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COMMON FIXED POINT OF MAPS IN COMPLETE PARTIAL METRIC SPACES

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Abstract. In this paper, we prove some common fixed point results for some mappings satisfying generalized contractive condition in complete partial metric space.

1. Introduction

In the last years, the extension of the theory of fixed point to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received a lot of attention. One of the most interesting is partial metric space. Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [10]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, give a modified version of the Banach contraction principle, more suitable in this context [10]. Subsequently, Valero [14], Oltra and Valero [12] and Altun et al [2] gave some generalizations of the result of Matthews. Romaguera [13] proved the Caristi type fixed point theorem on this space. The purpose of this paper is to present a general fixed point theorem for two pairs of mappings on two partial metric spaces satisfying implicit relations. Our result generalizes the main result from [7] and [11].

First, we recall some definitions and results needed in the sequel. The reader interested in fixed point theory in partial metric spaces is referred to the work of [1, 8, 10, 12, 13, 14] and references therein.

A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

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A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [5] and [10].

Let (X, d) and (X, p) be a metric space and partial metric space, respectively.

Lemma 1. *Mappings $\rho_i : X \times X \longrightarrow \mathbb{R}^+$ ($i \in \{1, 2, 3\}$) defined by*

$$\begin{aligned} \rho_1(x, y) &= d(x, y) + p(x, y) \\ \rho_2(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\} \\ \rho_3(x, y) &= d(x, y) + a \end{aligned}$$

define partial metrics on X, where $\omega : X \longrightarrow \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p- balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X, p) is said to (i) converge to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$ (ii) Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$ which is is finite.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Suppose that $\{x_n\}$ is a sequence in partial metric space (X, p), then we define $L(x_n) = \{x | x_n \longrightarrow x\}$.

The following example shows that every convergent sequence $\{x_n\}$ in a partial metric space X may not be Cauchy. In particular, it shows that the limit of a convergent sequence is not unique.

Example 1. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Let

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k + 1. \end{cases}$$

Then clearly it is convergent sequence and for every $x \geq 1$ we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$, therefore $L(x_n) = [1, \infty)$. But $\lim_{n,m\to\infty} p(x_n, x_m)$ does not exist.

The following Lemma shows that under certain conditions the limit is unique.

Lemma 2. *Let $\{x_n\}$ be a convergent sequence in partial metric space X such that $x_n \longrightarrow x$ and $x_n \longrightarrow y$. If*

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then $x = y$.

Proof. As
$$p(x,y) \le p(x,x_n) + p(x_n,y) - p(x_n,x_n),$$
therefore
$$p(x_n,x_n) \le p(x,x_n) + p(x_n,y) - p(x,y).$$
By given assumptions, we have $\lim_{n\to\infty} p(x_n,x) = p(x,x)$, $\lim_{n\to\infty} p(x_n,y) = p(y,y)$, and $\lim_{n\to\infty} p(x_n,x_n) = p(x,x)$. Therefore
$$p(x,x) \le p(x,x) + p(y,y) - p(x,y)$$
which shows that $p(y,y) \le p(x,y) \le p(y,y)$. Also,
$$p(x,y) \le p(y,x_n) + p(x_n,x) - p(x_n,x_n)$$
implies that
$$p(x_n,x_n) \le p(y,x_n) + p(x_n,x) - p(x,y)$$
which on taking limit as $n \to \infty$ gives
$$p(y,y) \le p(y,y) + p(x,x) - p(x,y)$$
and
$$p(x,x) \le p(x,y) \le p(x,x)$$
Thus $p(x,x) = p(x,y) = p(y,y)$, therefore $x = y$. □

Lemma 3. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in partial metric space* X *such that*
$$\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = p(x,x),$$
and
$$\lim_{n\to\infty} p(y_n,y) = \lim_{n\to\infty} p(y_n,y_n) = p(y,y),$$
then $\lim_{n\to\infty} p(x_n,y_n) = p(x,y)$. *In particular,* $\lim_{n\to\infty} p(x_n,z) = p(x,z)$ *for every* $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ converge to a $x \in X$ and $y \in X$ respectively, therefore for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that
$$p(x,x_n) < p(x,x) + \frac{\epsilon}{2},\ p(y,y_n) < p(y,y) + \frac{\epsilon}{2},\ p(x,x_n) < p(x_n,x_n) + \frac{\epsilon}{2}$$
and
$$p(y,y_n) < p(y_n,y_n) + \frac{\epsilon}{2}$$
for $n \ge n_0$. Now
$$\begin{aligned} p(x_n,y_n) &\le p(x_n,x) + p(x,y_n) - p(x,x) \\ &\le p(x_n,x) + p(x,y) + p(y,y_n) - p(y,y) - p(x,x) \\ &< p(x,y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = p(x,y) + \epsilon, \end{aligned}$$
and so we have
$$p(x_n,y_n) - p(x,y) < \epsilon.$$

Also,

$$\begin{aligned} p(x,y) &\leq p(x,x_n)+p(x_n,y)-p(x_n,x_n) \\ &\leq p(x,x_n)+p(x_n,y_n)+p(y_n,y)-p(y_n,y_n)-p(x_n,x_n) \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}+p(x_n,y_n)=p(x_n,y_n)+\epsilon \end{aligned}$$

implies that

$$p(x,y)-p(x_n,y_n)<\epsilon.$$

Hence for all $n \geq n_0$, we have $|p(x_n,y_n)-p(x,y)|<\epsilon$. Hence the result follows. □

Lemma 4. *If p is a partial metric on X, then mappings $p^s, p^m : X \times X \to \mathbb{R}^+$ given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

and

$$p^m(x,y)=\max\left\{\ p(x,y)-p(x,x), p(x,y)-p(y,y)\ \right\}$$

define equivalent metrics on X.

Proof. It is easy to see that p^s and p^m are metrics on X. Obviously,

$$p^m(x,y)\leq p^s(x,y)$$

for every $x,y\in X$. As for every positive real numbers a and b, we have $a+b\leq 2\max\{a,b\}$, therefore

$$\begin{aligned} p^s(x,y) &= 2p(x,y)-p(x,x)-p(y,y) \\ &\leq 2\max\left\{\ p(x,y)-p(x,x), p(x,y)-p(y,y)\ \right\}=2p^m(x,y). \end{aligned}$$

Hence

$$\frac{1}{2}p^s(x,y)\leq p^m(x,y)\leq p^s(x,y).$$

So p^s and p^m are equivalent. □

Lemma 5. ([10], [12]) *Let (X,p) be a partial metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x)=0$ if and only if

$$p(x,x)=\lim_{n\to\infty}p(x_n,x)=\lim_{n,m\to\infty}p(x_n,x_m).$$

Lemma 6. *If $\{x_n\}$ is a convergent sequence in (X,p^s), then it is a convergent sequence in the partial metric space (X,p).*

Proof. As, $\lim_{n\to\infty} p^s(x_n, x) = 0$, and $p(x_n, x_n) \le p(x_n, x)$ for every n and $x \in X$, therefore

$$p(x_n, x) - p(x, x) \le p^s(x_n, x)$$

implies that

$$\limsup_{n\to\infty} p(x_n, x) - p(x, x) \le \lim_{n\to\infty} p^s(x_n, x)$$

and consequently, $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. □

2. Main results

Theorem 1. *Let (X, p) be a complete partial metric space. Let $S, T : X \longrightarrow X$ be two selfmappings.Suppose that there exists $r \in [0, 1)$ such that*

$$\max\{p(S(x), TS(x)), p(T(x), ST(x))\} \le r \min\{p(x, S(x)), p(x, T(x))\} \quad (2.1)$$

for every $x \in X$ and that

$$\alpha(y) = \inf\{p(x, y) + \min\{p(x, S(x)), p(x, T(x))\} : x \in X\} > 0 \quad (2.2)$$

for every $y \in X$ with y is not a common fixed point of S and T. Then there exists $z \in X$ such that $z = S(z) = T(z)$. Moreover, if $v = S(v) = T(v)$, then $p(v, v) = 0$.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$\begin{aligned} x_n &= S(x_{n-1}),\ if\ n\ is\ odd \\ &= T(x_{n-1}),\ if\ n\ is\ even. \end{aligned}$$

Then if $n \in \mathbb{N}$ is odd, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(S(x_{n-1}), T(x_n)) \\ &= p(S(x_{n-1}), TS(x_{n-1})) \\ &\le \max\{p(S(x_{n-1}), TS(x_{n-1})), p(T(x_{n-1}), ST(x_{n-1}))\} \\ &\le r \min\{p(x_{n-1}, S(x_{n-1})), p(x_{n-1}, T(x_{n-1}))\},\ by\ (2.1) \\ &\le rp(x_{n-1}, S(x_{n-1})) \\ &= rp(x_{n-1}, x_n). \end{aligned}$$

If n is even, then by (2.1), we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(T(x_{n-1}), S(x_n)) \\ &= p(T(x_{n-1}), ST(x_{n-1})) \\ &\le \max\{p(T(x_{n-1}), ST(x_{n-1})), p(S(x_{n-1}), TS(x_{n-1}))\} \\ &\le r \min\{p(x_{n-1}, T(x_{n-1})), p(x_{n-1}, S(x_{n-1}))\}, \\ &\le rp(x_{n-1}, T(x_{n-1})) \\ &= rp(x_{n-1}, x_n). \end{aligned}$$

Thus for any positive integer n, it must be the case that

$$p(x_n, x_{n+1}) \le rp(x_{n-1}, x_n). \quad (2.3)$$

By repeated application of (2.3), we obtain

$$p(x_n, x_{n+1}) \leq r^n p(x_0, x_1).$$

Also, we have

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \leq r^n p(x_0, x_1)$$

and

$$p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}) \leq r^n p(x_0, x_1).$$

So, if $m > n$, then

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\leq [r^n + r^{n+1} + \cdots + r^{m-1}] p(x_0, x_1) \\ &\leq \frac{r^n}{1-r} p(x_0, x_1). \end{aligned}$$

By definition,

$$\begin{aligned} p^s(x_n, x_m) &= 2p(x_n, x_m) - p(x_m, x_m) - p(x_n, x_n) \\ &\leq 4\frac{r^n}{1-r}. \end{aligned}$$

Thus $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

That is $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete then from Lemma 5, the sequence $\{x_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, z) = 0$. Again from Lemma 5, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Assume that z is not a common fixed point of S and T. Then by hypothesis

$$\begin{aligned} 0 &< \inf\{p(x, z) + \min\{p(x, S(x)), p(x, T(x))\} : x \in X\} \\ &\leq \inf\{p(x_n, z) + \min\{p(x_n, S(x_n)), p(x_n, T(x_n))\} : n \in \mathbb{N}\} \\ &\leq \inf\left\{\frac{r^n}{1-r} p(x_0, x_1) + p(x_n, x_{n+1}) : n \in \mathbb{N}\right\} \\ &\leq \inf\left\{\frac{r^n}{1-r} p(x_0, x_1) + r^n p(x_0, x_1) : n \in \mathbb{N}\right\} = 0 \end{aligned}$$

which is a contradiction. Therefore, $z = S(z) = T(z)$.

If $v = S(v) = T(v)$ for some $v \in X$, then

$$\begin{aligned} p(v, v) &= \max\{p(S(v), TS(v)), p(T(v), ST(v))\} \\ &\leq r \min\{p(v, S(v)), p(v, T(v))\} \\ &= r \min\{p(v, v), p(v, v)\} \\ &= rp(v, v) \end{aligned}$$

which gives that, $p(v, v) = 0$. □

Example 2. Let (X,p) is a partial metric space where $X=\{\frac{1}{n}\}_{n=1}^{\infty}\cup\{0\}$ and $p(x,y)=\max\{x,y\}$. If define $S:X\longrightarrow X$ by $S(0)=0$, $S(\frac{1}{2n})=\frac{1}{4n+3}$, $S(\frac{1}{2n-1})=0$ and $T(0)=0$, $T(\frac{1}{2n-1})=\frac{1}{4n+4}$ and $T(\frac{1}{2n})=0$. Then for $x=\frac{1}{2n}$ we have

$$\begin{aligned}
&\max\{p(S(x),TS(x)),p(T(x),ST(x))\}\\
&\quad= \max\{p(S(\frac{1}{2n}),T(S(\frac{1}{2n}))),p(T(\frac{1}{2n}),S(T(\frac{1}{2n})))\}\\
&\quad= \max\{\frac{1}{4n+3},0)\}=\frac{1}{4n+3}\\
&\quad\le r\min\{p(x,S(x)),p(x,T(x))\}=r\min\{\frac{1}{2n},\frac{1}{2n}\}=r\frac{1}{2n}.
\end{aligned}$$

It is easy to see that the above inequality is true for $x=\frac{1}{2n-1}$ and $r=\frac{1}{2}$. Also,

$$\alpha(y)=\inf\{p(x,y)+\min\{p(x,S(x)),p(x,T(x))\}:x\in X\}>0 \qquad (2.4)$$

for every $y\in X$ with y is not a common fixed point of S and T for $\frac{1}{3}\le r<1$. These shows that the all conditions of Theorem 1 are satisfied and 0 is a fixed point for S,T.

Corollary 1. *Let (X,p) be a complete partial metric space and let $T:X\longrightarrow X$ be a mapping. Suppose that there exists $r\in[0,1)$ such that*

$$p(T(x),T^2(x))\le rp(x,T(x))\}$$

for every $x\in X$ and that

$$\alpha(y)=\inf\{p(x,y)+p(x,T(x)):x\in X\}>0$$

for every $y\in X$ with $y\neq T(y)$. Then there exists $z\in X$ such that $z=T(z)$. Moreover, if $v=T(v)$, then $p(v,v)=0$.

Proof. Taking S = T in Theorem 1, the conclusion of the Corollary follows. So Corollary 1 can be treated as a special case of Theorem 1. □

Theorem 2. *Let (X,p) be a complete partial metric space. Let S,T be mappings from X onto itself. Suppose that there exists $r>1$ such that*

$$\min\{p(TS(x),S(x)),p(ST(x),T(x))\}\ge r\max\{p(Sx,x),p(Tx,x)\} \qquad (2.5)$$

for every $x\in X$ and that

$$\alpha(y)=\inf\{p(x,y)+\min\{p(x,S(x)),p(x,T(x))\}:x\in X\}>0 \qquad (2.6)$$

for every $y\in X$ with y is not a common fixed point of S and T. Then there exists $z\in X$ such that $z=S(z)=T(z)$. Moreover, if $v=S(v)=T(v)$, then $p(v,v)=0$.

Proof. Let $x_0 \in X$ be arbitrary. Since S is onto, there is an element x_1 satisfying $x_1 \in S^{-1}(x_0)$. Since T is also onto, there is an element x_2 satisfying $x_2 \in T^{-1}(x_1)$. Proceeding in the same way, we can find $x_{2n+1} \in S^{-1}(x_{2n})$ and $x_{2n+2} \in S^{-1}(x_{2n+1})$ for $n = 1, 2, 3, \cdots$. Therefore, $x_{2n} = S(x_{2n+1}$ and $x_{2n+1} = S(x_{2n+2}$ for $n = 0, 1, 2, \cdots$. If $n = 2m$, then using (2.4)

$$\begin{aligned}
p(x_{n-1}, x_n) &= p(x_{2m-1}, x_{2m}) \\
&= p(Tx_{2m}, Sx_{2m+1}) \\
&= p(TSx_{2m+1}, Sx_{2m+1}) \\
&\geq \min\{p(TS(x_{2m+1}), S(x_{2m+1})), p(ST(x_{2m+1}), T(x_{2m+1}))\} \\
&\geq r \max\{p(Sx_{2m+1}, x_{2m+1}), p(Tx_{2m+1}, x_{2m+1})\} \\
&\geq rp(Sx_{2m+1}, x_{2m+1}) \\
&= rp(x_{2m}, x_{2m+1}) \\
&= rp(x_n, x_{n+1}).
\end{aligned}$$

If $n = 2m + 1$, then using (2.4)

$$\begin{aligned}
p(x_{n-1}, x_n) &= p(x_{2m}, x_{2m+1}) \\
&= p(Sx_{2m+1}, Tx_{2m+2}) \\
&= p(STx_{2m+2}, Tx_{2m+2}) \\
&\geq \min\{p(TS(x_{2m+2}), S(x_{2m+2})), p(ST(x_{2m+2}), T(x_{2m+2}))\} \\
&\geq r \max\{p(Sx_{2m+2}, x_{2m+2}), p(Tx_{2m+2}, x_{2m+2})\} \\
&\geq rp(Tx_{2m+2}, x_{2m+2}) \\
&= rp(x_{2m+1}, x_{2m+2}) \\
&= rp(x_n, x_{n+1}).
\end{aligned}$$

Thus for any positive integer n, it must be the case that

$$p(x_{n-1}, x_n) \geq rp(x_n, x_{n+1})$$

which implies that,

$$p(x_n, x_{n+1}) \leq \frac{1}{r} p(x_{n-1}, x_n) \leq \cdots \leq (\frac{1}{r})^n p(x_0, x_1). \tag{2.7}$$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < 1$ since $r > 1$.

Now, (2.6) becomes

$$p(x_n, x_{n+1}) \leq \alpha^n p(x_0, x_1).$$

Also, we have

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \leq \alpha^n p(x_0, x_1)$$

and

$$p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}) \leq \alpha^n p(x_0, x_1).$$

So, if $m > n$, then

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\leq [\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}] p(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} p(x_0, x_1). \end{aligned}$$

By definition,

$$\begin{aligned} p^s(x_n, x_m) &= 2p(x_n, x_m) - p(x_m, x_m) - p(x_n, x_n) \\ &\leq 4\frac{\alpha^n}{1-r}. \end{aligned}$$

Thus $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. That is $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete then from Lemma 5, the sequence $\{x_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, z) = 0$. Again from Lemma 5, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Assume that z is not a common fixed point of S and T. Then by hypothesis

$$\begin{aligned} 0 &< \inf\{p(x, z) + \min\{p(x, S(x)), p(x, T(x))\} : x \in X\} \\ &\leq \inf\{p(x_n, z) + \min\{p(x_n, S(x_n)), p(x_n, T(x_n))\} : n \in \mathbb{N}\} \\ &\leq \inf\left\{\frac{\alpha^n}{1-\alpha} p(x_0, x_1) + p(x_{n-1}, x_n) : n \in \mathbb{N}\right\} \\ &\leq \inf\left\{\frac{\alpha^n}{1-\alpha} p(x_0, x_1) + \alpha^{n-1} p(x_0, x_1) : n \in \mathbb{N}\right\} = 0 \end{aligned}$$

which is a contradiction. Therefore, $z = S(z) = T(z)$.

If $v = S(v) = T(v)$ for some $v \in X$, then

$$\begin{aligned} p(v, v) &= \min\{p(TS(v), S(v)), p(ST(v), T(v))\} \\ &\geq r \max\{p(S(v), v), p(T(v), v)\} \\ &= r \max\{p(v, v), p(v, v)\} \\ &= rp(v, v) \end{aligned}$$

which gives that, $p(v, v) = 0$. □

Corollary 2. *Let (X, p) be a complete partial metric space and let $T : X \longrightarrow X$ be an onto mapping. Suppose that there exists $r \in [0, 1)$ such that*

$$p(T^2(x), T(x)) \geq rp(T(x), x)\} \tag{2.8}$$

for every $x \in X$ and that

$$\alpha(y) = \inf\{p(x, y) + p(T(x), x) : x \in X\} > 0 \tag{2.9}$$

for every $y \in X$ with $y \neq T(y)$. Then there exists $z \in X$ such that $z = T(z)$. Moreover, if $v = T(v)$, then $p(v, v) = 0$.

Proof. Taking S = T in Theorem 2, we have the desired result. □

Corollary 3. *Let (X, p) be a complete partial metric space and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$p(T^2(x), T(x)) \geq rp(T(x), x)\}$$

for every $x \in X$, and T is onto continuous, then T has a fixed point.

Proof. Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf\{p(x, y) + p(T(x), x) : x \in X\} = 0.$$

Then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \{p(x_n, y) + p(T(x_n), x_n)\} = 0.$$

So, we have $p(x_n, y) \longrightarrow 0$ and $p(T(x_n), x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Now,

$$p(T(x_n), y) \leq p(T(x_n), x_n) + p(x_n, y) - p(x_n, x_n) \longrightarrow 0 \ as\ n \longrightarrow \infty.$$

Since T is continuous, we have

$$T(y) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = y.$$

This is a contradiction. Hence if $y \neq T(y)$, then

$$\inf\{p(x, y) + p(T(x), x) : x \in X\} > 0,$$

which is condition (2.8) of Corollary 2. By Corollary 2, there exists $z \in X$ such that $z = T(z)$. □

Now we give an example to support our result.

Example 3. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Define $T : X \longrightarrow X$ by $T(x) = 2x$.

Obviously T is onto and continuous. Also for each $x, y \in X$ we have

$$p(T^2x, Tx) = \max\{4x, 2x\} = 4x \geq r \max\{Tx, x\}$$

where $r = 2$. Thus T satisfy the conditions given in Corollary 3 and 0 is the unique common fixed point of T .

Corollary 4. *Let (X, p) be a complete partial metric space and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$p(T(x), T(y)) \geq r \min\{p(x, T(x)), p(T(y), y), p(x, y)\} \tag{2.10}$$

for every $x, y \in X$, and T is onto continuous, then T has a fixed point.

Proof. Replacing y by T(x) in (2.9), we obtain

$$p(T(x), T^2(x)) \geq r \min\{p(x, T(x)), p(T^2(x), T(x)), p(x, T(x))\} \tag{2.11}$$

for all $x \in X$.
Without loss of generality, we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. Since $r > 1$, it follows from (2.10) that

$$p(T^2(x), T(x)) \geq rp(T(x), x)$$

for every $x \in X$. By the argument similar to that used in Corollary 3, we can prove that, if $y \neq T(y)$, then

$$\inf\{p(x, y) + p(T(x), x) : x \in X\} > 0,$$

which is condition (2.8) of Corollary 2. So, Corollary 2 applies to obtain a fixed point of T. □

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