Bull. Korean Math. Soc. **50** (2013), No. 1, pp. 343–351
http://dx.doi.org/10.4134/BKMS.2013.50.1.343

# SINGULARITIES AND STRICTLY WANDERING DOMAINS OF TRANSCENDENTAL SEMIGROUPS

Zhi Gang Huang and Tao Cheng

Abstract. In this paper, the dynamics on a transcendental entire semigroup $G$ is investigated. We show the possible values of any limit function of $G$ in strictly wandering domains and Fatou components, respectively. Moreover, if $G$ is of class $\mathfrak{B}$, for any $z$ in a Fatou domain, there does not exist a sequence $\{g_k\}$ of $G$ such that $g_k(z) \to \infty$ as $k \to \infty$.

## 1. Introduction and main results

In a series of papers, Hinkkanen and Martin extended the classical theory of dynamics associated with the iteration of a single rational function to the more general setting of semigroups of rational functions, see [8, 9]. In 1998, Poon [11, 12] extended the study to transcendental semigroups and obtained some basic results. The dynamics of transcendental semigroups actually has some rather different properties than the dynamics of rational semigroups or the iteration of a single function.

Suppose $\{f_j : j = 1, 2, \ldots, m\}$ is a family of transcendental entire functions. We call the semigroup $G = \langle f_1, f_2, \ldots, f_m\rangle$ generated by $\{f_j\}$ under functional composition a transcendental semigroup.

Define the Fatou set of the semigroup $G$ by

$$F(G) = \{z \in \mathbb{C} : G \text{ is normal in some neighbourhood of } z\}$$

and the Julia set of $G$ by $J(G) = \mathbb{C} \setminus F(G)$.

If $G$ is generated by only one function $f$, then $F(G)$ and $J(G)$ are the Fatou set and Julia set respectively in the classical iteration theory of Fatou and Julia.

We say that a set $S$ is completely invariant under $f$ if $S$ is forward and backward invariant under $f$. It is well known that the Fatou set and Julia set of a single function $f$ are both completely invariant. But $F(G)$ and $J(G)$ need

Received September 5, 2011; Revised May 17, 2012.
2010 *Mathematics Subject Classification.* 37F10, 30D05.
*Key words and phrases.* transcendental semigroup, strictly wandering domain, limit function, singularity.
Supported by NSF of China (11001057,11001081,11071074), NSF of Jiangsu Province (BK2010234), Project of Qinglan of Jiangsu Province.

not be completely invariant. In fact, $F(G)$ is forward invariant and $J(G)$ is backward invariant.

Since $F(G)$ is forward invariant, for any component $U$ of $F(G)$ and any $g \in G$, we have $g(U) \subset F(G)$. We use $U_g$ to denote the component of $F(G)$ which contains $g(U)$.

**Definition 1.** A component $U$ of $F(G)$ is called a wandering domain of $G$ provided that the set $\{U_g : \forall g \in G\}$ is infinite. Otherwise, $U$ is called a non-wandering domain. Furthermore, we say that a component $U$ of $F(G)$ is strictly wandering if $U_g = U_h$ implies $g = h$.

For a single function $f$, the two definitions are the same. While for a semigroup, they are generally different. A wandering domain of $G$ may not be wandering under the iteration of each element of $G$ or may be a wandering domain of just one generator of $G$, cf. [8, Section 5].

Denote by sing $f^{-1}$ the set of singularities of $f^{-1}$, that is, the set of critical and asymptotic values of $f$. Let $E$ be $\cup_{n=0}^{\infty} f^n(\text{sing } f^{-1})$, $\overline{E}$ be the closure of $E$, and $E'$ be the derived set of $E$, i.e., the set of finite limit points of $E$. Let $\mathfrak{B}$ denote the class consisting of meromorphic functions $f$ such that sing $f^{-1}$ is bounded, and let $\mathfrak{S}$ denote the class of meromorphic functions $f$ which have only finitely many critical and asymptotic values. For a semigroup $G$, if all the elements belong to $\mathfrak{S}$ or $\mathfrak{B}$, we call $G$ a semigroup of class $\mathfrak{S}$ or $\mathfrak{B}$.

**Theorem 1.** *Suppose $G$ is a finitely generated transcendental semigroup. Then any limit function of $G$ on a strictly wandering domain $U$ of $F(G)$ is $\infty$ or lies in* $(\bigcup_{\phi \in G} \text{sing } \phi^{-1})'$.

**Theorem 2.** *Suppose $G$ is a finitely generated transcendental semigroup and $U$ is a component of $F(G)$. Let $q$ be a constant limit function of $G$ on $U$. Then either $q$ is $\infty$ or* $q \in \overline{(\bigcup_{\phi \in G} \text{sing } \phi^{-1})}$.

*Remark* 1. It is well known that all limit functions of $\{f^n|_U\}$ are constant if $U$ is a wandering domain. Baker [1] proved that constant limit functions in Fatou components $F(f)$ (not necessarily wandering) are in $\overline{E} \cup \{\infty\}$ if $f$ is a nonlinear entire function. Furthermore, Bergweiler et al. [5] proved that the constant limit functions in wandering domains are in $E' \cup \{\infty\}$. It is obvious that the results in [1, 5] are special cases of our theorems when $m = 1$. Moreover, our method, which mainly depends on hyperbolic metric, is different from the method in [1, 5]. There are also many papers on this subject for the case of meromorphic functions (see [3, 14, 15]).

In fact, if $f$ is an entire function in $\mathfrak{B}$, Eremenko and Lyubich [7] showed that for each $p > 0$ and $z \in F$, the orbit $\{f^{pn}(z)\}_{n=0}^{\infty}$ does not tend to $\infty$. Theorem 3 below extends the result to the case of transcendental semigroups. Obviously, our theorem contains this result in [7].

**Theorem 3.** *Suppose $G$ is a finitely generated transcendental semigroup of class $\mathfrak{B}$. Then for all $z \in F(G)$, there does not exist any sequence $\{g_k\}$ of $G$ such that $g_k(z) \to \infty$ as $k \to \infty$.*

As an application of Theorem 1 and Theorem 3, we can get a class of transcendental semigroups which has no strictly wandering domains.

**Theorem 4.** *Suppose $G$ is a finitely generated transcendental semigroup of class $\mathfrak{B}$. If there exists some $f_j$ $(j \in \{1, 2, \ldots, m\})$, which has no rationally indifferent cycles, such that $J(f_j) = J(G)$ and $J(G) \cap (\bigcup_{\psi \in G} \text{sing } (\psi^{-1}))'$ is finite, then $G$ has no strictly wandering domains.*

## 2. Proof of Theorem 1 and Theorem 2

First, we recall the basic knowledge of the hyperbolic metric on a hyperbolic domain. Let $\Omega$ be a hyperbolic domain in the complex plane $\mathbb{C}$, that is, $\mathbb{C} \setminus \Omega$ contains at least two points. We denote the hyperbolic density of $\Omega$ by $\lambda_\Omega(z)$. It is well known that the hyperbolic metric on the unit disk $\triangle$ is given by the density

$$\lambda_\triangle(z)|dz| = \frac{|dz|}{1 - |z|^2}.$$

Then the hyperbolic density $\lambda_\Omega(z)$ on $\Omega$ is determined by

$$\lambda_\Omega(p(z))|p'(z)| = \frac{1}{1 - |z|^2}, z \in \triangle,$$

where $p(z)$ is a holomorphic universal covering map of $\Omega$ from $\triangle$. Let $D(a, \delta) = \{z : |z - a| < \delta\}$ and $D^*(a, \delta) = \{z : |z - a| < \delta\} \setminus \{a\}$. It is easy to see that

$$\lambda_{D^*(0,\delta)}(z) = \frac{1}{2|z| \log(\frac{\delta}{|z|})}.$$

We need the following Schwarz-Pick Lemma which plays an important role in our proof.

**Lemma 1.** *Let $U$ and $\Omega$ both be hyperbolic domains, and let $f(z)$ be a holomorphic map on $U$ such that $f(U) \subset \Omega$. Then*

$$\lambda_\Omega(f(z))|f'(z)| \leq \lambda_U(z), \forall z \in U,$$

*and the equality holds if and only if $f$ is a covering map of $\Omega$ from $U$.*

Suppose $\Omega$ is a hyperbolic and simply connected domain. For a point $z_0 \in \Omega$, there exists a Riemann map $\phi : \triangle \to \Omega$ such that $\phi(0) = z_0$. By the Koebe $\frac{1}{4}$ Theorem, $\{|z - z_0| < \frac{1}{4}|\phi'(0)|\} \subseteq \Omega$, and then

$$\frac{1}{4}|\phi'(0)| \leq \delta_\Omega(z_0), \tag{1}$$

where $\delta_\Omega(z)$ is the Euclidean distance from $z$ to the boundary of $\Omega$. By Lemma 1 and the inequality (1),

$$\lambda_\Omega(z_0) = \lambda_\Omega(\phi(0)) \geq \frac{1}{4\delta_\Omega(z_0)}. \tag{2}$$

So

$$\lambda_\Omega(z)\delta_\Omega(z) \geq \frac{1}{4}, \quad z \in \Omega. \tag{3}$$

Let $U$ be a wandering domain of $G$. By Definition 1, $\{U_g : \forall g \in G\}$ is infinite. Then there exists a sequence of elements $\{g_j\}$ of $G$ such that $g_j(U) \subset U_{g_j}$ and $U_{g_i} \neq U_{g_j}$ if $i \neq j$. The following lemma which is Theorem 6.1 in [11] can be easily obtained.

**Lemma 2.** *Let $G$ be a transcendental semigroup. If $U$ is a wandering domain of $G$ and $\{g_j\}$ is a sequence of functions as above, then any limit function of $\{g_j\}$ on $U$ must be constant and belongs to $J(G)$.*

*Proof of Theorem 1.* By Lemma 2, all limit functions of $G$ on a strictly wandering domain $U$ are constant and belong to $J(G)$. Suppose, on the contrary, Theorem 1 does not hold. Thus, there exists a sequence $\{g_k\}$ of $G$ such that $g_k \to a$ as $k \to \infty$ where $a$ is a finite constant number. Without loss of generality, we may assume $a = 0$ and 0 does not lie in $(\text{sing } \phi^{-1})'$ for any $\phi \in G$. Now, we can take a sufficiently small $\varepsilon > 0$ such that $N = D^*(0, \varepsilon)$ does not meet sing $\phi^{-1}$ for any $\phi \in G$. Select a disc $D = D(z_0, r)$ with $\overline{D(z_0, r)} \subset U$. We have $w_k = g_k(z_0) \to 0$ as $k \to \infty$. Let $p_k$ be the branch of $g_k^{-1}$ such that $p_k(w_k) = z_0$. Write $u_k = \log(w_k)$, noting that $w_k \neq 0$. Then $h_k = p_k(\exp(t))$ is analytic at $t = u_k$, and $h_k(u_k) = z_0$. It follows that $h_k$ can be continued analytically to a single-valued function in $H = \{t : \text{Re}t < \log \varepsilon\}$. By Montel's Theorem, $\{q_k = h_k(u_k + (\log \varepsilon - \text{Re}u_k)v)\}$ is a normal family in $D(0, 1)$. Then by Marty's criterion, the spherical derivatives $q_k^{\#}(0) = |q_k'(0)|/(1 + |q_k(0)|^2)$ are bounded. So there exists a positive constant $B$ such that

$$\frac{(\log \varepsilon - \text{Re}u_k)|w_k p_k'(w_k)|}{1 + |z_0|^2} \leq B. \tag{4}$$

It follows that

$$|(g_k)'(z_0)| \geq \frac{|g_k(z_0)|(\log \varepsilon - \text{Re}u_k)}{B(1 + |z_0|^2)}, \tag{5}$$

where

$$\text{Re}u_k = \log |w_k| = \log |p_k(z_0)|.$$

On the other hand, since $g_k \to 0$ (as $k \to \infty$), we have $D_k = g_k(D) \subset N$ for all sufficiently large $k$. For convenience, we can assume $D_k = g_k(D) \subset N$ for all $k$. Since $D_k \cap \text{sing } g_k^{-1} = \emptyset$, $g_k|_D$ is proper (that is, the inverse image of any compact set in $D_k$ is compact in $D$). Noting that $D$ is simply connected, by the Riemann-Hurwitz formula, $D_k$ is also simply connected. By Lemma 1,

$$\lambda_{D_k}(g_k(z_0))|g_k'(z_0)| \leq \lambda_D(z_0). \tag{6}$$

Again by (3),

$$\lambda_{D_k}(g_k(z_0))\delta_{D_k}(g_k(z_0)) \geq \frac{1}{4}, \tag{7}$$

then

$$\lambda_{D_k}(g_k(z_0)) \geq \frac{1}{4\delta_{D_k}(g_k(z_0))} \geq \frac{1}{4|g_k(z_0)|}. \tag{8}$$

So by (6) and (8),

$$\frac{|g_k'(z_0)|}{4|g_k(z_0)|} \leq \lambda_D(z_0). \tag{9}$$

Combining (5) and (9),

$$4|g_k(z_0)|\lambda_D(z_0) \geq \frac{|g_k(z_0)|(\log\varepsilon - \mathrm{Re}u_k)}{B(1+|z_0|^2)}. \tag{10}$$

Since $\mathrm{Re}u_k \to -\infty$ as $k \to \infty$, we obtain a contradiction. □

*Proof of Theorem 2.* Suppose that Theorem 2 does not hold. There exists a disk neighborhood $V$ of $q$ which does not meet $\overline{\text{sing }\phi^{-1}}$ for any $\phi \in G$. Then for any $g \in G$, we can take all branches of $g^{-1}$ which are well defined on $V$. Denote by $\Lambda$ the family of transcendental entire functions where each element of $\Lambda$ is a branch of the inverse of an element of $G$. Since $\overline{\bigcup_{\phi\in G}\text{sing }\phi^{-1}}$ contains at least three points, it is easy to see that $\Lambda$ is normal in $V$. Let $\{g_j\}$ be a sequence with $g_j \to q$ locally uniformly on a compact subset $\widetilde{U}$ of $U$ and $g_j(\widetilde{U}) \subset V$ for sufficiently large $j$. Take a curve $\gamma \in \widetilde{U}$ containing at least two points. We can take sufficiently large $j$ and define a branch $h_j$ of $g_j^{-1}$ which is regular in $V$, such that it maps $g_j(\gamma)$ to $\gamma$. So $\{h_j\}$ is equicontinuous. On the other hand, $\{g_j(\gamma)\}$ converges to $q$, and for any neighborhood $W$ of $q$, there exists $j$ such that $h_j(W)$ contains $\gamma$. Thus, we get a contradiction. □

## 3. Proof of Theorem 3 and Theorem 4

**Lemma 3.** *Suppose $f \in \mathfrak{B}$ and $0 \notin \bigcup_{s=1}^{\infty} f^{-s}(\infty)$, then there exist a positive constant $R$ and a curve $\Gamma$ connecting 0 and $\infty$ such that $|f(z)| \leq R$ on $\Gamma$ and for all $z \in C \setminus \{0\}$ which are not poles of $f$,*

$$|f'(z)| \geq \frac{|f(z)|}{2\pi|z|}\log\frac{|f(z)|}{R}. \tag{11}$$

Lemma 3 is a combination of [4, Lemma 8] and [17, Lemma 2].

**Lemma 4.** *Let $q$ be an irrationally neutral fixed point of a transcendental entire function $f$. Then there is no point $z \in C$ such that $f^n(z) \to q$ (as $n \to \infty$), except when $f^n(z) = q$ for some $n$.*

*Remark* 2. Lemma 4 was proved by Perez-Marco [10] while Sullivan [13] got the same result for a single rational function $R$. Let $V$ be a component of $F(f)$ of a transcendental entire function $f$. Suppose that $f^{np}|_V \to q$ where $p$ is a positive integer and $q$ is a constant. It is well known that $f^{np}|_V$ can not tend to a repelling periodic point. And $q$ can not be a rationally neutral periodic point of $f$ if $V$ is not the corresponding parabolic domain. Therefore according to the classification theorem of periodic components, we have that $q$ is not a periodic point of $f$ if and only if $V$ is either (preimage of) a Baker domain or a wandering domain, in this case, $q \in J(f)$.

Next, Lemma 5 can be proved by the same method of [15, Theorem 3]; we give a brief proof here for completeness.

**Lemma 5.** *Let $f$ be a transcendental entire function, and suppose $a$ is not a (pre)periodic point. If the set $X$ of limit points of $\{f^n(a)\}$ is finite, then either $X = \{\infty\}$ or $X$ is a periodic cycle.*

*Proof.* Suppose that $X = \{a_1, a_2, \ldots, a_s\}$ contains a finite element $a_1$. Noting that $f^k(a_1)$ is also a limit point of $\{f^n(a)\}$ for any positive integer $k$. We see that $a_1$ is a (pre)periodic point and $X$ contains a periodic cycle. Without loss of generality, we assume $\{a_1, a_2, \ldots, a_t\}$, $t \le s$, is a periodic cycle. Take two positive numbers $d$ and $M$ such that $D(a_j, d)(j = 1, 2, \ldots, s)$ are mutually disjoint and lie inside $D(0, M)$. There exists a positive integer $N_0$ such that for $n \ge N_0$,

$$f^n(a) \in \bigcup_{j=1}^{s} D(a_j, d) \cup \{z : |z| > M\}.$$

Take a positive number $r < d$ and a positive integer $N_1 > N_0$, such that for any $n > N_1$, we have

$$f^n(a) \in \bigcup_{j=1}^{s} D(a_j, r) \cup \{z : |z| > M\} \tag{12}$$

and

$$f(D(a_j, r)) \subset D(f(a_j), d)(j = 1, 2, \ldots, s).$$

Assume that $f^{n_k}(a) \to a_1 (k \to \infty)$. For $k > N_1$, we have $f^{n_k}(a) \in D(a_1, r)$. So

$$f^{n_k+1}(a) \in D(f(a_1), d) = D(a_2, d).$$

Since $D(a_j, d)(j = 1, 2, \ldots, s)$ are mutually disjoint, by (6), we have $f^{n_k+1}(a) \in D(a_2, r)$. By induction, for $n > n_k$,

$$f^n(a) \in \bigcup_{j=1}^{t} D(a_j, r).$$

Thus $X$ is a periodic cycle. □

**Lemma 6.** *Let $f$ and $g$ be nonconstant entire functions. Then* $\operatorname{sing}((f\circ g)^{-1}) \subset \operatorname{sing} f^{-1} \cup f(\operatorname{sing} g^{-1})$.

Lemma 6 was proved by Bergweiler and Wang [6, Lemma 2].

**Lemma 7.** *Suppose $G$ is a finitely generated transcendental semigroup. Then*

$$J(G) = \overline{\bigcup_{g\in G} J(g)}.$$

Lemma 7 comes from [11, Theorem 4.2]. We denote by $\operatorname{ind}_\alpha\gamma$ the index of a curve $\gamma \subset C$ with respect to a point $\alpha$. The following lemma was proved by Baker [2, Theorem 3.1].

**Lemma 8.** *Let $f$ be a transcendental entire function, and let $U$ be a multiply connected component of the Fatou set $F(f)$. We denote by $\gamma$ a Jordan curve that is not contractible in $U$. Then*

1. $f^n \to \infty$ *uniformly on compact subsets of $U$, and so the distance between $f^n(\gamma)$ and $0$ is large*;
2. $\operatorname{ind}_0 f^n(\gamma) > 0$ *for sufficiently large $n$ and* $\operatorname{ind}_0 f^n(\gamma) \to \infty$ *as $n \to \infty$.*

Now we are in the position to prove Theorem 3 and Theorem 4.

*Proof of Theorem 3.* By Lemma 3, there exist $R$ and curves $\Gamma_j (1 \le j \le m)$ connecting $0$ and $\infty$ such that $|f_j(z)| \le R$ on $\Gamma_j (1 \le j \le m)$ and for all $z \in \mathbb{C} \setminus \{0\}$,

$$|f_j'(z)| \ge \frac{|f_j(z)|}{2\pi|z|}\log\frac{|f_j(z)|}{R}, \quad 1 \le j \le m. \tag{13}$$

Suppose there exist a point $z_0 \in F(G)$ and a sequence $\{g_k\}$ of $G$ such that $g_k \to \infty$ as $k \to \infty$. Since $\{g_k\}$ is normal at $z_0$, we can take a fixed positive number $R_0$ and a disk $D = D(z_0, R_0)$ such that $g_k|_D \to \infty$ as $k \to \infty$. Denote $D_k = g_k(D)$. $D_k$ lies in the component $U_{g_k}$ which is a component of $F(G)$. Noting each $f_j \in \mathcal{B}$, from Lemma 6, it is easy to see that an element of $G$ must be in Class $\mathfrak{B}$.

We claim that every component must be simply connected. Suppose a component $V$ of $F(G)$ is multiply-connected. We draw a simple closed Jordan curve $\gamma$ which is not homotopic to point there. By Lemma 7, we have $J(G) = \overline{\bigcup_{g\in G} J(g)}$. Therefore, there exists an $h \in G$ such that $J(h)$ intersects the bounded interior surrounded by $\gamma$. Since $\gamma \subset V \subset F(h)$, $\gamma$ is not null-homotopic with respect to $F(h)$. Otherwise $\infty \in F(h)$, which is impossible, and the component $\widetilde{V}$ of $F(h)$ which contains $V$ is multiply-connected. By Lemma 8, $h^n$ tends to $\infty$ locally uniformly on $\widetilde{V}$ as $n \to \infty$. But $h$ is in class $\mathfrak{B}$, so there is a contradiction. Thus $U_{g_k}$ is simply connected.

By Lemma 1, we have

$$\lambda_{U_{g_k}}(g_k(z_0))|g_k'(z_0)| \le \frac{1}{R_0}.$$

Since $U_{g_k}$ is simply connected, by (3), for sufficiently large $k$,

$$|(g_k)'(z_0)| \leq \frac{4}{R_0}\delta_{U_{g_k}}(g_k(z_0)) \leq \frac{4|g_k(z_0)|}{R_0}. \tag{14}$$

On the other hand, write $g_k = h_{k_n} \circ h_{k_{n-1}} \circ \cdots \circ h_{k_1}$, where $h_{k_j}$ is chosen from$\{f_1, \ldots, f_m\}$. Note that $n \to \infty$ when $k \to \infty$. Write $w_p = h_{k_p} \circ h_{k_{p-1}} \circ \cdots \circ h_{k_1}(z_0)$, $p = 1, 2, \ldots, n$ and $w_0 = z_0$. By (13), we have

$$|g_k'(z_0)| = \prod_{s=0}^{n-1} |h'_{k_{s+1}}(w_s)| \geq \prod_{s=0}^{n-1} \frac{|w_{s+1}|}{2\pi|w_s|} \log\frac{|w_{s+1}|}{R} = \frac{|g_k(z_0)|}{|z_0|} \prod_{s=0}^{n-1} \frac{1}{2\pi} \log\frac{|w_{s+1}|}{R}.$$

This inequality contradicts (14) since $\prod_{s=0}^{n-1} \frac{1}{2\pi} \log \frac{|w_{s+1}|}{R} \to \infty (k \to \infty)$. This completes the proof. □

*Proof of Theorem 4.* Suppose $G$ has a strictly wandering domain $U$. By Theorem 1, all the limit points of $G$ lie in $J(G) \cap (\bigcup_{\psi \in G} \operatorname{sing}(\psi^{-1}))'$. Thus, the limit set $X$ of $\{f_j^n|_U\}$ contains finitely many elements. Let $X = \{a_1, a_2, \ldots, a_t\}$. By Theorem 3 and Lemma 5, $X$ is a periodic cycle. Since $a_1 \in J(G) = J(f_j)$ and $f_j$ has no rationally indifferent cycles, $a_1$ is a repelling periodic point or non-Siegel point of $f_j$. By Lemma 4 and Remark 2, we obtain a contradiction. □

**Acknowledgement.** The authors would like to thank the referees for their valuable suggestions and comments.

## References


[1] I. N. Baker, *Limit functions and sets of non-normality in iteration theory*, Ann. Acad. Sci. Fenn. Ser. A I. **467** (1970), 2–11.

[2] ———, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49** (1984), no. 3, 563–576.

[3] ———, *Limit functions in wandering domains of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 499–505.

[4] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151–188.

[5] W. Bergweiler, M. Haruke, H. Kriete, H. G. Meier, and N. Terglane, *On the limit functions of iterates in wandering domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), no. 2, 369–375.

[6] W. Bergweiler and Y. F. Wang, *On the dynamics of composite entire functions*, Ark. Mat. **36** (1998), no. 1, 31–39.

[7] A. E. Eremenko and M. Y. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[8] A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions. I*, Proc. London Math. Soc. (3) **73** (1996), no. 2, 358–384.

[9] ———, *Julia sets of rational semigroups*, Math. Z. **222** (1996), no. 2, 161–169.

[10] M. R. Perez, *Sur une question de Dulac et Fatou*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 8, 1045–1048.

[11] K. K. Poon, *Fatou-Julia theory on transcendental semigroups*, Bull. Austral. Math. Soc. **58** (1998), no. 3, 403–410.

[12] ———, *Fatou-Julia theory on transcendental semigroups. II*, Bull. Austral. Math. Soc. **59** (1999), no. 2, 257–262.

[13] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418.
[14] J. H. Zheng, *Singularities and wandering domains in iteration of meromorphic functions*, Illinois J. Math. **44** (2000), no. 3, 520–530.
[15] ———, *Singularities and limit functions in iteration of meromorphic functions*, J. London Math. Soc. (2) **67** (2003), no. 1, 195–207.
[16] ———, *Iteration of functions which are meromorphic outside a small set*, Tohoku Math. J. (2) **57** (2005), no. 1, 23–43.
[17] ———, *On transcendental meromorphic functions which are geometrically finite*, J. Aust. Math. Soc. **72** (2002), no. 1, 93–107.

Zhi Gang Huang
Department of Mathematics
Suzhou University of Science and Technology
Suzhou, 215009, P. R. China
*E-mail address*: alexehuang@yahoo.com.cn

Tao Cheng
Department of Mathematics
East China Normal University
Shanghai, 200062, P. R. China
*E-mail address*: chengtaorex@sohu.com