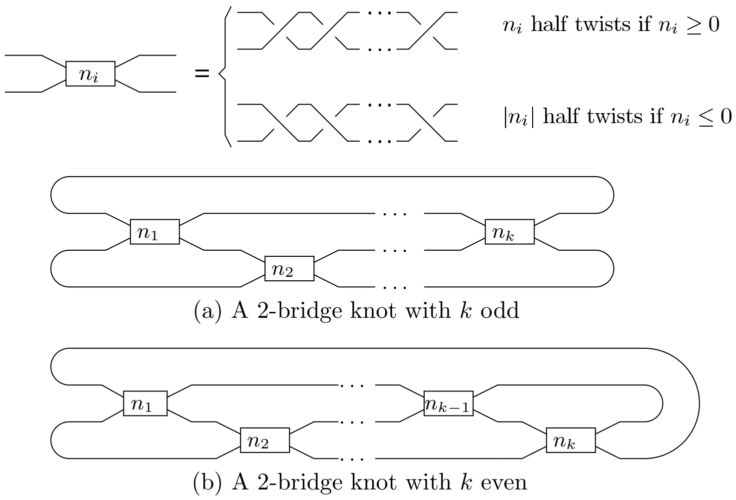

Figure 2: A 2-bridge knot $C(n_1, n_2, \cdots, n_k)$

It is a 4-plat whose defining braid is

$$\sigma_2^{n_1}\sigma_1^{-n_2}\sigma_2^{n_3}\sigma_1^{-n_4}\cdots\sigma_1^{-n_k} \quad \text{if } k \text{ is even},$$

$$\sigma_2^{n_1}\sigma_1^{-n_2}\sigma_2^{n_3}\sigma_1^{-n_4}\cdots\sigma_2^{n_k} \quad \text{if } k \text{ is odd}.$$

Here σ_i is a standard braid generator as in Figure 10.3 of [1]. We now denote

$$D(n_1, n_2, \cdots, n_k) = C(2n_1, 2n_2, \cdots, 2n_k).$$

Theorem 3.2. *Let K be any fibered 2-bridge knot of genus $g > 0$ and*

$$K_g = D(\underbrace{-1, -1, \cdots, -1}_{2g}).$$

Then $E(2)_K$ and $E(2)_{K_g}$ admit marked monodromy factorizations whose monodromy groups are the same.

Proof. It is known [5, 10] that any fibered 2-bridge knot of genus g is of the form

$$D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})$$

where each $\epsilon_i \in \{+1, -1\}$. Since a Seifert surface of $D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})$ with $\epsilon_i \in \{+1, -1\}$ can be obtained by a sequence of plumbings of positive or negative Hopf band corresponding to $\epsilon_i = +1$ or $\epsilon_i = -1$ and since a positive Hopf band corresponds to a right handed Dehn twist along the core circle of Hopf band and negative Hopf band corresponds to a left handed Dehn twist, we get

$$\Phi_{D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})} = t_{c_{2g}}^{\epsilon_{2g}} t_{c_{2g-1}}^{\epsilon_{2g-1}} t_{c_{2g-2}}^{\epsilon_{2g-2}} \cdots t_{c_2}^{\epsilon_2} t_{c_1}^{\epsilon_1}$$

where simple closed curves c_i are as in Figure 3.

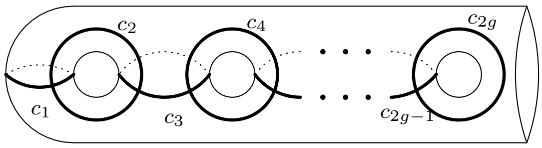

Figure 3: Simple closed curves for monodromy of 2-bridge knot

Let H be the subgroup of $\mathrm{Mod}(\Sigma_{2g+1})$ which is generated by

$$\{t_{B_i}, t_{c_j}, t_{b_{g+1}}, t_{b'_{g+1}} \mid i = 0, 1, 2, \cdots, 2g+1,\ j = 1, 2, \cdots, 2g\}$$

by using the notation as in Figure 1.

Let us first show that $G_F(\Phi_{K_g}(\eta_g^2) \cdot \eta_g^2) = H$.

Since $\Phi_{K_g} = t_{c_{2g}}^{-1} t_{c_{2g-1}}^{-1} t_{c_{2g-2}}^{-1} \cdots t_{c_2}^{-1} t_{c_1}^{-1} \in H$, we get

$$\Phi_{K_g}(t_{B_i}) = \Phi_{K_g} t_{B_j} \Phi_{K_g}^{-1} \in H \text{ for each } i = 0, 1, \cdots, 2g+1$$

and $\Phi_{K_g}(t_{b_{g+1}}) = t_{b_{g+1}}$, $\Phi_{K_g}(t_{b'_{g+1}}) = t_{b'_{g+1}}$. So $G_F(\Phi_{K_g}(\eta_g^2) \cdot \eta_g^2) \leq H$.

Now we will prove that $H \leq G_F(\Phi_{K_g}(\eta_g^2) \cdot \eta_g^2)$. To do this, we need to check that

$$t_{c_j} \in G_F(\Phi_{K_g}(\eta_g^2) \cdot \eta_g^2)$$

for each $j = 1, 2, \cdots, 2g$. Let us observe from Figure 4 that

$$\begin{aligned} (3.1) \qquad \Phi_{K_g}(B_j) &= t_{c_{2g}}^{-1} t_{c_{2g-1}}^{-1} t_{c_{2g-2}}^{-1} \cdots t_{c_2}^{-1} t_{c_1}^{-1}(B_j) \\ &= t_{c_{2g}}^{-1} t_{c_{2g-1}}^{-1} \cdots t_{c_{j+1}}^{-1} t_{c_j}^{-1}(B_j) \text{ because } c_i \cap B_j = \emptyset \text{ for } i < j \\ &= t_{c_{2g}}^{-1} t_{c_{2g-1}}^{-1} \cdots t_{c_{j+1}}^{-1}(t_{c_j}^{-1}(B_j)) \\ &= t_{c_j}^{-1}(B_j) \text{ because } t_{c_j}^{-1}(B_j) \cap c_i = \emptyset \text{ for } j < i. \end{aligned}$$

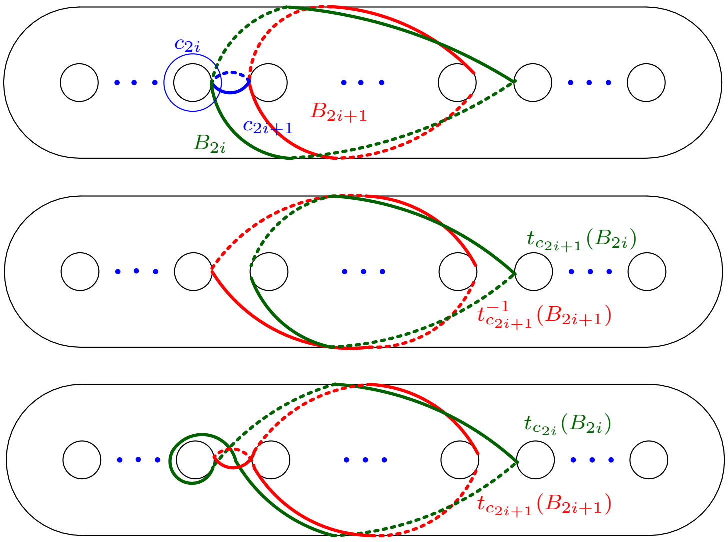

Figure 4: $t_{c_{2i}}(B_{2i})$, $t_{c_{2i+1}}(B_{2i})$ and $t^{\pm 1}_{c_{2i+1}}(B_{2i+1})$

Therefore we get

$$c_j = t^{-1}_{B_j}(t^{-1}_{c_j}(B_j)) = t^{-1}_{B_j}(\Phi_{K_g}(B_j)) \tag{3.2}$$

as in Figure 5. Since $t^{\pm 1}_{B_j}$, $\Phi_{K_g}(B_j) \in G_F(\Phi_{K_g}(\eta_g^2)\cdot\eta_g^2)$, it implies

$$t_{c_j} = t^{-1}_{B_j}(\Phi_{K_g}(t_{B_j})) = t^{-1}_{B_j}\Phi_{K_g}(t_{B_j})t_{B_j} \in G_F(\Phi_{K_g}(\eta_g^2)\cdot\eta_g^2) \tag{3.3}$$

for each $j = 1, 2, \cdots, 2g$. So we get

$$H \leq G_F(\Phi_{K_g}(\eta_g^2)\cdot\eta_g^2).$$

Now we will show that

$$G_F(\Phi_K(\eta_g^2)\cdot\eta_g^2) = H$$

for any fibered 2-bridge knot $K = D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})$.

If $\epsilon_i = -1$, then by the same method of equation (3.1) we get $\Phi_K(B_i) = t^{-1}_{c_i}(B_i)$ and $c_i = t^{-1}_{B_i}(\Phi_K(B_i))$. So by the same way as in equation (3.3) we get

$$t_{c_i} \in G_F(\Phi_K(\eta_g^2)\cdot\eta_g^2) \tag{3.4}$$

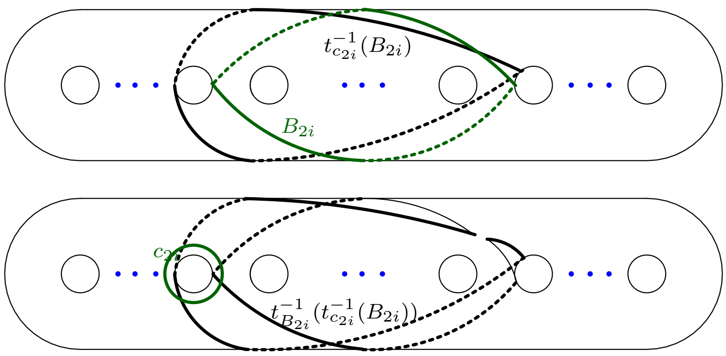

Figure 5: $t_{B_j}^{-1}(t_{c_j}^{-1}(B_j))$ is isotopic to c_j for $j = 2i$ (same for $j = 2i+1$)

whenever $\epsilon_i = -1$.

Now let us consider the case $\epsilon_{i_0} = +1$ and $\epsilon_j = -1$ for each $i_0 + 1 \leq j \leq 2g$. Then $t_{c_j} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$ for each $j = i_0+1, i_0+2, \cdots, 2g$ by equation (3.4). Therefore

$$(t_{c_{i_0}} t_{c_{i_0-1}}^{\epsilon_{i_0-1}} \cdots t_{c_2}^{\epsilon_2} t_{c_1}^{\epsilon_1})(t_{B_\ell}) \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$$

for each $\ell = 0, 1, \cdots, 2g+1$. Because $c_j \cap B_{i_0} = \emptyset$ for each $j = 1, 2, \cdots, i_0 - 1$, we get

$$(t_{c_{i_0}} t_{c_{i_0-1}}^{\epsilon_{i_0-1}} \cdots t_{c_2}^{\epsilon_2} t_{c_1}^{\epsilon_1})(B_{i_0}) = t_{c_{i_0}}(B_{i_0})$$

and

$$c_{i_0} = t_{B_{i_0}}(t_{c_{i_0}}(B_{i_0})) = t_{B_{i_0}}(t_{c_{i_0}} t_{c_{i_0-1}}^{\epsilon_{i_0-1}} \cdots t_{c_2}^{\epsilon_2} t_{c_1}^{\epsilon_1}(B_{i_0}))$$

as in Figure 6.

Therefore

$$t_{c_{i_0}} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2). \tag{3.5}$$

Now we will use mathematical induction argument. Suppose that $\epsilon_i = +1$ for $i = i_0, i_1, \cdots, i_N$ which satisfies $1 \leq i_N < \cdots < i_1 < i_0 \leq 2g$ and all other $\epsilon_i = -1$. Then by Equations (3.4) and (3.5),

$$t_{c_i} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$$

for each $j = i_1 + 1, i_1 + 2, \cdots, 2g$. So we can apply the same method as before and we get $t_{c_{i_1}} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$.

By repeating the same method, we get

$$t_{c_{i_j}} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2) \tag{3.6}$$

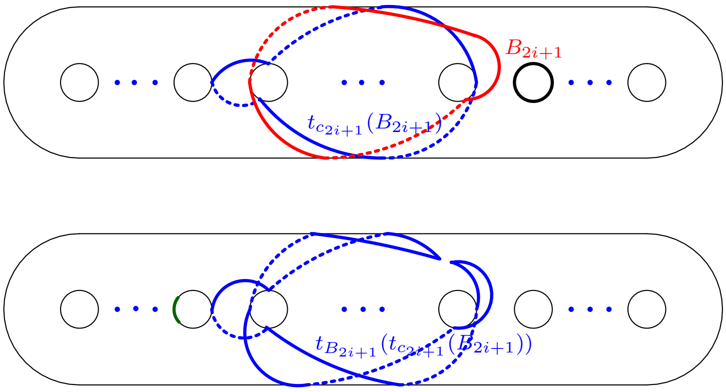

Figure 6: $t_{B_j}(t_{c_j}(B_j))$ is isotopic to c_j for $j = 2i+1$ (same for $j = 2i$)

for each $j = 0, 1, 2, \cdots, N$ at which $\epsilon_{i_j} = +1$.

So Equations (3.4) and (3.6) imply that $t_{c_i} \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$ for each $i = 1, 2, \cdots, 2g$. Therefore we get

$$H = G_F(\Phi_{D(\epsilon_1,\epsilon_2,\cdots,\epsilon_{2g-1},\epsilon_{2g})}(\eta_g^2) \cdot \eta_g^2).$$

$\square$

Remark 3.3. Note that smooth 4-manifolds with the same Seiberg-Witten invariants are very hard to prove whether they are diffeomorphic or not in general. Regarding this direction, R. Fintushel and R. Stern first constructed a pair of non-diffeomorphic 4-manifolds which share the same Seiberg-Witten invariants [3] by using covering method at the price of big fundamental group and recently we extended such family of examples [15]. A special family of 2-bridge knots in Definition 3.4. are the main ingredient when we constructed such examples.

Definition 3.4. [15] Let us define inductively a family of 2-bridge knots as follows:

(a) Set $W(0,0) = 1, 1$ and $K(0,0) = D(W(0,0))$.

(b) For each integer $n > 0$ and $i = \sum_{j=0}^{n-1} \varepsilon_j 2^j$ with $\varepsilon_j \in \{0,1\}$, define a list $W(n,i)$ by

$$W(n-1, \sum_{j=0}^{n-2} \varepsilon_j 2^j), (-1)^{\varepsilon_{n-1}+1}, -W(n-1, \sum_{j=0}^{n-2} \varepsilon_j 2^j), (-1)^{\varepsilon_{n-1}+1}, W(n-1, \sum_{j=0}^{n-2} \varepsilon_j 2^j)$$

and $K(n,i) = D(W(n,i))$.

Corollary 3.5. *For each integer $n \geq 1$ and $i = 0, 1, 2, \cdots, 2^n - 1$, every knot surgery 4-manifold $E(2)_{K(n,i)}$ admits a marked monodromy factorization which shares the same monodromy group.*

Proof. Since each $K(n, i)$ is a fibered 2-bridge knot, we get the result directly from Theorem 3.2. □

Remark 3.6. Even though we could not distinguish $E(2)_{K(n,i)}$ by using monodromy group, we expect that these 4-manifolds can be distinguished in smooth category by using other new invariants.

Corollary 3.7. *For any fibered 2-bridge knots K and K' of the same genus $g > 0$,*

$$Y(2; K, K') = E(2)_K \sharp_{\Sigma_{2g+1}} E(2)_{K'}$$

are all diffeomorphic to each other.

Proof. It is shown in [17] that if $\varphi \in G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2)$, then

$$\Phi_K(\eta_g^2) \cdot \eta_g^2 \cdot \Psi(\eta_g^2) \cdot \eta_g^2 \sim \Phi_K(\eta_g^2) \cdot \eta_g^2 \cdot (\varphi^{\pm 1}\Psi)(\eta_g^2) \cdot \eta_g^2$$

for any $\Psi \in \mathrm{Mod}(\Sigma_{2g+1})$. Theorem 3.2 implies that

$$\Phi_{K_g},\ \Phi_K,\ \Phi_{K'} \in H = G_F(\Phi_K(\eta_g^2) \cdot \eta_g^2) = G_F(\Phi_{K_g}(\eta_g^2) \cdot \eta_g^2)$$

for any fibered 2-bridge knot K and K', so we get

$$\begin{aligned}\Phi_K(\eta_g^2) \cdot \eta_g^2 \cdot \Phi_{K'}(\eta_g^2) \cdot \eta_g^2 &\sim \Phi_K(\eta_g^2) \cdot \eta_g^2 \cdot \eta_g^2 \cdot \eta_g^2 \\ &\sim \Phi_K(\eta_g^2) \cdot \eta_g^2 \cdot \Phi_{K_g}(\eta_g^2) \cdot \eta_g^2 \\ &\sim \eta_g^2 \cdot \eta_g^2 \cdot \Phi_{K_g}(\eta_g^2) \cdot \eta_g^2 \\ &\sim \Phi_{K_g}(\eta_g^2) \cdot \eta_g^2 \cdot \eta_g^2 \cdot \eta_g^2.\end{aligned}$$

If two 4-manifolds have isomorphic Lefschetz fibration structures, then they are diffeomorphic because they are related by a sequence of 2-handle moves. Therefore we get the conclusion. □

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