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A Sufficient Condition on Optimal Berry-Esseen Bounds of Functionals of Gaussian Fields

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Abstract

We find a sufficient condition for optimal Berry-Esseen bounds in the normal approximation of functionals of Gaussian fields studied by Nourdin and Peccati (2009b).

Keywords: Malliavin calculus, Berry-Esseen bound, Stein's method, multiple stochastic integral.

1. Introduction

Let $\{F_n\}$ be a sequence of zero-mean real-valued random variables with the form of a functional of an infinite dimensional Gaussian field. Nourdin and Peccati (2009a), by combining Malliavin calculus with Stein's method, obtain explicit bounds of the type

$$\sup_{z\in R} |P(F_n \le z) - \Phi(z)| \le \alpha(n), \quad n \ge 1, \tag{1.1}$$

where $\Phi(z) = \int_{-\infty}^{z} (1/\sqrt{2\pi}) e^{-x^2} 2dx$, and $\alpha(n)$ is some positive sequence converging to zero.

We describe the approach used in Nourdin and Peccati (2009a) as follows. Fix $z \in R$ and consider the *Stein equation*

$$\mathbf{1}_{(-\infty,z]}(x) - \Phi(x) = f'(x) - xf(x), \quad x \in R. \tag{1.2}$$

Then it is well known that for every $z \in R$, the equation (1.2) has a solution f_z such that $\|f_z\|_\infty \le \sqrt{2\pi}/4$ and $\|f_z'\|_\infty \le 1$. Denote by DF_n the Malliavin derivative F_n and by L^{-1} the *pseudo-inverse of the Ornstein-Ulhenbeck generator*. DF_n is a random element in an appropriate Hilbert space $\mathbb{H}$. By the integration by parts of a Malliavin calculus,

$$\begin{aligned} P(F_n \le z) - \Phi(z) &= E\left[f_z'(F_n) - F_n f_z(F_n)\right] \\ &= E\left[f_z'(F_n)\left(1 - \left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}\right)\right]. \end{aligned} \tag{1.3}$$

The estimate $\|f_z'\|_\infty \le 1$ and the Cauchy-Schwartz inequality yield , from (1.3),

$$\sup_{z\in R} |P(F_n \le z) - \Phi(z)| \le \sqrt{E\left[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathbb{H}})^2\right]}\,. \tag{1.4}$$

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Hence the upper bound $\alpha(n)$ appearing in (1.1) is given by

$$\alpha(n) = \sqrt{E\left[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathbb{H}})^2\right]}.$$

The upper bound $\alpha(n)$ is said to be *optimal* for the sequence $\{F_n\}$ if there exists a constant $c \in (0, 1)$ such that for all sufficiently large n,

$$c < \frac{\sup_{z\in R} |P(F_n \le z) - \Phi(z)|}{\alpha(n)} \le 1. \tag{1.5}$$

Nourdin and Peccati (2009b) show the existence of a constant c appearing in (1.5) by considering the convergence of quantities in (1.3). For this, they assume that the random vectors

$$\left(F_n, \frac{1 - \left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}}{\alpha(n)}\right), \quad n \ge 1, \tag{1.6}$$

converge, in distribution, to a two-dimensional Gaussian vector with non-zero covariance.

Let $\mathbb{D}^{p,q}$ be a class of random variables defined in Section 2. We describe the main result of Nourdin and Peccati (2009b).

Theorem 1. (Nourdin and Peccati) *Let $\{F_n\}$ be a sequence of centered and square integrable functional of some Gaussian process $X = \{X(h) : h \in \mathbb{H}\}$ such that $E(F_n^2) \to 1$ as $n \to \infty$. Suppose that the following three conditions hold:*

(i) for every n, $F_n \in \mathbb{D}^{1,2}$ and F_n has an absolutely continuous law with respect to Lebesgue measure.

(ii) the quantity $\alpha(n) = \sqrt{E[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathbb{H}})^2]}$ is such that (a) $\alpha(n)$ is finite for all n, (b) as $n \to \infty$, $\alpha(n)$ converges to zero and (c) there exists $m \ge 1$ such that $\alpha(n) > 0$ for $n \ge m$.

(iii) as $n \to \infty$, the two-dimensional random vectors

$$\left(F_n, \frac{1 - \left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}}{\alpha(n)}\right)$$

converge, in distribution, to a centered two-dimensional Gaussian vector (N_1, N_2) such that $E(N_1^2) = E(N_2^2) = 1$ and $E(N_1N_2) = \rho$.

Then, the following upper bound holds:

$$\sup_{z\in R} |P(F_n \le z) - \Phi(z)| \le \sqrt{E[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathbb{H}})^2]}. \tag{1.7}$$

Moreover, for every $z \in R$,

$$\frac{P(F_n \le z) - \Phi(z)}{\alpha(n)} \to \frac{\rho^2}{3}(z^2 - 1)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}. \tag{1.8}$$

In this paper, by using Malliavin calculus, we find a sufficient condition on the sequence $\{F_n\}$ to hold the assumption (iii) of Theorem 1.

2. Preliminaries

In this section, we briefly review some basic facts about Malliavin calculus for Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that $\mathbb{H}$ is a real separable Hilbert space with a scalar product denoted by $< \cdot, \cdot >_{\mathbb{H}}$. Let $X = \{X(h), h \in \mathbb{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $E(X(h)X(g)) = \langle h, g\rangle_{\mathbb{H}}$. For every $q \geq 1$, let $\mathcal{H}_q$ be the q^{th} Wiener chaos of X, that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_q(X(h)) : h \in \mathbb{H}, \|h\|_{\mathbb{H}} = 1\}$, where H_q is the n^{th} Hermite polynomial. We define a linear isometric mapping $I_q : \mathbb{H}^{\odot q} \to \mathcal{H}_q$ by $I_q(h^{\otimes q}) = q!H_q(X(h))$, where $\mathbb{H}^{\odot n}$ is the symmetric tensor product. The following duality formula holds

$$\mathbb{E}\left[FI_q(h)\right] = \mathbb{E}\left[\langle D^q F, h\rangle_{\mathbb{H}^{\otimes q}}\right], \tag{2.1}$$

for any element $h \in \mathbb{H}^{\odot q}$ and any random variable $F \in \mathbb{D}^{q,2}$. Here $\mathbb{D}^{q,2}$ is the closure of the set of smooth random variables with respect to the norm

$$\|F\|^2_{q,2} = \mathbb{E}\left[F^2\right] + \sum_{k=1}^{q} \mathbb{E}\left[\left\|D^k F\right\|^2_{\mathbb{H}^{\otimes k}}\right],$$

where D^k is the iterative Malliavin derivative.

If $f \in \mathbb{H}^{\odot p}$, the Malliavin derivative of the multiple stochastic integrals is given by

$$D_z I_q\left(f_q\right) = qI_{q-1}\left(f_q(\cdot\,, z)\right), \quad \text{for } z \in [0,1]^2. \tag{2.2}$$

Let $\{e_l, l \geq 1\}$ be a complete orthonormal system in $\mathbb{H}$.

The operator L, acting on square integrable random variables, is defined through the projection operators $\{J_q\}_{q\geq 0}$ as $L = \sum_{q=0}^{\infty} -qJ_q$, being called the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. It has the following properties: F is an element of $\mathrm{Dom}(L)(= \mathbb{D}^{2,2})$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}(\delta)$ and in this case $\delta DF = -LF$. We also define the operator L^{-1}, which is *pseudo-inverse* of L, as follows:

$$L^{-1}F = \sum_{q=1}^{\infty} -\frac{1}{q} J_q(F), \quad \text{for every } F \in \mathbb{L}^2(X).$$

Recall that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and that $LL^{-1}F = F - E[F]$ for all $F \in \mathbb{L}^2(X)$.

3. Main Results

We describe our main result in the following Theorem.

Theorem 2. *Let $\{F_n\}$ be a sequence of centered and square integrable functional of some Gaussian process $X = \{X(h) : h \in \mathbb{H}\}$ such that $E(F_n^2) \to 1$ as $n \to \infty$ and the condition (i) in Theorem 1 holds. If the following conditions hold,*

(i) as n tends to infinity,

$$\left\|(I - L)^{-\frac{1}{2}}(DF_n)\right\|^2_{\mathbb{H}} \to 1 \quad \textit{in } L^2(\Omega).$$

(ii) as n tends to infinity,

$$\frac{1}{\alpha(n)}\left\langle (I-L)^{-1}\left(\left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} + \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}}\right), DF_n\right\rangle_{\mathbb{H}} \to \rho \quad in \;\; L^2(\Omega),$$

where

$$\alpha(n) = \sqrt{E\left[(1-\langle DF_n, -DL^{-1}F_n\rangle_{\mathbb{H}})^2\right]}\,.$$

(iii) as n tends to infinity,

$$\frac{1}{\alpha(n)^2}\left\|(I-L)^{-\frac{1}{2}}\left(\left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} + \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}}\right)\right\|_{\mathbb{H}}^2 \to 1 \quad in \;\; L^2(\Omega),$$

then as $n \to \infty$, the two-dimensional random vectors

$$\left(F_n, \frac{1-\left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}}{\alpha(n)}\right)$$

converge, in distribution, to a centered two-dimensional Gaussian vector (N_1, N_2) such that $E(N_1^2) = E(N_2^2) = 1$ and $E(N_1N_2) = \rho$.

Proof: If (N_1, N_2) is a centered two-dimensional Gaussian vector (N_1, N_2) such that $E(N_1^2) = E(N_2^2) = 1$ and $E(N_1N_2) = \rho$, then the characteristic function of bivariate normal is given by

$$\varphi(s,t) = \exp\left\{-\frac{1}{2}\left(s^2 + 2\rho st + t^2\right)\right\}.$$

We will show that as n tends to infinity,

$$E\left[e^{isF_n+itG_n}\right] \to \varphi(s,t). \tag{3.1}$$

Let us set

$$G_n = \frac{1-\left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}}{\alpha(n)}.$$

For every $n \geq 1$, define $\varphi_n(s,t) = E(e^{isF_n+itG_n})$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial s}\varphi_n(s,t) &= iE\left[F_n e^{isF_n+itG_n}\right],\\
\frac{\partial}{\partial t}\varphi_n(s,t) &= iE\left[G_n e^{isF_n+itG_n}\right],\\
\frac{\partial^2}{\partial s\partial t}\varphi_n(s,t) &= -E\left[F_nG_n e^{isF_n+itG_n}\right].
\end{aligned}$$

Since $E(F_n^2) \to 1$ as $n \to \infty$ and $E(G_n^2) = 1$ for all $n \geq 1$, we have, from Chebyshev's inequality, that for any $\epsilon > 0$ there exists a constant δ_ϵ such that

$$P\left(\sqrt{F_n^2 + G_n^2} \geq \delta_\epsilon\right) \leq \frac{E\left(F_n^2\right)+1}{\delta_\epsilon^2} \leq \frac{C}{\delta_\epsilon^2}.$$

If we take $\delta_\epsilon = \sqrt{\epsilon/C}$, then

$$\sup_n P\left(\sqrt{F_n^2 + G_n^2} \geq \delta_\epsilon\right) \leq \epsilon.$$

Hence the sequence of random vectors (F_n, G_n) is tight. By Prohorov's theorem (Theorem 6.1 in Billingsley (1968)), the sequence (F_n, G_n) is relatively compact, and it suffices to show that the limit of any subsequence converging in distribution is the bivariate normal

$$\mathcal{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right).$$

We assume that the sequence of random vectors (F_n, G_n) converges, in distribution, to (F, G), and it suffices to show that

$$(F, G) \sim \mathcal{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right).$$

Let us set

$$\begin{aligned}
\frac{\partial}{\partial s}\varphi(s,t) &= iE\left[Fe^{isF+itG}\right], \\
\frac{\partial}{\partial t}\varphi(s,t) &= iE\left[Ge^{isF+itG}\right], \\
\frac{\partial^2}{\partial s \partial t}\varphi(s,t) &= -E\left[FGe^{isF+itG}\right].
\end{aligned}$$

It is clear that $G_n e^{isF_n+itG_n}$ converges, in distribution, to $Ge^{isF+itG}$. Since $E(G_n^2) = 1$ for all $n \geq 1$, we get, as n goes to infinity,

$$\frac{\partial}{\partial t}\varphi_n(s,t) \to \frac{\partial}{\partial t}\varphi(s,t).$$

By using the definition of the operator L^{-1}, $L = -\delta D$ and divergence operator δ, we get

$$\begin{aligned}
\frac{\partial \varphi_n}{\partial t}(s,t) &= iE\left[L^{-1}L(G_n)e^{isF_n+itG_n}\right] \\
&= -iE\left[\sum_{q\geq 1}\frac{1}{q}J_q\left(LG_n\right)e^{isF_n+itG_n}\right] \\
&= -iE\left[\sum_{q\geq 1}\frac{1}{q}L\left(J_qG_n\right)e^{isF_n+itG_n}\right] \\
&= i\sum_{q\geq 1}\frac{1}{q}E\left[\delta D\left(J_qG_n\right)e^{isF_n+itG_n}\right] \\
&= i\sum_{q\geq 1}\frac{1}{q}E\left[\left\langle D\left(J_qG_n\right), D\left(e^{isF_n+itG_n}\right)\right\rangle_{\mathcal{H}}\right].
\end{aligned}$$

Using the equality $D(J_qF) = J_{q-1}(DF)$ for all $q \geq 1$, we write

$$\begin{aligned}\frac{\partial\varphi_n}{\partial t}(s,t) &= -sE\left[\left\langle \sum_{q=0} \frac{1}{q+1} J_q(DG_n), DF_n \right\rangle_{\mathbb{H}} e^{isF_n+itG_n}\right] - tE\left[\left\langle \sum_{q\geq 1} \frac{1}{q+1} J_q(DG_n), DG_n \right\rangle_{\mathbb{H}} e^{isF_n+itG_n}\right] \\ &= -sE\left[\left\langle (I-L)^{-1}(DG_n), DF_n \right\rangle_{\mathbb{H}} e^{isF_n+itG_n}\right] - tE\left[\left\langle (I-L)^{-1}(DG_n), DG_n \right\rangle_{\mathbb{H}} e^{isF_n+itG_n}\right].\end{aligned}$$

The above two assumptions (ii) and (iii) give

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_n(s,t) \to &- s\rho E\left[e^{isF+itG}\right] - tE\left[e^{isF+itG}\right] \\ = &- s\rho\varphi(s,t) - t\varphi(s,t).\end{aligned}$$

Hence we have

$$\frac{\partial\varphi}{\partial t}(s,t) = -(s\rho + t)\varphi(s,t). \tag{3.2}$$

From the differential equation (3.2), we obtain

$$\varphi(s,t) = \exp\left(-\frac{1}{2}\left(t^2 + 2st\rho\right)\right)\varphi(s,0). \tag{3.3}$$

However, similarly as for $(\partial/\partial t)\varphi_n(s,t)$, we get

$$\frac{\partial}{\partial s}\varphi_n(s,0) = -sE\left[\left\langle (I-L)^{-1}(DF_n), DF_n \right\rangle_{\mathcal{H}} e^{isF_n}\right].$$

The condition (i) yields that as n tends to infinity,

$$\frac{\partial}{\partial s}\varphi_n(s,0) \to -s\varphi(s,0). \tag{3.4}$$

Clearly, $F_n e^{itF_n}$ converges, in distribution, to Fe^{itF} and the boundedness of $L^2(\Omega)$ prove that as n tends to infinity,

$$\frac{\partial}{\partial s}\varphi_n(s,0) \to \frac{\partial}{\partial s}\varphi(s,0). \tag{3.5}$$

From (3.4) and (3.5), it follows, with $\varphi(0,0) = 1$, that

$$\varphi(s,0) = e^{-\frac{1}{2}s^2}. \tag{3.6}$$

Combining (3.3) with (3.6), we obtain

$$\varphi(s,t) = \exp\left(-\frac{1}{2}\left(t^2 + 2st\rho + s^2\right)\right). \tag{3.7}$$

Therefore we complete the proof of (3.1). □

We give a sufficient condition on $\{F_n\}$ corresponding to (i), (ii) and (iii) in Theorem 2 in the case when $F_n = I_q(f_n)$.

Corollary 1. *For $q \geq 2$, consider a sequence $\{F_n = I_q(f_n), n \geq 1\}$ of square integrable random variables belonging to n^{th} Wiener chaos such that*

$$E(F_n^2) = q!\|f_n\|^2_{\mathbb{H}^{\otimes q}} \to 1 \quad as \ \ n \to \infty.$$

If the following conditions hold,

(i) as n tends to infinity,

$$q\|I_{q-1}(f_n)\|^2_{\mathbb{H}} \to 1 \quad in \ \ L^2(\Omega)$$

(ii) as n tends to infinity,

$$\frac{-2q^2(q-1)}{\alpha(n)} \left\langle (I-L)^{-1}\left(\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right), I_{q-1}(f_n)\right\rangle_{\mathbb{H}} \to \rho \quad in \ \ L^2(\Omega),$$

where

$$\alpha(n) = \sqrt{E\left[\left(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathcal{H}}\right)^2\right]}.$$

(iii) as n tends to infinity,

$$\frac{4q^2(q-1)^2}{\alpha(n)^2} \left\|(I-L)^{-\frac{1}{2}}\left(\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right)\right\|^2_{\mathbb{H}} \to 1 \quad in \ \ L^2(\Omega),$$

then, as n tends to infinity, the two-dimensional random vectors

$$\left(F_n, \frac{1 - \left\langle DF_n, -DL^{-1}F_n\right\rangle_{\mathbb{H}}}{\alpha(n)}\right)$$

converge, in distribution, to a centered two-dimensional Gaussian vector (N_1, N_2) such that $E(N_1^2) = E(N_2^2) = 1$ and $E(N_1N_2) = \rho$.

Proof: The equations $(I-L)^{-1}(DI_q(f_n)) = I_{q-1}(f_n)$ and $DI_q(f_n) = qI_{q-1}(f_n)$ give

$$\left\langle (I-L)^{-1}(DF_n), DF_n\right\rangle_{\mathbb{H}} = q\left\|I_{q-1}(f_n)\right\|^2_{\mathbb{H}}.$$

By using the equations $DL^{-1}(I_q(f_n)) = -I_{q-1}(f_n)$, $D^2I_q(f_n) = q(q-1)I_{q-2}(f_n)$ and $D^2L^{-1}(I_q(f_n)) = -(q-1)I_{q-2}(f_n)$, we obtain

$$\left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} = \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}} = -q(q-1)\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}. \tag{3.8}$$

From (3.8), it follows that

$$\begin{aligned}&\frac{1}{\alpha(n)}\left\langle (I-L)^{-1}\left(\left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} + \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}}\right), DF_n\right\rangle_{\mathbb{H}}\\ &= \frac{-2q^2(q-1)}{\alpha(n)}\left\langle (I-L)^{-1}\left(\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right), I_{q-1}(f_n)\right\rangle_{\mathbb{H}},\end{aligned}$$

which implies the condition (ii). As for the condition (iii), we obtain

$$\begin{aligned}
&\frac{1}{\alpha(n)^2}\left\langle (I-L)^{-1}\left(\left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} + \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}}\right), \left\langle D^2F_n, DL^{-1}F_n\right\rangle_{\mathbb{H}} + \left\langle DF_n, D^2L^{-1}F_n\right\rangle_{\mathbb{H}}\right\rangle_{\mathbb{H}} \\
&= \frac{4q^2(q-1)^2}{\alpha(n)^2}\left\langle (I-L)^{-1}\left(\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right), \left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right\rangle_{\mathbb{H}} \\
&= \frac{4q^2(q-1)^2}{\alpha(n)^2}\left\|(I-L)^{-\frac{1}{2}}\left(\left\langle I_{q-1}(f_n), I_{q-2}(f_n)\right\rangle_{\mathbb{H}}\right)\right\|^2_{\mathbb{H}}.
\end{aligned}$$

Therefore, we complete of the proof of this Corollary. □

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