New Bootstrap Method for Autoregressive Models

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Abstract

A new bootstrap method combined with the stationary bootstrap of Politis and Romano (1994) and the classical residual-based bootstrap is applied to stationary autoregressive (AR) time series models. A stationary bootstrap procedure is implemented for the ordinary least squares estimator (OLSE), along with classical bootstrap residuals for estimated errors, and its large sample validity is proved. A finite sample study numerically compares the proposed bootstrap estimator with the estimator based on the classical residual-based bootstrapping. The study shows that the proposed bootstrapping is more effective in estimating the AR coefficients than the residual-based bootstrapping.

Keywords: Autoregressive model, stationary bootstrap, residual-based bootstrap, asymptotic property.

1. Introduction

There have been lasting interests in bootstrapping methods for ARMA models. A residual-based bootstrap for AR processes was proposed by Freedman (1981, 1984) and its higher-order property was investigated by Bose (1988). Bootstrap prediction intervals for AR models were developed by Thombs and Schucany (1990), Kabaila (1993), Grigoletto (1998) and Clements and Kim (2007) while a bootstrap prediction interval for vector autoregression was constructed by Kim (2004). Politis (2003) reviewed various bootstrap methods for AR models. Pascual *et al.* (2004) proposed a flexible and simple bootstrap procedure for the prediction of ARIMA processes. Goncalves and Kilian (2004, 2007) established asymptotic validity of fixed-design wild bootstrap and pairwise bootstrap for AR processes with conditional heteroskedasticity. Hwang and Shin (2011) applied the stationary bootstrapping for nonparametric estimators of nonlinear AR models.

In this paper, a new bootstrap method combined with the stationary bootstrap of Politis and Romano (1994) and the classical residual-based bootstrap is applied to linear AR models. The stationary bootstrap method generates stationary bootstrapped data, and these data are used to develop the stationary bootstrap version of the least square estimators, where classical bootstrap residuals are used for estimated errors. We show that, for bootstrap estimation of AR models, our proposed bootstrapping is superior to residual-based bootstrapping. The stationary bootstrap is a powerful resampling method for stationary time series processes. It is a block-resampling technique characterized by geometrically distributed random block length, and is very useful because the bootstrapped process is conditionally stationary and preserves the weak dependence structure of the original process. Recent applications of the stationary bootstrap were made by Swensen (2003), Paparoditis and Politis (2005) and Parker *et*

al. (2006) for unit root tests and by Hwang and Shin (2011, 2012a, 2012b) for nonparametric analysis. Lahiri (1999) and Nordman (2009) analyzed new properties of the stationary bootstrap.

As a theoretical result, the first-order asymptotic validity of our proposed bootstrap for AR processes is established by showing that the bootstrap OLSE has the same limiting distribution as the OLSE. A finite sample simulation experiment compares the proposed bootstrapping with the usual residual-based bootstrapping for AR(1) model. The experiment reveals that the proposed bootstrapping is better than the residual-based bootstrapping for parameter estimation and for confidence interval estimation.

The remaining of the paper is organized as follows. In Section 2, the stationary bootstrap procedure is described for the OLSE of AR models and a large sample asymptotic result is established. In Section 3, a Monte Carlo study is provided. In Section 4, concluding remarks are given.

2. The Stationary Bootstrap and Large Sample Asymptotics

We consider a stationary AR(p) process of known order p, defined by

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + a_t, \quad t = \ldots, -2, -1, 0, 1, 2, \ldots \tag{2.1}$$

with the following assumption:

(**C1**) $\{a_t\}$ is a sequence of zero-mean, independent random variable with common distribution F_a such that $E[a_t^2] = \sigma_a^2 < \infty$; $\phi = (\phi_0, \phi_1, \ldots, \phi_p)'$ is a vector of unknown parameters such that all the roots of $1 - \phi_1 B - \cdots - \phi_p B^p = 0$ lie outside the unit circle.

We construct the stationary bootstrap estimator of the unknown parameters, for which the stationary bootstrap procedure is reviewed in Section 2.1. In Section 2.2, an algorithm is developed for the OLSE and its large sample validity is established.

2.1. The stationary bootstrap procedure

Suppose that $\{\mathbf{Y}_t\}$ is a stationary weakly dependent time series taking values in $\mathbb{R}^k$ for some $k \geq 1$. Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be observed. First we define a new time series $\{\mathbf{Y}_{ni} : i \geq 1\}$ by a periodic extension of the observed data set as follows. For each $i \geq 1$, define $\mathbf{Y}_{ni} := \mathbf{Y}_j$ where j is such that $i = qn + j$ for some integer $q \geq 0$. The sequence $\{\mathbf{Y}_{ni} : i \geq 1\}$ is obtained by wrapping the data $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ around a circle, and relabelling them as $\mathbf{Y}_{n1}, \mathbf{Y}_{n2}, \ldots$. Next, for a positive integer l, define the blocks $B(i, l)$, $i \geq 1$ as $B(i, l) = \{\mathbf{Y}_{ni}, \ldots, \mathbf{Y}_{n(i+l-1)}\}$ consisting of l observations starting from $\mathbf{Y}_{ni}$. Bootstrap observations under the stationary bootstrap method are obtained by selecting a random number of blocks from collection $\{B(i, l) : i = 1, \ldots, n, l \geq 1\}$. To do this, we generate random variables $I_1, \ldots, I_n$ and $L_1, \ldots, L_n$ such that conditional on the observations $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$,

(i) $I_1, \ldots, I_n$ are *i.i.d.* discrete uniform on $\{1, \ldots, n\}$: $P^*(I_1 = i) = 1/n$, for $i = 1, \ldots, n$,

(ii) $L_1, \ldots, L_n$ are *i.i.d.* random variables having the geometric distribution with a parameter $\rho \in (0, 1)$: $P^*(L_1 = l) = \rho(1 - \rho)^{l-1}$, for $l = 1, 2, \ldots$, where $\rho = \rho(n)$ depends on the sample size n,

(iii) the collections $\{I_1, \ldots, I_n\}$ and $\{L_1, \ldots, L_n\}$ are independent.

Here and in the following, P^* and E^* denote, respectively, the conditional probability and the conditional expectation, given $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$. For notational simplicity, we suppress dependence of the variables $I_1, \ldots, I_n, L_1, \ldots, L_n$ and of the parameter ρ on n. We assume that ρ goes to 0 as $n \to \infty$.

Under the stationary bootstrap the block length variables $L_1, \ldots, L_n$ are random and the expected block length $E^* L_1$ is ρ^{-1}, which tends to ∞ as $n \to \infty$. Now, a pseudo-time series $Y_1^*, \ldots, Y_n^*$ is generated in the following way. Let $\tau = \inf\{k \geq 1 : L_1 + \cdots + L_k \geq n\}$ and select the τ blocks $B(I_1, L_1), \ldots, B(I_\tau, L_\tau)$. Note that there are $L_1 + \cdots + L_\tau$ elements in the resampled blocks $B(I_1, L_1), \ldots, B(I_\tau, L_\tau)$. Arranging these elements in a series and deleting the last $L_1 + \cdots + L_\tau - n$ elements, we get the bootstrap observations $Y_1^*, \ldots, Y_n^*$. Conditionally on $\{Y_1, \ldots, Y_n\}$, $\{Y_t^*\}$ is stationary.

2.2. Bootstrap estimator and large sample validity

We develop a bootstrap for the OLSE of ϕ. Let $\{y_{1-p}, \ldots, y_0, y_1, \ldots, y_n\}$ be a realization of the AR(p) process. Let $Y_t = (1, y_{t-1}, \ldots, y_{t-p})$ for $t = 1, 2, \ldots, n$. For a matrix A, A' denotes the transpose of A. Let $y = (y_1, y_2, \ldots, y_n)'$, $a = (a_1, a_2, \ldots, a_n)'$ and

$$
\mathbf{X} = \begin{pmatrix} 1 & y_0 & y_{-1} & \cdots & y_{1-p} \\ 1 & y_1 & y_0 & \cdots & y_{2-p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_{n-1} & y_{n-2} & \cdots & y_{n-p} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_n \end{pmatrix}.
$$

Then we have $y = \mathbf{X}\phi + a$ and the OLSE of $\hat{\phi}$ is

$$
\hat{\phi} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'y. \tag{2.2}
$$

Our bootstrap estimator is of the form $\hat{\phi}^* = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\tilde{y}^*$ where $\mathbf{X}^*$ is constructed from $\mathbf{X}$ by the stationary bootstrapping and $\tilde{y}^* = \mathbf{X}^*\hat{\phi} + \tilde{a}^*$ with $\tilde{a}^* = (\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*)'$ and $\{\hat{a}_t^* : t = 1, 2, \ldots, n\}$ being a random draw from the OLS residuals. More details are described in the following algorithm.

Algorithm 1.

Step 1. Estimate ϕ by the OLSE in (2.2).

Step 2. Compute the OLS-residuals $\hat{a}_t$:

$$
\hat{a}_t = y_t - \hat{\phi}_0 - \hat{\phi}_1 y_{t-1} - \cdots - \hat{\phi}_p y_{t-p}, \quad t = p + 1, \ldots, n
$$

and $\hat{a}_t = 0$ for $t = 1, \ldots, p$. Let $\hat{F}_a$ be the empirical distribution function of the residuals, *i.e.*, $\hat{F}_a(x) = 1/n \sum_{i=1}^{n} 1_{(-\infty, \hat{a}_i]}(x)$.

Step 3. Apply the stationary bootstrap method to $\{Y_1, \ldots, Y_n\}$, and obtain stationary bootstrap observations $\{Y_1^*, \ldots, Y_n^*\}$ with $Y_t^* = (1, y_{t-1}^*, \ldots, y_{t-p}^*)$.

Step 4. Calculate $\hat{\phi}^* = (\hat{\phi}_0^*, \hat{\phi}_1^*, \ldots, \hat{\phi}_p^*)'$, the stationary bootstrap estimator of ϕ with the stationary bootstrap observations in Step 3 as follows:

$$
\hat{\phi}^* = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1} \mathbf{X}^{*\prime}\tilde{y}^*,
$$

where $\mathbf{X}^* = (Y_1^{*\prime}, Y_2^{*\prime}, \ldots, Y_n^{*\prime})'$ is the $n \times (p + 1)$ matrix with the stationary bootstrap observations, and $\tilde{y}^* = \mathbf{X}^*\hat{\phi} + \tilde{a}^*$ with $\tilde{a}^* = (\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*)'$ where $\{\hat{a}_t^* : t = 1, 2, \ldots, n\}$ is a sequence of *i.i.d.* random draw from $\hat{F}_a$.

In the following theorem, we present our main theoretical result whose proof is given in Appendix. In the sequel, we write $X_n \xrightarrow{p} X$ if X_n converges in probability to X; $X_n \xrightarrow{p^*} X$ if X_n converges to X in conditional probability given the observations $\{y_{1-p}, \ldots, y_n\}$; $X_n \xrightarrow{d} X$ if X_n converges in distribution to X; and $X_n \xrightarrow{d^*} X$ if X_n converges in distribution to X in conditional probability given observations $\{y_{1-p}, \ldots, y_n\}$.

Theorem 1. *Consider (2.1) with assumption (C1). If $\rho \to 0$ and $n\rho \to \infty$ as $n \to \infty$, then we have, as $n \to \infty$,*

$$\sup_x \left| P^* \left(\sqrt{n} \left[\hat{\phi}^* - \hat{\phi} \right] \le x \right) - P \left(\sqrt{n} \left[\hat{\phi} - \phi \right] \le x \right) \right| \xrightarrow{p} 0.$$

3. Monte Carlo Study

Our proposed bootstrapping combined with the stationary bootstrap (SB, hereafter) is compared with classical residual-based bootstrapping (RB) for inference on an AR(1) model given by

$$y_t = \phi_0 + \phi_1 y_{t-1} + a_t, \quad t = 1, \ldots, n. \tag{3.1}$$

Parameters are set to: $\phi_0 = 0$; $\phi_1 = 0.95, 0.7, -0.7$; $n = 25, 50, 100, 200$. Data $\{y_t, t = 1, \ldots, n\}$ are simulated with pseudo *i.i.d.* $N(0, 1)$ errors $a_1, \ldots, a_n$ generated by RNNOA, an IMSL FORTRAN subroutine. The initial value y_0 is set to 0. For ϕ_1, bootstrap estimators and 90% bootstrap confidence intervals are constructed using RB and SB. We briefly outline RB.

RB. Step 1 : Fit (3.1) by OLS obtaining the OLSE $\bar{\phi} = (\bar{\phi}_0, \bar{\phi}_1)'$ and the OLS residuals $\bar{a}_t, t = 2, \ldots, n$ with $\bar{a}_1 = 0$. Step 2 : Compute $y_t^* = \bar{\phi}_0 + \bar{\phi}_1 y_{t-1}^* + a_t^*$, $t = 2, \ldots, n$ with $y_1^* = y_1$ where $\{a_1^*, \ldots, a_n^*\}$ is a set of random draw from $\{\bar{a}_t, t = 1, \ldots, n\}$. Step 3 : Compute bootstrap OLSE $\bar{\phi}^* = (\bar{\phi}_0^*, \bar{\phi}_1^*)'$ by OLS-fitting of (3.1) with the bootstrap data $\{y_1^*, \ldots, y_n^*\}$.

An RB OLSE of ϕ_1 for a given sample $\{y_1, \ldots, y_n\}$ is the average of B, say, bootstrap OLSE $\bar{\phi}^{*(b)}$, $b = 1, \ldots, B$ obtained by repeating Step 2–Step 3 of RB B times. A 90% RB confidence interval of ϕ_1 is $[L_B^*, U_B^*]$, where L_B^* and U_B^* are 5% and 95% empirical percentiles of $\bar{\phi}^{*(b)}, b = 1, \ldots, B$, respectively.

An SB OLSE and confidence interval of ϕ_1 are constructed by the procedure described in Section 2.2. In order to implement SB, we use the block length parameter $\rho = 0.05 i_\rho (n/100)^{-1/3}, i_\rho = 1, 2, 3$. The order $n^{-1/3}$ is chosen because Politis and White (2004) and Patton *et al.* (2009) reported that optimal order of ρ for estimating simple mean is $n^{-1/3}$. Even though we are estimating the AR(1) coefficient rather than the simple mean, the order $n^{-1/3}$ would be a reasonable one.

Table 1 reports means and variances of the bootstrap estimates as well as coverage probabilities and average lengths of 90% bootstrap confidence intervals. The table is constructed by 10,000 independent replications of the bootstrap estimating procedures with $B = 1,000$. In the table, "Eff." denotes the relative efficiency $\text{MSE}_{RB}/\text{MSE}_{SB}$ of the SB estimator relative to RB estimator, where $\text{MSE} = (\text{Mean} - \phi_1)^2 + \text{Variance}$ is the mean square error of a corresponding estimator of ϕ_1.

Table 1 reveals many interesting results. The most conspicuous feature is that values of Eff. are generally greater than 1, implying that SB produces better estimator than RB. Especially for $\phi_1 = 0.95$, the efficiency is greater than 1.9 for all $n = 25, 50, 100, 200$ considered here. For $\phi_1 = 0.7$, SB retains efficiency advantage over RB having Eff value ≥ 1.2. On the other hand, for $\phi = -0.7$, SB loses the efficiency advantage but is still almost as efficient as RB with Eff values around 1.

Table 1: Bootstrap parameter estimates and confidence intervals for AR(1) model.

ϕ_1	n	ρ	Parameter estimation, ϕ_1					90% Confidence Interval, ϕ_1			
			Mean		Variance		Eff.	Cov. Prob.		Avg. Len.	
			RB	SB	RB	SB		RB	SB	RB	SB
.95	25	.08	.63	.74	.020	.022	1.91	15.	61.	.52	.40
.95	25	.16	.63	.74	.019	.022	1.91	14.	62.	.52	.41
.95	25	.24	.63	.74	.020	.022	1.91	15.	62.	.52	.41
.95	50	.06	.78	.85	.007	.008	2.04	24.	73.	.31	.24
.95	50	.13	.78	.85	.007	.007	2.05	24.	74.	.31	.24
.95	50	.19	.78	.85	.007	.008	2.04	25.	74.	.31	.24
.95	100	.05	.86	.90	.002	.003	2.07	44.	82.	.18	.14
.95	100	.10	.86	.90	.002	.003	2.05	42.	81.	.18	.14
.95	100	.15	.86	.90	.002	.003	2.07	43.	81.	.18	.14
.95	200	.04	.91	.93	.001	.001	1.90	60.	86.	.10	.09
.95	200	.08	.91	.93	.001	.001	1.92	61.	86.	.10	.09
.95	200	.12	.91	.93	.001	.001	1.91	60.	85.	.10	.09
.70	25	.08	.45	.56	.024	.031	1.69	62.	83.	.59	.55
.70	25	.16	.46	.57	.023	.031	1.70	63.	85.	.59	.56
.70	25	.24	.45	.56	.024	.032	1.68	62.	84.	.59	.56
.70	50	.06	.57	.63	.011	.013	1.55	73.	87.	.39	.37
.70	50	.13	.58	.64	.012	.013	1.54	73.	88.	.39	.37
.70	50	.19	.57	.63	.011	.013	1.56	73.	88.	.39	.38
.70	100	.05	.64	.67	.006	.006	1.35	82.	89.	.26	.25
.70	100	.10	.64	.67	.006	.006	1.36	81.	88.	.26	.25
.70	100	.15	.64	.67	.006	.006	1.36	81.	88.	.26	.25
.70	200	.04	.67	.69	.003	.003	1.20	86.	89.	.17	.17
.70	200	.08	.67	.69	.003	.003	1.19	85.	89.	.17	.17
.70	200	.12	.67	.69	.003	.003	1.20	86.	90.	.17	.17
$-.70$	25	.08	$-.63$	$-.66$	.019	.023	.97	91.	90.	.50	.48
$-.70$	25	.16	$-.63$	$-.66$	.020	.023	.97	90.	90.	.50	.48
$-.70$	25	.24	$-.63$	$-.66$	.019	.023	.97	90.	90.	.50	.48
$-.70$	50	.06	$-.66$	$-.68$	.010	.011	.99	91.	91.	.35	.34
$-.70$	50	.13	$-.66$	$-.68$	.010	.011	.99	90.	91.	.35	.35
$-.70$	50	.19	$-.66$	$-.68$	.010	.011	1.00	90.	90.	.35	.35
$-.70$	100	.05	$-.68$	$-.69$	.005	.005	1.00	90.	90.	.24	.24
$-.70$	100	.10	$-.68$	$-.69$	.005	.005	.99	91.	90.	.24	.24
$-.70$	100	.15	$-.68$	$-.69$	.005	.005	.99	91.	90.	.24	.24
$-.70$	200	.04	$-.69$	$-.69$	.003	.003	1.00	90.	90.	.17	.17
$-.70$	200	.08	$-.69$	$-.69$	.003	.003	1.00	90.	91.	.17	.17
$-.70$	200	.12	$-.69$	$-.70$	.003	.003	1.00	90.	90.	.17	.17

Note: Number of replications = 10,000, number of bootstrap repetitions = 1,000.

Investigating Table 1, we see that the efficiency advantage of SB originates from less-biased estimation. Mean of SB estimator is closer to ϕ_1 than that of RB especially for $\phi_1 = 0.95$. On the other hand, variance of SB estimator is slightly greater than that of RB. We see that SB is also more effective for the confidence interval than RB: compared with the RB confidence interval, the SB confidence interval has a better coverage probability that is closer to the nominal value 90% and has shorter average length especially for $\phi_1 = 0.95$ and ($\phi_1 = 0.7; n = 25, 50$). Performance of SB seems to be insensitive to ρ: results for $i_\rho = 1, 2, 3$ are similar. In this Monte Carlo experiment, we may conclude that SB is better than RB for small n and large ϕ_1 and SB is as good as RB for other n and ϕ_1.

4. Concluding Remarks

We have established a large sample validity of the stationary bootstrap method for the OLSE of AR(1)

model. A small sample study showed that the stationary bootstrapping is better than the residual-based bootstrapping. The developed theory would be useful in establishing a large sample validity of the OLSE-based stationary bootstrapping prediction interval. This issue will be investigated for a more general ARMA(p, q) model in a future study.

A choice of the tuning parameter is important because performance heavily depends on the choice. In our proposed bootstrap estimator combined with the stationary bootstrap, one way for tuning the parameter ρ of the geometric distribution for the block length in the stationary bootstrap can be done as the data-driven choice ρ^* which is a function of the sample size n and of the finite order p of AR(p) model, given $\{y_{1-p}, \ldots, y_n\}$, as follows:

$$\rho^* = \arg\min E \left\| \text{Var}^* \left(\hat{\phi}^* - \hat{\phi} \right) \right\| = \arg\min E \left\| E^* \left(\mathbf{X}^{*\prime} \mathbf{X}^* \right)^{-1} \right\|,$$

where $\|A\| = (\lambda_{\max}(A'A))^{1/2}$ is the maximum eigenvalue of a matrix A and where the last equality can be implied in the proof of Theorem 1 in Appendix. Derivation of ρ^* would not be simple and may be a challenging work.

Appendix: Proof of Theorem 1

It is well-known that $\sqrt{n}[\hat{\phi} - \phi] \xrightarrow{d} N(0, \sigma_a^2 \Gamma^{-1})$ where Γ is the $(p + 1) \times (p + 1)$ positive definite covariance matrix. It is enough to show that $\sqrt{n}[\hat{\phi}^* - \hat{\phi}]$ has the same limiting distribution.

We recall that the least squares estimator $\hat{\phi}$ satisfies

$$\sqrt{n} \left[\hat{\phi} - \phi \right] = \left(\frac{\mathbf{X}'\mathbf{X}}{n} \right)^{-1} \left(\frac{1}{\sqrt{n}} \right) \mathbf{X}'a \xrightarrow{d} N \left(0, \sigma_a^2 \Gamma^{-1} \right)$$

along with

$$\left(\frac{\mathbf{X}'\mathbf{X}}{n} \right)^{-1} \xrightarrow{p} \Gamma^{-1} \quad \text{and} \quad \left(\frac{1}{\sqrt{n}} \right) \mathbf{X}'a \xrightarrow{d} N \left(0, \sigma_a^2 \Gamma \right). \tag{A.1}$$

Now in order to show that $\sqrt{n}[\hat{\phi}^* - \hat{\phi}]$ has the same limiting distribution, observe that

$$\sqrt{n} \left[\hat{\phi}^* - \hat{\phi} \right] = \left(\frac{\mathbf{X}^{*\prime} \mathbf{X}^*}{n} \right)^{-1} \left(\frac{1}{\sqrt{n}} \right) \mathbf{X}^{*\prime} \tilde{a}^*.$$

Since $\mathbf{Y}_i$ and $\mathbf{Y}_i^*$ are the ith rows of $\mathbf{X}$ and $\mathbf{X}^*$, respectively, we have

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} \mathbf{Y}_i{}'\mathbf{Y}_i \quad \text{and} \quad \mathbf{X}^{*\prime}\mathbf{X}^* = \sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^*.$$

First, we show that $\sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^*/n$ and $\sum_{i=1}^{n} \mathbf{Y}_i'\mathbf{Y}_i/n$ have the same limiting in probability.

Let $U_{i,r}$ be the sum of observations in block $B(i, r) = \{\mathbf{Y}_{nj} : i \leq j \leq i + r - 1\}$, i.e., $U_{i,r} = \sum_{j=i}^{i+r-1} \mathbf{Y}_{nj}'\mathbf{Y}_{nj}$. Let $s_\tau = L_1 + L_2 + \cdots + L_\tau$ and use

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* - \frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i'\mathbf{Y}_i \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* - \frac{1}{n} \sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* \right| + \left| \frac{1}{n} \sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* - \frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i'\mathbf{Y}_i \right|. \tag{A.2}$$

It will be shown in Lemma 1 below that the first term of the right-hand side of (A.2) tends to 0 in (conditional) probability.

Lemma 1.

$$\frac{1}{n} \sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime} \mathbf{Y}_i^* \xrightarrow{p^*} 0.$$

Proof: Recalling the definition of τ, and letting $s_{\tau-1} = L_1 + \cdots + L_{\tau-1}$, $R_1 = n - s_{\tau-1}$ and $R = L_\tau - R_1$, we have that $\sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime} \mathbf{Y}_i^*$ is the sum of observations in $B(I_\tau, L_\tau)$, after deleting the first $R_1(= n - s_{\tau-1})$ of them. Note that R, conditional on $(R_1, s_{\tau-1})$, has a geometric distribution with mean $1/\rho$. This follows from the memoryless property of the geometric distribution. Hence, $(1/n) \sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime} \mathbf{Y}_i^*$ is equal in distribution to $(1/n)U_{I,R}$, where I is uniform on $\{1, \ldots, n\}$. It is enough to show that the (conditional) mean and variance of $(1/n)U_{I,R}$ tends to 0.

We have

$$E^*[U_{I,R}|R] = \sum_{i=1}^{n} \frac{1}{n} \left[\sum_{j=i}^{i+R-1} \mathbf{Y}_{nj}^{\prime} \mathbf{Y}_{nj} \right] = R\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right), \tag{A.3}$$

and thus

$$\frac{1}{n} E^*[U_{I,R}] = \frac{1}{n} E^*[E^*[U_{I,R}|R]] = \frac{1}{n\rho} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) = O_p\left(\frac{1}{n\rho} \right) \xrightarrow{p} 0 \tag{A.4}$$

by (A.1) and since $n\rho \to \infty$. Also we have

$$\frac{1}{n^2} \mathrm{Var}^*[U_{I,R}] = \frac{1}{n^2} E^* \left[\mathrm{Var}^*(U_{I,R}|R) \right] + \frac{1}{n^2} \mathrm{Var}^* \left[E^*(U_{I,R}|R) \right].$$

Its second term $(1/n^2)\mathrm{Var}^*[E^*(U_{I,R}|R)]$ is equal to

$$\frac{1}{n^2} \mathrm{Var}^* \left[R\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) \right] = \frac{1-\rho}{n^2 \rho^2} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right)^{\prime} = O_p\left(\frac{1}{n^2 \rho^2} \right) \xrightarrow{p} 0.$$

For the first term $(1/n^2)E^*[\mathrm{Var}^*(U_{I,R}|R)]$, consider

$$\mathrm{Var}^*(U_{I,R}|R) = E^* \left[U_{I,R} U_{I,R}^{\prime} | R \right] - E^* \left[U_{I,R}|R \right] \left(E^* \left[U_{I,R}|R \right] \right)^{\prime}.$$

We observe

$$E^* \left[U_{I,R} U_{I,R}^{\prime} | R \right] = \sum_{i=1}^{n} \frac{1}{n} \left(\sum_{j=i}^{i+R-1} \mathbf{Y}_{nj}^{\prime} \mathbf{Y}_{nj} \right) \left(\sum_{j=i}^{i+R-1} \mathbf{Y}_{nj}^{\prime} \mathbf{Y}_{nj} \right)^{\prime} = \frac{R}{n} \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right)^{\prime}$$

and then, by this and by (A.3), we have

$$E^* \left[\mathrm{Var}^*(U_{I,R}|R) \right] = \frac{1}{n\rho} \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right)^{\prime} - \frac{1}{n^2 \rho^2} \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right) \left(\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i \right)^{\prime}.$$

Thus, by (A.1)

$$\frac{1}{n^2} E^*[\text{Var}^*(U_{I,R}|R)] = O_p\left(\frac{1}{n\rho}\right) + O_p\left(\frac{1}{n^2\rho^2}\right) \xrightarrow{p} 0.$$

Therefore the (conditional) variance of $(1/n)U_{I,R}$ tends to 0, and the first term of the right-hand side of (A.2) tends to 0 in (conditional) probability.   □

Now we show in Lemma 2 below that the second term of the right-hand side of (A.2) tends to 0 in (conditional) probability.

Lemma 2.
$$\left|\frac{1}{n}\sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* - \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^\prime\mathbf{Y}_i\right| \xrightarrow{p^*} 0.$$

Proof: Recalling the definition of $U_{i,r} = \sum_{j=i}^{i+r-1} \mathbf{Y}_{nj}^\prime\mathbf{Y}_{nj}$, we have

$$\frac{1}{n}\sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime}\mathbf{Y}_i^* = \frac{1}{n}\sum_{i=1}^{\tau} U_{I_i,L_i}.$$

Just as in Politis and Romano (1994) and Hwang and Shin (2012a), since $\tau = n\rho + O_p(\sqrt{n\rho})$, we consider a sequence $m = m_n$ with $m/(n\rho) \to 1$, and it suffices to show that

$$\left|\frac{1}{n}\sum_{i=1}^{m} U_{I_i,L_i} - \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^\prime\mathbf{Y}_i\right| \xrightarrow{p^*} 0.$$

The left-hand side is less than or equal to

$$\left|\frac{1}{n}\sum_{i=1}^{m} U_{I_i,L_i} - \frac{\rho}{m}\sum_{i=1}^{m} U_{I_i,L_i}\right| + \left|\frac{\rho}{m}\sum_{i=1}^{m} U_{I_i,L_i} - \rho E^*[U_{I_1,L_1}]\right| + \left|\rho E^*[U_{I_1,L_1}] - \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^\prime\mathbf{Y}_i\right|. \qquad (A.5)$$

The first term of (A.5) is less than or equal to

$$\left|\sum_{i=1}^{m} U_{I_i,L_i}\right|\left|\frac{1}{n} - \frac{\rho}{m}\right| = \left|\frac{1}{nm}\sum_{i=1}^{m} U_{I_i,L_i}\right||m - n\rho| = o_p\left(\frac{1}{n\rho}\right) \xrightarrow{p^*} 0,$$

the last equality holds by (A.4). The second term of (A.5),

$$\rho\left|\frac{1}{m}\sum_{i=1}^{m} U_{I_i,L_i} - E^*[U_{I_1,L_1}]\right| = o_p(\rho) \xrightarrow{p^*} 0,$$

where the last equality holds by the weak law of large numbers of *i.i.d.* sequence $\{U_{I_i,L_i} : i = 1, 2, \dots\}$ as $m \to \infty$. For the third term of (A.5), we calculate $E^*[U_{I_1,L_1}]$. Similarly to above, we have

$$E^*[U_{I_1,L_1}] = E^*\left[E^*(U_{I_1,L_1}|L_1)\right] = E^*\left[L_1\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^\prime\mathbf{Y}_i\right)\right] = \frac{1}{\rho}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^\prime\mathbf{Y}_i\right).$$

Thus the third term of (A.5) is zero, and the desired result in Lemma 2 holds. $\qquad\square$

Therefore, according to Lemma 1 and Lemma 2 along with (A.2), $\sum_{i=1}^{n} \mathbf{Y}_i^{*\prime} \mathbf{Y}_i^{*}/n$ and $\sum_{i=1}^{n} \mathbf{Y}_i^{\prime} \mathbf{Y}_i/n$ have the same limiting Γ in probability.

Now we will verify in Lemma 3 below the convergence of $(1/\sqrt{n})\mathbf{X}^{*\prime}\tilde{a}^{*}$ in distribution.

Lemma 3.

$$\left(\frac{1}{\sqrt{n}}\right)\mathbf{X}^{*\prime}\tilde{a}^{*} \xrightarrow{d^{*}} N\left(0, \sigma_a^2 \Gamma\right).$$

Proof: We write

$$\mathbf{X}^{*\prime}\tilde{a}^{*} = (\mathbf{Y}_1^{*\prime}, \mathbf{Y}_2^{*\prime}, \ldots \mathbf{Y}_n^{*\prime})(\hat{a}_1^{*}, \hat{a}_2^{*}, \ldots, \hat{a}_n^{*})' = \sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\hat{a}_i^{*}$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{Y}_i^{*\prime}\hat{a}_i^{*} = \frac{1}{\sqrt{n}}\sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime}\hat{a}_i^{*} - \frac{1}{\sqrt{n}}\sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime}\hat{a}_i^{*}.$$

Let $V_{i,r} = \sum_{j=i}^{i+r-1} \mathbf{Y}_{nj}'\hat{a}_j^{*}$, which is related to the observations in block $B(i, r)$. By the same argument as above, $(1/\sqrt{n})\sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime}\hat{a}_i^{*}$ is equal in distribution to $(1/\sqrt{n})V_{I,R}$, and we have

$$E^{*}[V_{I,R}] = \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}E^{*}[V_{i,r}] = \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}\sum_{j=i}^{i+r-1} \mathbf{Y}_{nj}'E^{*}\left[\hat{a}_j^{*}\right] = 0$$

since $E^{*}[\hat{a}_j^{*}] = (1/n)\sum_{i=1}^{n}\hat{a}_i = 0$, and that

$$\begin{aligned}
\mathrm{Var}^{*}[V_{I,R}] &= E^{*}\left[V_{I,R}V_{I,R}'\right] = \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}E^{*}\left[V_{i,r}V_{i,r}'\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}E^{*}\left[\left(\sum_{j=i}^{i+r-1}\mathbf{Y}_{nj}'\hat{a}_j^{*}\right)\left(\sum_{l=i}^{i+r-1}\mathbf{Y}_{nl}'\hat{a}_l^{*}\right)'\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}\left(\sum_{j=i}^{i+r-1}\sum_{l=i}^{i+r-1}\mathbf{Y}_{nj}'\mathbf{Y}_{nl}E^{*}\left[\hat{a}_j^{*}\hat{a}_l^{*}\right]\right).
\end{aligned}$$

Letting $\hat{\sigma}_a^2 = (1/n)\sum_{i=1}^{n}\hat{a}_i^2$, and noting that $E^{*}[\hat{a}_j^{*}\hat{a}_l^{*}] = \hat{\sigma}_a^2$ if $j = l$, and 0 otherwise, we have

$$\mathrm{Var}^{*}[V_{I,R}] = \sum_{r=1}^{\infty} \rho(1-\rho)^{r-1}r\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i'\mathbf{Y}_i\hat{\sigma}_a^2\right) = E[R]\hat{\sigma}_a^2\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i'\mathbf{Y}_i\right) = \frac{\hat{\sigma}_a^2}{\rho}\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right) \tag{A.6}$$

and thus

$$\frac{1}{n}\mathrm{Var}^{*}[V_{I,R}] = O_p\left(\frac{1}{n\rho}\right) \xrightarrow{p} 0$$

by (A.1) and by the fact that $\hat{\sigma}_a^2 \to \sigma_a^2$ in probability. Therefore,

$$\frac{1}{\sqrt{n}} \sum_{i=n+1}^{s_\tau} \mathbf{Y}_i^{*\prime} \hat{a}_i^* \xrightarrow{p^*} 0.$$

Recalling the definition of $V_{i,r} = \sum_{j=i}^{i+r-1} \mathbf{Y}_{nj}^\prime \hat{a}_j^*$, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{s_\tau} \mathbf{Y}_i^{*\prime} \hat{a}_i^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau} V_{I_i, L_i}.$$

Similarly to above, for a sequence $m = m_n$ with $m/(n\rho) \to 1$, it suffices to show that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{m} V_{I_i, L_i} \xrightarrow{d^*} N\left(0, \sigma_a^2 \Gamma\right). \tag{A.7}$$

Note that $E^*((1/\sqrt{n}) \sum_{i=1}^{m} V_{I_i, L_i}) = 0$ and $\{V_{I_i, L_i} : i = 1, 2, \ldots\}$ is *i.i.d.* sequence since $\{(I_i, L_i) : i = 1, 2, \ldots\}$ are *i.i.d.* Thus, by (A.6), we have

$$\text{Var}^*\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{m} V_{I_i, L_i}\right) = \frac{m}{n} \text{Var}^*(V_{I_1, L_1}) = \frac{m\hat{\sigma}_a^2}{n\rho} \left(\frac{1}{n} \mathbf{X}^\prime \mathbf{X}\right) \xrightarrow{p} \sigma_a^2 \Gamma.$$

The left-hand side of (A.7) is equal to

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{m} V_{I_i, L_i} - \sqrt{\frac{\rho}{m}} \sum_{i=1}^{m} V_{I_i, L_i}\right) + \sqrt{\frac{\rho}{m}} \sum_{i=1}^{m} V_{I_i, L_i}. \tag{A.8}$$

The first term of (A.8) is less than or equal to

$$\left|\sum_{i=1}^{m} V_{I_i, L_i}\right| \left|\frac{1}{\sqrt{n}} - \sqrt{\frac{\rho}{m}}\right| = \left|\frac{1}{m} \sum_{i=1}^{m} V_{I_i, L_i}\right| \left|\sqrt{m} - \sqrt{n\rho}\right| \sqrt{\frac{m}{n}} = o_p(1) \xrightarrow{p^*} 0.$$

To verify the asymptotic normal distribution of the second term of (A.8) (and thus that of $(1/\sqrt{n}) \mathbf{X}^{*\prime} \tilde{a}^*$), let Z_j^* be the $(j+1)^{th}$ component of $(1/\sqrt{n}) \mathbf{X}^{*\prime} \tilde{a}^*$ for $j = 0, 1, \ldots, p$, and let $z_{i,j}^*$ be the $(j+1)^{th}$ component of V_{I_i, L_i} for $j = 0, 1, \ldots, p$ and for each $i = 1, \ldots, m$. That is, we observe

$$\left(Z_0^*, Z_1^*, \ldots, Z_p^*\right)^\prime = \frac{1}{\sqrt{n}} \mathbf{X}^{*\prime} \tilde{a}^*, \quad \text{and} \quad \sqrt{\frac{\rho}{m}} \sum_{i=1}^{m} V_{I_i, L_i} = \sqrt{\frac{\rho}{m}} \sum_{i=1}^{m} \left(z_{i,0}^*, z_{i,1}^*, \ldots, z_{i,p}^*\right)^\prime.$$

By above convergence, $(Z_0^*, Z_1^*, \ldots, Z_p^*)^\prime$ and $\sqrt{\rho/m} \sum_{i=1}^{m} (z_{i,0}^*, z_{i,1}^*, \ldots, z_{i,p}^*)^\prime$ have the same limiting.

For $(s_0, s_1, \ldots, s_p) \in \mathbb{R}^{p+1}$, and for each $i = 1, \ldots, m$, we have $E^*(\sum_{j=0}^{p} s_j z_{i,j}^*) = 0$ and

$$\text{Var}^*\left(\sum_{j=0}^{p} s_j z_{i,j}^*\right) = \sum_{j=0}^{p} s_j^2 \text{Var}^*\left(z_{i,j}^*\right) + 2 \sum_{j<l} s_j s_l \text{Cov}^*\left(z_{i,j}^*, z_{i,l}^*\right) \cong \sum_{j=0}^{p} s_j^2 \frac{\sigma_a^2 \gamma_{jj}}{\rho} + 2 \sum_{j<l} s_j s_l \frac{\sigma_a^2 \gamma_{jl}}{\rho}$$

in probability by (A.6), where γ_{jl} is the (j, l)-component of Γ.

Since $\{V_{I_i,L_i} = (z_{i,0}^*, z_{i,1}^*, \ldots, z_{i,p}^*)' \mid 1 \le i \le m\}$ are *i.i.d.* vectors with mean zero under P^*, we obtain (for $\iota = \sqrt{-1}$),

$$
\begin{aligned}
E^*\left[e^{\iota t\left(\sum_{j=0}^p s_j Z_j^*\right)}\right] &\cong E^*\left[e^{\iota(t/\sqrt{m})\sum_{i=1}^m \sqrt{\rho}\left(\sum_{j=0}^p s_j z_{i,j}^*\right)}\right] = \left(E^*\left[e^{\iota(t/\sqrt{m})\sqrt{\rho}\left(\sum_{j=0}^p s_j z_{1,j}^*\right)}\right]\right)^m \\
&= \left[1 + \frac{\iota t\sqrt{\rho}}{\sqrt{m}}E^*\left(\sum_{j=0}^p s_j z_{1,j}^*\right) - \frac{t^2\rho}{2m}(1+o(1))E^*\left(\sum_{j=0}^p s_j z_{1,j}^*\right)^2\right]^m \\
&\cong \left[1 - \frac{t^2}{2m}(1+o(1))\left(\sum_{j=0}^p s_j^2 \sigma_a^2 \gamma_{jj} + 2\sum_{j<l} s_j s_l \sigma_a^2 \gamma_{jl}\right)\right]^m,
\end{aligned}
$$

where "$\cong$" is such that $a_n \cong b_n$ denotes $a_n/b_n \to 1$ as $n \to \infty$, *i.e.*, a_n and b_n have the same limiting. The last term tends to $\exp\left\{-(t^2/2)\sigma_a^2\left(\sum_{j=0}^p s_j^2 \gamma_{jj} + 2\sum_{j<l} s_j s_l \gamma_{jl}\right)\right\}$ as $n \to \infty$ ($m \to \infty$). Therefore,

$$
s_0 Z_0^* + \cdots + s_p Z_p^* \xrightarrow{d^*} N\left(0, \sigma_a^2\left(\sum_{j=0}^p s_j^2 \gamma_{jj} + 2\sum_{j<l} s_j s_l \gamma_{jl}\right)\right),
$$

and thus by the Cramér-Wold device, $(1/\sqrt{n})\mathbf{X}^{*\prime}\tilde{a}^* \xrightarrow{d^*} N(0, \sigma_a^2\Gamma)$, and the proof of Lemma 3 is completed. $\square$

The proof of Theorem 1 is obtained by Lemma 1, Lemma 2 and Lemma 3.

Acknowledgement

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