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A POSTERIORI $L^\infty(L^2)$-ERROR ESTIMATES OF SEMIDISCRETE MIXED FINITE ELEMENT METHODS FOR HYPERBOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we discuss the a posteriori error estimates of the semidiscrete mixed finite element methods for quadratic optimal control problems governed by linear hyperbolic equations. The state and the co-state are discretized by the order k Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise polynomials of order $k(k \geq 0)$. Using mixed elliptic reconstruction method, a posteriori $L^\infty(L^2)$-error estimates for both the state and the control approximation are derived. Such estimates, which are apparently not available in the literature, are an important step towards developing reliable adaptive mixed finite element approximation schemes for the control problem.

1. Introduction

In recent years, there is a growing demand for designing reliable and efficient space-time algorithms for numerical computations of both linear and nonlinear time dependent partial differential equations. Most of these algorithms are based on a posteriori error estimators which provide appropriate tools for adaptive mesh refinements. The theory of a posteriori analysis of finite element methods for parabolic problems is well-developed (see, e.g., [3, 4, 19, 22, 27, 29, 36, 40]). Surprisingly, there has been considerably less work on the error control of finite element methods for second order hyperbolic problems, despite the substantial amount of research in the design of finite element methods for the wave problem (see, e.g., [6, 7, 8, 11, 20]). A posteriori bounds for standard implicit time-stepping finite element approximations to the linear wave equation have been proposed and analyzed (but only in very specific situations) by Adjerid [1]. Also, Bernardi and Süli [12] derive rigorous a posteriori $L^\infty(H^1)$-error bounds, using energy arguments. We note that goal-oriented-error estimation for wave problems (via duality techniques) is also available [9, 10], while some earlier work on a posteriori estimates for first order

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hyperbolic systems have been studied in the time semidiscrete setting [37], as well as in the fully discrete one [26].

It is well known that finite element approximation of the optimal control problems has been an important and hot topic in engineering design work, and has been extensively studied in literature [14, 24, 25, 31, 34, 39]. For the optimal control problems governed by elliptic or parabolic state equations, a priori error estimates of finite element approximations were studied in, for example, [2, 23, 28, 30, 33, 35, 38]. There also exist lots of works concentrating on the adaptivity of various optimal control problems (see, e.g., [14, 23, 30, 33, 35, 34]).

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods (see, for example, [13]). When the objective functional contains the gradient of the state variable, mixed finite element methods should be used for discretization of the state equation with which both the scalar variable and its flux variable can be approximated in the same accuracy. Recently, in [16, 17, 18] the authors have done some primary works on a priori, superconvergence and a posteriori error estimates error estimates for linear elliptic optimal control problems by mixed finite element methods. However, there doesn't seem to exist any work on a posteriori error analysis of mixed finite element approximation for hyperbolic problems in the literature, especially for hyperbolic optimal control problems.

In this article, we shall investigate a posteriori error estimates of the semidiscrete mixed finite element approximation for hyperbolic optimal control problems. Combining the idea about the elliptic construction of [36] with our hyperbolic optimal control problems, we define the mixed elliptic construction for the state and the co-state variables. Using the mixed elliptic construction method, we derive a posteriori $L^\infty(L^2)$-error estimates for both the state and the control approximation.

The optimal control problem that we are interested in is as follows:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\} \tag{1}$$

$$y_{tt}(x,t)+\mathrm{div}\boldsymbol{p}(x,t)=f(x,t)+u(x,t),\quad x\in\Omega,\ t\in J, \tag{2}$$

$$\boldsymbol{p}(x,t)=-A(x)\nabla y(x,t),\quad x\in\Omega,\ t\in J, \tag{3}$$

$$y(x,t)=0,\quad x\in\partial\Omega,\ t\in J, \tag{4}$$

$$y(x,0)=y_0(x),\quad x\in\Omega, \tag{5}$$

$$y_t(x,0)=y_1(x),\quad x\in\Omega, \tag{6}$$

where the bounded open set $\Omega\subset\mathbf{R}^2$ is a convex polygon with the boundary $\partial\Omega$, $J=[0,T]$. Let K be a closed convex set in $U=L^2(J;L^2(\Omega))$, $f,y_d\in$

$L^2(J;L^2(\Omega))$, $\boldsymbol{p}_d \in (L^2(J;L^2(\Omega)))^2$, $y_0 \in H^2(\Omega)$ and $y_1 \in H^1(\Omega)$. We assume that the coefficient matrix $A(x) = (a_{ij}(x))_{2\times 2} \in W^{1,\infty}(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric 2×2-matrix and there are constants $c_1, c_2 > 0$ satisfying for any vector $\mathbf{X} \in \mathbf{R}^2$, $c_1\|\mathbf{X}\|^2_{\mathbf{R}^2} \le \mathbf{X}^t A\mathbf{X} \le c_2\|\mathbf{X}\|^2_{\mathbf{R}^2}$. K is a set defined by

$$K = \left\{ u \in U : \int_0^T \int_\Omega u \; dxdt \ge 0 \right\}.$$

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p} = \sum_{|\alpha|=m} \|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T ||v||^s_{W^{m,p}(\Omega)} dt \right)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^1(J;W^{m,p}(\Omega))$ and $C^k(J;W^{m,p}(\Omega))$. The details can be found in [32]. In addition C denotes a general positive constant independent of h.

The plan of this paper is as follows. In Section 2, we shall construct the semidiscrete mixed finite element approximation for the optimal control problems (1)-(6), then we introduce some projection operators and define mixed elliptic constructions. Using mixed elliptic reconstructions, we derive a posteriori error estimates of mixed finite element approximation for the control problem in Section 3. Finally, we give a conclusion and some future works.

2. Mixed methods of optimal control problems

In this section, we shall construct the semidiscrete mixed finite element approximation for the hyperbolic optimal control problem (1)-(6). To fix the idea, we shall take the state spaces $\boldsymbol{L} = L^2(J;\boldsymbol{V})$ and $Q = L^2(J;W)$, where $\boldsymbol{V}$ and W are defined as follows:

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \left\{ \boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega) \right\}, \quad W = L^2(\Omega).$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{v}\|^2_{0,\Omega} \right)^{1/2}.$$

Let $\alpha = A^{-1}$, we recast (1)-(6) as the following weak form: find $(\mathbf{p}, y, u) \in \boldsymbol{L} \times Q \times K$ such that

$$\min_{u\in K\subset U} \left\{ \frac{1}{2} \int_0^T \left(\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right) dt \right\}, \tag{7}$$

$$(\alpha\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ t \in J, \tag{8}$$

$$(y_{tt}, w) + (\mathrm{div}\boldsymbol{p}, w) = (f + u, w), \quad \forall\, w \in W,\ t \in J, \tag{9}$$

$$y(x,0) = y_0(x), \quad \forall\, x \in \Omega, \tag{10}$$

$$y_t(x,0) = y_1(x), \quad \forall\, x \in \Omega. \tag{11}$$

It follows from [31] that the optimal control problem (7)-(11) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (7)-(11) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{L} \times Q$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(\alpha \boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ t \in J, \tag{12}$$

$$(y_{tt}, w) + (\mathrm{div}\boldsymbol{p}, w) = (f + u, w), \quad \forall\, w \in W,\ t \in J, \tag{13}$$

$$y(x,0) = y_0(x), \quad \forall\, x \in \Omega, \tag{14}$$

$$y_t(x,0) = y_1(x), \quad \forall\, x \in \Omega, \tag{15}$$

$$(\alpha \boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ t \in J, \tag{16}$$

$$(z_{tt}, w) + (\mathrm{div}\boldsymbol{q}, w) = (y - y_d, w), \quad \forall\, w \in W,\ t \in J, \tag{17}$$

$$z(x,T) = 0, \quad \forall\, x \in \Omega, \tag{18}$$

$$z_t(x,T) = 0, \quad \forall\, x \in \Omega, \tag{19}$$

$$\int_0^T (u + z, \tilde{u} - u)dt \geq 0, \quad \forall\, \tilde{u} \in K, \tag{20}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

Due to the special structure of our control constraint K, we are able to derive an important relationship between the optimal control u and the optimal co-state z. This relationship is a key to our analysis.

Lemma 2.1. *Let $(y, \boldsymbol{p}, z, \boldsymbol{q}, u)$ be the solution of* (12)-(20)*. Then we have $u = \max\{0, \bar{z}\} - z$, where*

$$\bar{z} = \frac{\int_0^T \int_\Omega z dx dt}{\int_0^T \int_\Omega 1 dx dt}$$

denotes the integral average on $\Omega \times J$ of the function z.

Let $\mathcal{T}_h$ be regular triangulations of Ω. h_τ is the diameter of τ and $h = \max h_\tau$. Further, let $\mathcal{E}_h$ be the set of element sides of the triangulation $\mathcal{T}_h$ with $\Gamma_h = \cup\, \mathcal{E}_h$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote the Raviart-Thomas space [21] associated with the triangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k $(k \geq 0)$. Let $\boldsymbol{V}(\tau) = \{\boldsymbol{v} \in P_k^2(\tau) + x \cdot P_k(\tau)\}$, $W(\tau) = P_k(\tau)$. We define

$$\begin{aligned}
&\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall\, \tau \in \mathcal{T}_h, \boldsymbol{v}_h|_\tau \in \boldsymbol{V}(\tau)\},\\
&W_h := \{w_h \in W : \forall\, \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\},\\
&K_h := L^2(J; W_h) \cap K.
\end{aligned}$$

The mixed finite element discretization of (7)-(11) is as follows: compute $(\boldsymbol{p}_h, y_h, u_h) \in L^2(J;\boldsymbol{V}_h) \times L^2(J;W_h) \times K_h$ such that

$$\min_{u_h\in K_h} \left\{ \frac{1}{2}\int_0^T \left(\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2\right)dt \right\}, \tag{21}$$

$$(\alpha\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\ t \in J, \tag{22}$$

$$(y_{h,tt}, w_h) + (\mathrm{div}\boldsymbol{p}_h, w_h) = (f + u_h, w_h), \quad \forall\, w_h \in W_h,\ t \in J, \tag{23}$$

$$y_h(x,0) = y_0^h(x), \quad \forall\, x \in \Omega, \tag{24}$$

$$y_{h,t}(x,0) = y_1^h(x), \quad \forall\, x \in \Omega, \tag{25}$$

where $y_0^h(x) \in W_h$ and $y_1^h(x) \in W_h$ are two approximations of y_0 and y_1. The optimal control problem (21)-(25) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (21)-(25) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in L^2(J;\boldsymbol{V}_h) \times L^2(J;W_h)$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(\alpha\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\ t \in J, \tag{26}$$

$$(y_{h,tt}, w_h) + (\mathrm{div}\boldsymbol{p}_h, w_h) = (f + u_h, w_h), \quad \forall\, w_h \in W_h,\ t \in J, \tag{27}$$

$$y_h(x,0) = y_0^h(x), \quad \forall\, x \in \Omega, \tag{28}$$

$$y_{h,t}(x,0) = y_1^h(x), \quad \forall\, x \in \Omega, \tag{29}$$

$$(\alpha\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\ t \in J, \tag{30}$$

$$(z_{h,tt}, w_h) + (\mathrm{div}\boldsymbol{q}_h, w_h) = (y_h - y_d, w_h), \quad \forall\, w_h \in W_h,\ t \in J, \tag{31}$$

$$z_h(x,T) = 0, \quad \forall\, x \in \Omega, \tag{32}$$

$$z_{h,t}(x,T) = 0, \quad \forall\, x \in \Omega, \tag{33}$$

$$\int_0^T (u_h + z_h, \tilde{u}_h - u_h)dt \geq 0, \quad \forall\, \tilde{u}_h \in K_h. \tag{34}$$

Similar to Lemma 2.1, we can derive the following relationship between the control approximation u_h and the co-state approximation z_h:

$$u_h = \max\{0, \overline{z_h}\} - z_h, \tag{35}$$

where $\overline{z_h} = \frac{\int_0^T \int_\Omega z_h dxdt}{\int_0^T \int_\Omega 1 dxdt}$ denotes the integral average on $\Omega \times J$ of the function z_h.

In the rest of the paper, we shall use some intermediate variables. For any control function $u_h \in K_h$, we define the state solution

$$(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h))$$

satisfies

$$(\alpha\boldsymbol{p}(u_h), \boldsymbol{v}) - (y(u_h), \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ t \in J, \tag{36}$$

$$(y_{tt}(u_h), w) + (\mathrm{div}\boldsymbol{p}(u_h), w) = (f + u_h, w), \quad \forall\, w \in W,\ t \in J, \tag{37}$$

$$y(u_h)(x,0) = y_0(x), \quad \forall\, x \in \Omega, \tag{38}$$

$$y_t(u_h)(x,0) = y_1(x), \quad \forall\, x \in \Omega, \tag{39}$$

$$(\alpha \boldsymbol{q}(u_h), \boldsymbol{v}) - (z(u_h), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(u_h) - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ t \in J, \tag{40}$$

$$(z_{tt}(u_h), w) + (\mathrm{div}\boldsymbol{q}(u_h), w) = (y(u_h) - y_d, w), \quad \forall\, w \in W,\ t \in J, \tag{41}$$

$$z(u_h)(x,T) = 0, \quad \forall\, x \in \Omega, \tag{42}$$

$$z_t(u_h)(x,T) = 0, \quad \forall\, x \in \Omega, \tag{43}$$

where the exact solutions $y(u_h)$ and $z(u_h)$ satisfy the zero boundary condition.

Define the errors as follows:

$$e_y = y(u_h) - y_h, \;\; e_{\boldsymbol{p}} = \boldsymbol{p}(u_h) - \boldsymbol{p}_h,$$
$$e_z = z(u_h) - z_h, \;\; e_{\boldsymbol{q}} = \boldsymbol{q}(u_h) - \boldsymbol{q}_h.$$

Then, from (26)-(27), (30)-(31), (36)-(37) and (40)-(41), the above errors satisfy the following equations

$$(\alpha e_{\boldsymbol{p}}, \boldsymbol{v}) - (e_y, \mathrm{div}\boldsymbol{v}) = -\mathbf{r}_1(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{44}$$

$$(e_{y,tt}, w) + (\mathrm{div} e_{\boldsymbol{p}}, w) = -\mathbf{r}_2(w), \quad \forall\, w \in W, \tag{45}$$

$$(\alpha e_{\boldsymbol{q}}, \boldsymbol{v}) - (e_z, \mathrm{div}\boldsymbol{v}) = -(e_{\boldsymbol{p}}, \boldsymbol{v}) - \mathbf{r}_3(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{46}$$

$$(e_{z,tt}, w) + (\mathrm{div} e_{\boldsymbol{q}}, w) = (e_y, w) - \mathbf{r}_4(w), \quad \forall\, w \in W, \tag{47}$$

where the residuals $\mathbf{r}_1$-$\mathbf{r}_4$ are given as follows:

$$\mathbf{r}_1(\boldsymbol{v}) := (\alpha \boldsymbol{p}_h, \boldsymbol{v}) - (y_h, \mathrm{div}\boldsymbol{v}), \tag{48}$$

$$\mathbf{r}_2(w) := (y_{h,tt}, w) + (\mathrm{div}\boldsymbol{p}_h, w) - (f + u_h, w), \tag{49}$$

$$\mathbf{r}_3(\boldsymbol{v}) := (\alpha \boldsymbol{q}_h, \boldsymbol{v}) + (\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}) - (z_h, \mathrm{div}\boldsymbol{v}), \tag{50}$$

$$\mathbf{r}_4(w) := (z_{h,tt}, w) + (\mathrm{div}\boldsymbol{q}_h, w) - (y_h - y_d, w). \tag{51}$$

We now introduce mixed elliptic reconstructions $\tilde{y}(t), \tilde{z}(t) \in H_0^1(\Omega)$ and $\tilde{\boldsymbol{p}}(t), \tilde{\boldsymbol{q}}(t) \in \boldsymbol{V}$ of y_h, z_h and $\boldsymbol{p}_h, \boldsymbol{q}_h$ for $t \in [0,T]$, respectively.

For given y_h, z_h, $\boldsymbol{p}_h$ and $\boldsymbol{q}_h$, let mixed elliptic reconstructions $\tilde{y}(t), \tilde{z}(t) \in H_0^1(\Omega)$ and $\tilde{\boldsymbol{p}}(t), \tilde{\boldsymbol{q}}(t) \in \boldsymbol{V}$ satisfy

$$(\alpha(\tilde{\boldsymbol{p}} - \boldsymbol{p}_h), \boldsymbol{v}) - (\tilde{y} - y_h, \mathrm{div}\boldsymbol{v}) = -\mathbf{r}_1(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{52}$$

$$(\mathrm{div}(\tilde{\boldsymbol{p}} - \boldsymbol{p}_h), w) = -\mathbf{r}_2(w), \quad \forall\, w \in W, \tag{53}$$

$$(\alpha(\tilde{\boldsymbol{q}} - \boldsymbol{q}_h), \boldsymbol{v}) - (\tilde{z} - z_h, \mathrm{div}\boldsymbol{v}) = -(\tilde{\boldsymbol{p}} - \boldsymbol{p}_h, \boldsymbol{v}) - \mathbf{r}_3(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{54}$$

$$(\mathrm{div}(\tilde{\boldsymbol{q}} - \boldsymbol{q}_h), w) = (\tilde{y} - y_h, w) - \mathbf{r}_4(w), \quad \forall\, w \in W. \tag{55}$$

Since $\mathbf{r}_1(\boldsymbol{v}_h) = \mathbf{r}_3(\boldsymbol{v}_h) = 0\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h$, $\mathbf{r}_2(w_h) = \mathbf{r}_4(w_h) = 0\ \forall\, w_h \in W_h$, we note that y_h and $\boldsymbol{p}_h$ are standard mixed elliptic projection of $\tilde{y}$ and $\tilde{\boldsymbol{p}}$, respectively, z_h and $\boldsymbol{q}_h$ are nonstandard mixed elliptic projection of $\tilde{z}$ and $\tilde{\boldsymbol{q}}$.

Using mixed elliptic reconstructions, we now rewrite:

$$e_{\boldsymbol{p}} = (\tilde{\boldsymbol{p}} - \boldsymbol{p}_h) - (\tilde{\boldsymbol{p}} - \boldsymbol{p}(u_h)) := \eta_{\boldsymbol{p}} - \xi_{\boldsymbol{p}},$$
$$e_y = (\tilde{y} - y_h) - (\tilde{y} - y(u_h)) := \eta_y - \xi_y,$$
$$e_{\boldsymbol{q}} = (\tilde{\boldsymbol{q}} - \boldsymbol{q}_h) - (\tilde{\boldsymbol{q}} - \boldsymbol{q}(u_h)) := \eta_{\boldsymbol{q}} - \xi_{\boldsymbol{q}},$$

$$e_z = (\tilde{z} - z_h) - (\tilde{z} - z(u_h)) := \eta_z - \xi_z.$$

Let $P_h : W \to W_h$ be the orthogonal $L^2(\Omega)$-projection into W_h [5], which satisfies:

$$(P_h w - w, \chi) = 0, \quad w \in W, \ \chi \in W_h, \tag{56}$$

$$\|P_h w - w\|_{0,q} \le C\|w\|_{t,q} h^t, \quad 0 \le t \le k+1, \text{ if } w \in W \cap W^{t,q}(\Omega), \tag{57}$$

$$\|P_h w - w\|_{-r} \le C\|w\|_t h^{r+t}, \quad 0 \le r, \ t \le k+1, \text{ if } w \in H^t(\Omega). \tag{58}$$

Next, recall the Fortin projection (see [13] and [21]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\mathrm{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \quad \forall\, \boldsymbol{q} \in \boldsymbol{V}, \ w_h \in W_h, \tag{59}$$

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\|_{0,q} \le C h^r \|\boldsymbol{q}\|_{r,q}, \quad 1/q < r \le k+1, \ \forall\, \boldsymbol{q} \in \boldsymbol{V} \cap (W^{r,q}(\Omega))^2, \tag{60}$$

$$\|\mathrm{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q})\|_0 \le C h^r \|\mathrm{div}\boldsymbol{q}\|_r, \quad 0 \le r \le k+1, \ \forall\ \mathrm{div}\boldsymbol{q} \in H^r(\Omega). \tag{61}$$

We have the commuting diagram property

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{62}$$

where and after, I denotes identity operator.

3. A posteriori error estimates

In this section we study a posteriori error estimates for the mixed finite element approximation to the hyperbolic optimal control problems.

Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solutions of (12)-(20) and (26)-(34), respectively. We decompose the errors as follows:

$$\begin{aligned}
\boldsymbol{p} - \boldsymbol{p}_h &= \boldsymbol{p} - \boldsymbol{p}(u_h) + \boldsymbol{p}(u_h) - \boldsymbol{p}_h := r_{\boldsymbol{p}} + e_{\boldsymbol{p}}, \\
y - y_h &= y - y(u_h) + y(u_h) - y_h := r_y + e_y, \\
\boldsymbol{q} - \boldsymbol{q}_h &= \boldsymbol{q} - \boldsymbol{q}(u_h) + \boldsymbol{q}(u_h) - \boldsymbol{q}_h := r_{\boldsymbol{q}} + e_{\boldsymbol{q}}, \\
z - z_h &= z - z(u_h) + z(u_h) - z_h := r_z + e_z.
\end{aligned}$$

From (12)-(13), (16)-(17), (36)-(37) and (40)-(41), we derive the error equations:

$$(\alpha r_{\boldsymbol{p}}, \boldsymbol{v}) - (r_y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{63}$$

$$(r_{y,tt}, w) + (\mathrm{div} r_{\boldsymbol{p}}, w) = (u - u_h, w), \quad \forall\, w \in W, \tag{64}$$

$$(\alpha r_{\boldsymbol{q}}, \boldsymbol{v}) - (r_z, \mathrm{div}\boldsymbol{v}) = -(r_{\boldsymbol{p}}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{65}$$

$$(r_{z,tt}, w) + (\mathrm{div} r_{\boldsymbol{q}}, w) = (r_y, w), \quad \forall\, w \in W. \tag{66}$$

Lemma 3.1. *Let $r_{\boldsymbol{p}}$, r_y, $r_{\boldsymbol{q}}$ and r_z satisfy* (63)-(66). *Then there is a constant $C > 0$ independent of h such that*

$$\|r_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))} + \|r_{y,t}\|_{L^\infty(J;L^2(\Omega))} \le C\|u - u_h\|_{L^2(J;L^2(\Omega))}, \tag{67}$$

$$\|r_y\|_{L^\infty(J;L^2(\Omega))} + \|r_z\|_{L^\infty(J;L^2(\Omega))} \le C\|u - u_h\|_{L^2(J;L^2(\Omega))}. \tag{68}$$

Proof. Letting $t = 0$ and $\boldsymbol{v} = r_{\boldsymbol{p}}(0)$ in (63), since $r_y(0) = 0$, consequently we find that $r_{\boldsymbol{p}}(0) = 0$. Differentiate (63) with respect to t, we obtain

$$(\alpha r_{\boldsymbol{p},t}, \boldsymbol{v}) - (r_{y,t}, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}. \tag{69}$$

Setting $\boldsymbol{v} = r_{\boldsymbol{p}}$ and $w = r_{y,t}$ as the test functions and add the two relations of (69) and (64), we have

$$(\alpha r_{\boldsymbol{p},t}, r_{\boldsymbol{p}}) + (r_{y,tt}, r_{y,t}) = (u - u_h, r_{y,t}). \tag{70}$$

Then, using ϵ-Cauchy inequality, we derive

$$\frac{1}{2}\frac{d}{dt}(\|\alpha^{\frac{1}{2}} r_{\boldsymbol{p}}\|^2 + \|r_{y,t}\|^2) \le C\|u - u_h\|^2 + C\|r_{y,t}\|^2. \tag{71}$$

On integrating (71) with respect to time from 0 to t, using the Gronwall's lemma, the assumption on A and $r_{\boldsymbol{p}}(0) = 0$, we have

$$\|r_{y,t}\|_{L^\infty(J;L^2(\Omega))} + \|r_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))} \le C\|u - u_h\|_{L^2(J;L^2(\Omega))}. \tag{72}$$

Since $r_y - r_y(0) = \int_0^t r_{y,s} ds$, using (72), we have

$$\|r_y\| \le C\|r_{y,t}\|_{L^2(J;L^2(\Omega))} \le C\|u - u_h\|_{L^2(J;L^2(\Omega))}. \tag{73}$$

We integrate (66) with respect to time from t to T and use the symbol:

$$\check{\psi}(t) := \int_t^T \psi(s) ds \tag{74}$$

to obtain

$$-(r_{z,t}, w) + (\mathrm{div}\check{r}_{\boldsymbol{q}}, w) = (\check{r}_y, w), \quad \forall\ w \in W. \tag{75}$$

Choose $w = r_z$ in (75) and $\boldsymbol{v} = \check{r}_{\boldsymbol{q}}$ in (65) respectively, then add the resulting equations to get

$$-\frac{1}{2}\frac{d}{dt}(\|r_z\|^2 + \|\alpha^{\frac{1}{2}} \check{r}_{\boldsymbol{q}}\|^2) = (\check{r}_y, r_z) - (r_{\boldsymbol{p}}, \check{r}_{\boldsymbol{q}}). \tag{76}$$

Note that

$$\|\check{r}_y\| \le C\|r_y\|_{L^2(J;L^2(\Omega))}. \tag{77}$$

Integrating (76) with respect to time from t to T, using Cauchy inequality and Gronwall's lemma, we arrive at

$$\|r_z\|_{L^\infty(J;L^2(\Omega))} \le C\|r_y\|_{L^2(J;L^2(\Omega))} + C\|r_{\boldsymbol{p}}\|_{L^2(J;L^2(\Omega))}. \tag{78}$$

By (72), (73) and (78), we derive (67) and (68). $\square$

Now, let us derive the a posteriori error estimates for the control u.

Lemma 3.2. *Let* $(y, \boldsymbol{p}, z, \boldsymbol{q}, u)$ *and* $(y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h, u_h)$ *be the solutions of* (12)-(20) *and* (26)-(34)*, respectively. Assume that* $(u_h + z_h)|_\tau \in H^1(\tau)$ *and that exists* $w \in K_h$ *such that*

$$\left| \int_0^T (u_h + z_h, w - u) dt \right| \le C \int_0^T \sum_\tau h_\tau |u_h + z_h|_{H^1(\tau)} \|u - u_h\|_{L^2(\tau)} dt.$$

Then we have

$$\|u-u_h\|_{L^2(J;L^2(\Omega))} \le C\eta_1 + C\|z_h - z(u_h)\|_{L^2(J;L^2(\Omega))}, \tag{79}$$

where

$$\eta_1 = \left(\int_0^T \sum_\tau h_\tau^2 |u_h + z_h|^2_{H^1(\tau)} dt\right)^{\frac{1}{2}}.$$

Proof. It follows from (20) and (34) that

$$\begin{aligned}
\|u-u_h\|^2_{L^2(J;L^2(\Omega))} &= \int_0^T (u-u_h, u-u_h)dt \\
&= \int_0^T (u+z, u-u_h)dt + \int_0^T (u_h+z_h, u_h-u)dt \\
&\quad + \int_0^T (z_h - z(u_h), u-u_h)dt + \int_0^T (z(u_h)-z, u-u_h)dt \\
&\le \int_0^T (u_h+z_h, w-u)dt + \int_0^T (z_h - z(u_h), u-u_h)dt \\
&\quad + \int_0^T (z(u_h)-z, u-u_h)dt \\
&=: I_1 + I_2 + I_3. \qquad (80)
\end{aligned}$$

From the assumption above, it easy to see that

$$\begin{aligned}
I_1 &= \int_0^T (u_h+z_h, w-u)dt \\
&\le C(\delta)\eta_1^2 + \delta\|u-u_h\|^2_{L^2(J;L^2(\Omega))}, \qquad (81)
\end{aligned}$$

where δ is an arbitrary small positive number, $C(\delta)$ is dependent on δ^{-1}. Moreover, it is clear that

$$\begin{aligned}
I_2 &= \int_0^T (z_h - z(u_h), u-u_h)dt \\
&\le C(\delta)\|z_h - z(u_h)\|^2_{L^2(J;L^2(\Omega))} + \delta\|u-u_h\|^2_{L^2(J;L^2(\Omega))}. \qquad (82)
\end{aligned}$$

Now we turn to I_3. Note that

$$y(x,0) - y(u_h)(x,0) = y_t(x,0) - y_t(u_h)(x,0) = 0$$

and

$$z(x,T) - z(u_h)(x,T) = z_t(x,T) - z_t(u_h)(x,T) = 0.$$

Then from (12)-(13), (16)-(17), (36)-(37) and (40)-(41), we have

$$\begin{aligned}
I_3 &= \int_0^T (u-u_h, z(u_h)-z)dt \\
&= \int_0^T \Big(((y-y(u_h))_{tt}, z(u_h)-z) + (\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}(u_h)), z(u_h)-z)\Big)dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \frac{d}{dt}((y-y(u_h))_t, z(u_h)-z)dt - \int_0^T ((y-y(u_h))_t, (z(u_h)-z)_t)dt \\
&\quad + \int_0^T (\alpha(\boldsymbol{q}(u_h)-\boldsymbol{q}), \boldsymbol{p}-\boldsymbol{p}(u_h))dt + \int_0^T (\boldsymbol{p}(u_h)-\boldsymbol{p}, \boldsymbol{p}-\boldsymbol{p}(u_h))dt \\
&= \int_0^T \frac{d}{dt}((z(u_h)-z)_t, y-y(u_h))dt - \int_0^T ((z(u_h)-z)_t, (y-y(u_h))_t)dt \\
&\quad + \int_0^T (\alpha(\boldsymbol{p}-\boldsymbol{p}(u_h)), \boldsymbol{q}(u_h)-\boldsymbol{q})dt + \int_0^T (\boldsymbol{p}(u_h)-\boldsymbol{p}, \boldsymbol{p}-\boldsymbol{p}(u_h))dt \\
&= \int_0^T \Big(((z(u_h)-z)_{tt}, y-y(u_h)) + (y-y(u_h), \operatorname{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}))\Big)dt \\
&\quad + \int_0^T (\boldsymbol{p}(u_h)-\boldsymbol{p}, \boldsymbol{p}-\boldsymbol{p}(u_h))dt \\
&= \int_0^T \Big((\boldsymbol{p}(u_h)-\boldsymbol{p}, \boldsymbol{p}-\boldsymbol{p}(u_h)) + (y(u_h)-y, y-y(u_h))\Big)dt \\
&\leq 0. \qquad (83)
\end{aligned}
$$

Thus, we obtain from (80)-(83) that which proves (79). □

Remark 3.1. Let w in Lemma 3.2 be such that $w = \pi^c u$, where

$$\pi^c v(t)|_{x\in\tau} = \int_\tau v(x,t)dx/|\tau|, \quad \forall\ v \in U,$$

where $|\tau|$ is the measure of the element τ. Then it follows that $w \in K_h$ and

$$
\begin{aligned}
&\left|\int_0^T (u_h + z_h, w-u)dt\right| \\
&= \left|\int_0^T (u_h + z_h, \pi^c u - u)dt\right| \\
&= \left|\int_0^T (u_h + z_h - \pi^c(u_h+z_h), \pi^c(u-u_h) - (u-u_h))dt\right| \\
&\leq C\int_0^T \sum_\tau h_\tau |u_h + z_h|_{H^1(\tau)} \|u-u_h\|_{L^2(\tau)} dt.
\end{aligned}
$$

Hence, the assumption in Lemma 3.2 is satisfied.

From the equations (52)-(55), we can see that:

Lemma 3.3. *Let mixed elliptic reconstructions $\tilde{y}$, $\tilde{\boldsymbol{p}}$, $\tilde{z}$ and $\tilde{\boldsymbol{q}}$ satisfy* (52)-(55). *Then the following properties hold true*:

$$\alpha\tilde{\boldsymbol{p}} = -\nabla\tilde{y}, \quad \alpha\tilde{\boldsymbol{q}} + \tilde{\boldsymbol{p}} - \boldsymbol{p}_d = -\nabla\tilde{z}. \qquad (84)$$

Using (52)-(55) in (44)-(47), we derive the error equations:

$$(\alpha\xi_{\boldsymbol{p}}, \boldsymbol{v}) - (\xi_y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{85}$$

$$(\xi_{y,tt}, w) + (\mathrm{div}\xi_{\boldsymbol{p}}, w) = (\eta_{y,tt}, w), \quad \forall\ w \in W, \tag{86}$$

$$(\alpha\xi_{\boldsymbol{q}}, \boldsymbol{v}) - (\xi_z, \mathrm{div}\boldsymbol{v}) = -(\xi_{\boldsymbol{p}}, \boldsymbol{v}), \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{87}$$

$$(\xi_{z,tt}, w) + (\mathrm{div}\xi_{\boldsymbol{q}}, w) = (\xi_y, w) + (\eta_{z,tt}, w), \quad \forall\ w \in W. \tag{88}$$

Lemma 3.4. *Let* ξ_y *and* $\xi_{\boldsymbol{p}}$ *satisfy* (85)-(86). *Then we have the following estimates*:

$$\begin{aligned}
&\|\xi_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{y,t}\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|\eta_{y,t}(0)\| + \|\eta_{y,tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_{\boldsymbol{p}}(0)\| \\
&\quad + \|y_1 - y_1^h\| + \|A\nabla y_0 + \boldsymbol{p}_h(0)\|),
\end{aligned} \tag{89}$$

$$\begin{aligned}
&\|\xi_y\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|y_0 - y_0^h\| + \|y_1 - y_1^h\| + \|\eta_y(0)\| + \|\eta_{y,t}\|_{L^2(J;L^2(\Omega))}),
\end{aligned} \tag{90}$$

$$\begin{aligned}
&\|\xi_{\boldsymbol{p},t}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{y,tt}\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|\mathrm{div}(A\nabla y_0) + \mathrm{div}\boldsymbol{p}_h(0)\| + \|\mathrm{div}\eta_{\boldsymbol{p}}(0)\| \\
&\quad + \|A\nabla y_1 + \boldsymbol{p}_{h,t}(0)\| + \|\eta_{\boldsymbol{p},t}(0)\| \\
&\quad + \|\eta_{y,tt}(0)\| + \|\eta_{y,ttt}\|_{L^2(J;L^2(\Omega))}).
\end{aligned} \tag{91}$$

Proof. Firstly, we differentiate the equation (85) with respect to t, and obtain

$$(\alpha\xi_{\boldsymbol{p},t}, \boldsymbol{v}) - (\xi_{y,t}, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}. \tag{92}$$

Choose $\boldsymbol{v} = \xi_{\boldsymbol{p}}$ and $w = \xi_{y,t}$ as the test functions and add the two relations of (92) and (86). Then, using ϵ-Cauchy inequality, we derive

$$\frac{1}{2}\frac{d}{dt}(\|\alpha^{\frac{1}{2}}\xi_{\boldsymbol{p}}\|^2 + \|\xi_{y,t}\|^2) \leq \frac{1}{2}\|\eta_{y,tt}\|^2 + \frac{1}{2}\|\xi_{y,t}\|^2. \tag{93}$$

On integrating (93) with respect to time from 0 to t, using the assumption on A, we find that

$$\begin{aligned}
\|\xi_{y,t}\|^2 + \|\xi_{\boldsymbol{p}}\|^2 \leq C\int_0^t \|\eta_{y,tt}\|^2 ds + C\int_0^t \|\xi_{y,t}\|^2 ds \\
+ C\|\xi_{y,t}(0)\|^2 + C\|\xi_{\boldsymbol{p}}(0)\|^2.
\end{aligned} \tag{94}$$

Applying the Gronwall's lemma to (94), we get

$$\begin{aligned}
&\|\xi_{y,t}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|\xi_{y,t}(0)\| + \|\xi_{\boldsymbol{p}}(0)\| + \|\eta_{y,tt}\|_{L^2(J;L^2(\Omega))}).
\end{aligned} \tag{95}$$

Note that

$$\begin{aligned}
\|\xi_{\boldsymbol{p}}(0)\| &\leq \|\boldsymbol{p}(u_h)(0) - \boldsymbol{p}_h(0)\| + \|\eta_{\boldsymbol{p}}(0)\| \\
&\leq \|A\nabla y_0 + \boldsymbol{p}_h(0)\| + \|\eta_{\boldsymbol{p}}(0)\|.
\end{aligned} \tag{96}$$

Integrate (86) with respect to time from 0 to t and use the symbol:

$$\hat{\phi}(t) := \int_0^t \phi(s)ds \tag{97}$$

to find that

$$(\xi_{y,t}, w) + (\text{div}\hat{\xi}_{\boldsymbol{p}}, w) = (y_1^h - y_1, w) + (\eta_{y,t}, w), \quad \forall\, w \in W. \tag{98}$$

Set $w = \eta_y$ in (98) and $\boldsymbol{v} = \hat{\xi}_{\boldsymbol{p}}$ in (85). Then add the resulting equations and use Cauchy-Schwarz and Young's inequalities to obtain

$$\frac{d}{dt}(\|\xi_y\|^2 + \|\alpha^{\frac{1}{2}}\hat{\xi}_{\boldsymbol{p}}\|^2) \leq C(\|y_1^h - y_1\|^2 + \|\eta_{y,t}\|^2 + \|\xi_y\|^2). \tag{99}$$

Integrating with respect to time from 0 to t, we arrive at

$$\|\xi_y\| \leq C(\|\xi_y(0)\| + \|y_1 - y_1^h\| + \|\eta_{y,t}\|_{L^2(J;L^2(\Omega))}). \tag{100}$$

Let $t = 0$ and $w = \xi_{y,tt}(0)$ in (86), we can derive

$$\|\xi_{y,tt}(0)\| \leq C(\|\text{div}\xi_{\boldsymbol{p}}(0)\| + \|\eta_{y,tt}(0)\|). \tag{101}$$

Differentiating the equations (85) and (86) respect to t, we get

$$(\alpha\xi_{\boldsymbol{p},tt}, \boldsymbol{v}) - (\xi_{y,tt}, \text{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{102}$$

$$(\xi_{y,ttt}, w) + (\text{div}\xi_{\boldsymbol{p},t}, w) = (\eta_{y,ttt}, w), \quad \forall\, w \in W. \tag{103}$$

Choosing $\boldsymbol{v} = \xi_{\boldsymbol{p},t}$ and $w = \xi_{y,tt}$ as the test functions and add the two relations of (102)-(103), using Cauchy inequality, we find that

$$\frac{1}{2}\frac{d}{dt}(\|\alpha^{\frac{1}{2}}\xi_{\boldsymbol{p},t}\|^2 + \|\xi_{y,tt}\|^2) \leq \frac{1}{2}\|\eta_{y,ttt}\|^2 + \frac{1}{2}\|\xi_{y,tt}\|^2. \tag{104}$$

Integrating (104) with respect to time from 0 to t, using Gronwall's lemma, we arrive at

$$\begin{aligned} &\|\xi_{\boldsymbol{p},t}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{y,tt}\|_{L^\infty(J;L^2(\Omega))} \\ &\leq C(\|\xi_{y,tt}(0) + \|\xi_{\boldsymbol{p},t}(0)\| + \|\eta_{y,ttt}\|_{L^2(J;L^2(\Omega))}). \end{aligned} \tag{105}$$

Note that

$$\begin{aligned} \|\text{div}\xi_{\boldsymbol{p}}(0)\| &\leq \|\text{div}(\boldsymbol{p}(u_h)(0)) - \text{div}\boldsymbol{p}_h(0)\| + \|\text{div}\eta_{\boldsymbol{p}}(0)\| \\ &\leq \|\text{div}(A\nabla y_0) + \text{div}\boldsymbol{p}_h(0)\| + \|\text{div}\eta_{\boldsymbol{p}}(0)\| \end{aligned} \tag{106}$$

and

$$\begin{aligned} \|\xi_{\boldsymbol{p},t}(0)\| &\leq \|\boldsymbol{p}_t(u_h)(0) - \boldsymbol{p}_{h,t}(0)\| + \|\eta_{\boldsymbol{p},t}(0)\| \\ &\leq \|A\nabla y_1 + \boldsymbol{p}_{h,t}(0)\| + \|\eta_{\boldsymbol{p},t}(0)\|. \end{aligned} \tag{107}$$

Combining (95)-(96), (100)-(101) with (105)-(107), we complete the proof. $\square$

Lemma 3.5. *Let* ξ_z *and* $\xi_{\boldsymbol{q}}$ *satisfy* (87)-(88). *Then we have the following estimates*:

$$\begin{aligned}
&\|\xi_{\boldsymbol{q}}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{z,t}\|_{L^\infty(J;L^2(\Omega))}\\
&\leq C(\|\xi_{z,t}(T)\| + \|A\boldsymbol{p}_h(T) - A\boldsymbol{p}_d(T) + \boldsymbol{q}_h(T)\|\\
&\quad + \|\xi_{\boldsymbol{p}}(T)\| + \|\eta_{\boldsymbol{p}}(T)\| + \|\xi_{\boldsymbol{p},t}\|_{L^2(J;L^2(\Omega))}\\
&\quad + \|\xi_y\|_{L^2(J;L^2(\Omega))} + \|\eta_{z,tt}\|_{L^2(J;L^2(\Omega))}),
\end{aligned}\tag{108}$$

$$\begin{aligned}
&\|\xi_z\|_{L^\infty(J;L^2(\Omega))}\\
&\leq C(\|\xi_{\boldsymbol{p}}\|_{L^2(J;L^2(\Omega))} + \|\eta_{z,t}\|_{L^2(J;L^2(\Omega))}\\
&\quad + \|\eta_z(T)\| + \|\xi_y\|_{L^2(J;L^2(\Omega))}).
\end{aligned}\tag{109}$$

Proof. We differentiate the equation (87) with respect to t, and obtain

$$(\alpha\xi_{\boldsymbol{q},t}, \boldsymbol{v}) - (\xi_{z,t}, \mathrm{div}\boldsymbol{v}) = -(\xi_{\boldsymbol{p},t}, \boldsymbol{v}), \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}.\tag{110}$$

Choose $\boldsymbol{v} = -\xi_{\boldsymbol{q}}$ and $w = -\xi_{z,t}$ as the test functions and add the two relations of (110) and (88). Then, using ϵ-Cauchy inequality, we derive

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}(\|\alpha^{\frac{1}{2}}\xi_{\boldsymbol{q}}\|^2 + \|\xi_{z,t}\|^2) &\leq C(\|\eta_{z,tt}\|^2 + \|\xi_{z,t}\|^2 + \|\xi_{\boldsymbol{q}}\|^2\\
&\quad + \|\xi_{\boldsymbol{q}}\|^2 + \|\xi_{\boldsymbol{p},t}\|^2 + \|\xi_y\|^2).
\end{aligned}\tag{111}$$

On integrating (111) with respect to time from t to T, using the assumption on A, we find that

$$\begin{aligned}
\|\xi_{z,t}\|^2 + \|\xi_{\boldsymbol{q}}\|^2 &\leq C\int_t^T (\|\xi_{\boldsymbol{p},t}\|^2 + \|\xi_{\boldsymbol{q}}\|^2 + \|\xi_y\|^2 + \|\eta_{z,tt}\|^2 + \|\xi_{z,t}\|^2)ds\\
&\quad + C(\|\xi_{z,t}(T)\|^2 + \|\xi_{\boldsymbol{q}}(T)\|^2).
\end{aligned}\tag{112}$$

Applying the Gronwall's lemma to (112), we get

$$\begin{aligned}
&\|\xi_{y,t}\|_{L^\infty(J;L^2(\Omega))} + \|\xi_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))}\\
&\leq C(\|\xi_{z,t}(T)\| + \|\xi_{\boldsymbol{q}}(T)\| + \|\eta_{z,tt}\|_{L^2(J;L^2(\Omega))}\\
&\quad + \|\eta_{\boldsymbol{p},t}\|_{L^2(J;L^2(\Omega))} + \|\eta_y\|_{L^2(J;L^2(\Omega))}).
\end{aligned}\tag{113}$$

Observe that

$$\begin{aligned}
\|\xi_{\boldsymbol{q}}(T)\| &\leq \|\boldsymbol{q}(u_h)(T) - \boldsymbol{q}_h(T)\| + \|\eta_{\boldsymbol{q}}(T)\|\\
&\leq \|A(\boldsymbol{p}_h(T) - \boldsymbol{p}_d(T)) + \boldsymbol{q}_h(T)\| + \|\eta_{\boldsymbol{q}}(T)\|\\
&\quad + C(\|\xi_{\boldsymbol{p}}(T)\| + \|\eta_{\boldsymbol{p}}(T)\|).
\end{aligned}\tag{114}$$

Integrate (88) with respect to time from t to T and use the symbol (74) to get

$$-(\xi_{z,t}, w) + (\mathrm{div}\check{\xi}_{\boldsymbol{q}}, w) = (\check{\xi}_y, w) - (\eta_{z,t}, w), \quad \forall\ w \in W.\tag{115}$$

Set $w=\eta_z$ in (115) and $\boldsymbol{v}=\check{\xi}_{\boldsymbol{q}}$ in (87). Then add the resulting equations to obtain

$$-\frac{1}{2}\frac{d}{dt}(\|\xi_z\|^2+\|\alpha^{\frac{1}{2}}\check{\xi}_{\boldsymbol{q}}\|^2)=(\check{\xi}_y,\xi_z)-(\xi_{\boldsymbol{p}},\check{\xi}_{\boldsymbol{q}})-(\eta_{z,t},\xi_z). \tag{116}$$

Integrating with respect to time from t to T, similar to (78), we get

$$\begin{aligned}\|\xi_z\| \le\ & C(\|\xi_y\|_{L^2(J;L^2(\Omega))}+\|\xi_{\boldsymbol{p}}\|_{L^2(J;L^2(\Omega))}\\ &+\|\xi_z(T)\|+\|\eta_{z,t}\|_{L^2(J;L^2(\Omega))}).\end{aligned} \tag{117}$$

By use of (113), (114) and (117), we derive (108) and (109). □

From (52)-(55), we derive the error equations:

$$(\alpha\eta_{\boldsymbol{p}},\boldsymbol{v}_h)-(\eta_y,\mathrm{div}\boldsymbol{v}_h)=0,\quad \forall\ \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{118}$$

$$(\mathrm{div}\eta_{\boldsymbol{p}},w_h)=0,\quad \forall\ w_h\in W_h, \tag{119}$$

$$(\alpha\eta_{\boldsymbol{q}},\boldsymbol{v}_h)-(\eta_z,\mathrm{div}\boldsymbol{v}_h)=-(\eta_{\boldsymbol{p}},\boldsymbol{v}_h),\quad \forall\ \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{120}$$

$$(\mathrm{div}\eta_{\boldsymbol{q}},w_h)=(\eta_y,w_h),\quad \forall\ w_h\in W_h. \tag{121}$$

To prove the main theorem, we need the following a posteriori estimates of η_y, $\eta_{y,t}$, $\eta_{y,tt}$, $\eta_{y,ttt}$, $\eta_{\boldsymbol{p}}$, $\eta_{\boldsymbol{p},t}$, $\eta_{\boldsymbol{p},tt}$, $\mathrm{div}\eta_{\boldsymbol{p}}$, η_z, $\eta_{z,t}$, $\eta_{z,tt}$ and $\eta_{\boldsymbol{q}}$ related to the mixed elliptic reconstructions (52)-(55).

Lemma 3.6. *For Raviart-Thomas elements, there exists a positive constant C which depends only on the coefficient matrix A, the domain Ω, the shape regularity of the elements and polynomial degree k such that*

$$\begin{aligned}\|\eta_y\|^2\le C\bigg(&\|h^{1+\min\{1,k\}}(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h-f-u_h)\|^2\\ &+\min_{w_h\in W_h}\|h(\alpha\boldsymbol{p}_h-\nabla_h w_h)\|^2\bigg),\end{aligned} \tag{122}$$

$$\begin{aligned}\|\eta_{y,t}\|^2\le C\bigg(&\|h^{1+\min\{1,k\}}(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h-f-u_h)_t\|^2\\ &+\min_{w_h\in W_h}\|h(\alpha\boldsymbol{p}_{h,t}-\nabla_h w_h)\|^2\bigg),\end{aligned} \tag{123}$$

$$\begin{aligned}\|\eta_{y,tt}\|^2\le C\bigg(&\|h^{1+\min\{1,k\}}(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h-f-u_h)_{tt}\|^2\\ &+\min_{w_h\in W_h}\|h(\alpha\boldsymbol{p}_{h,tt}-\nabla_h w_h)\|^2\bigg),\end{aligned} \tag{124}$$

$$\begin{aligned}\|\eta_{y,ttt}\|^2\le C\bigg(&\|h^{1+\min\{1,k\}}(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h-f-u_h)_{ttt}\|^2\\ &+\min_{w_h\in W_h}\|h(\alpha\boldsymbol{p}_{h,ttt}-\nabla_h w_h)\|^2\bigg),\end{aligned} \tag{125}$$

$$\|\eta_{\boldsymbol{p}}\|^2 \leq C\bigg(\|h^{\frac{1}{2}}J(\alpha\boldsymbol{p}_h\cdot\boldsymbol{t})\|^2_{0,\Gamma_h} + \|h\ \mathrm{curl}_h(\alpha\boldsymbol{p}_h)\|^2 + \|h(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h - f - u_h)\|^2\bigg), \tag{126}$$

$$\|\eta_{\boldsymbol{p},t}\|^2 \leq C\bigg(\|h^{\frac{1}{2}}J(\alpha\boldsymbol{p}_{h,t}\cdot\boldsymbol{t})\|^2_{0,\Gamma_h} + \|h\ \mathrm{curl}_h(\alpha\boldsymbol{p}_{h,t})\|^2 + \|h(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h - f - u_h)_t\|^2\bigg), \tag{127}$$

$$\|\eta_{\boldsymbol{p},tt}\|^2 \leq C\bigg(\|h^{\frac{1}{2}}J(\alpha\boldsymbol{p}_{h,tt}\cdot\boldsymbol{t})\|^2_{0,\Gamma_h} + \|h\ \mathrm{curl}_h(\alpha\boldsymbol{p}_{h,tt})\|^2 + \|h(y_{h,tt}+\mathrm{div}\boldsymbol{p}_h - f - u_h)_{tt}\|^2\bigg), \tag{128}$$

$$\|\mathrm{div}\eta_{\boldsymbol{p}}\|^2 \leq C\|y_{h,tt}+\mathrm{div}\boldsymbol{p}_h - f - u_h\|^2, \tag{129}$$

$$\|\eta_z\|^2 \leq C\bigg(\|h^{1+\min\{1,k\}}(z_{h,tt}+\mathrm{div}\boldsymbol{q}_h - y_h + y_d)\|^2 + \|\eta_y\|^2 + \|\eta_{\boldsymbol{p}}\|^2 + \min_{w_h\in W_h}\|h(\alpha\boldsymbol{q}_h+\boldsymbol{p}_h-\boldsymbol{p}_d-\nabla_h w_h)\|^2\bigg), \tag{130}$$

$$\|\eta_{z,t}\|^2 \leq C\bigg(\|h^{1+\min\{1,k\}}(z_{h,tt}+\mathrm{div}\boldsymbol{q}_h - y_h + y_d)_t\|^2 + \|\eta_{y,t}\|^2 + \|\eta_{\boldsymbol{p},t}\|^2 + \min_{w_h\in W_h}\|h(\alpha\boldsymbol{q}_{h,t}+\boldsymbol{p}_{h,t}-\boldsymbol{p}_{d,t}-\nabla_h w_h)\|^2\bigg), \tag{131}$$

$$\|\eta_{z,tt}\|^2 \leq C\bigg(\|h^{1+\min\{1,k\}}(z_{h,tt}+\mathrm{div}\boldsymbol{q}_h - y_h + y_d)_{tt}\|^2 + \|\eta_{y,tt}\|^2 + \|\eta_{\boldsymbol{p},tt}\|^2 + \min_{w_h\in W_h}\|h(\alpha\boldsymbol{q}_{h,tt}+\boldsymbol{p}_{h,tt}-\boldsymbol{p}_{d,tt}-\nabla_h w_h)\|^2\bigg), \tag{132}$$

$$\|\eta_{\boldsymbol{q}}\|^2 \leq C\bigg(\|h(z_{h,tt}+\mathrm{div}\boldsymbol{q}_h - y_h + y_d)\|^2 + \|h\ \mathrm{curl}_h(\alpha\boldsymbol{q}_h+\boldsymbol{p}_h-\boldsymbol{p}_d)\|^2 + \|\eta_y\|^2 + \|\eta_{\boldsymbol{p}}\|^2 + \|h^{\frac{1}{2}}J((\alpha\boldsymbol{q}_h+\boldsymbol{p}_h-\boldsymbol{p}_d)\cdot\boldsymbol{t})\|^2_{0,\Gamma_h}\bigg), \tag{133}$$

where $J(\boldsymbol{v}\cdot\boldsymbol{t})$ denotes the jump of $\boldsymbol{v}\cdot\boldsymbol{t}$ across element edge E for all $\boldsymbol{v}\in\boldsymbol{V}$ with $\boldsymbol{t}$ being the tangential unit vector along the edge $E\in\Gamma_h$.

Proof. Based on the tools developed in [15, 18], it is straight forward to derive a posteriori error estimates for η_y, $\eta_{y,t}$, $\eta_{y,tt}$, $\eta_{y,ttt}$, $\eta_{\boldsymbol{p}}$, $\eta_{\boldsymbol{p},t}$, $\eta_{\boldsymbol{p},tt}$, $\mathrm{div}\eta_{\boldsymbol{p}}$, η_z, $\eta_{z,t}$, $\eta_{z,tt}$ and $\eta_{\boldsymbol{q}}$. Here we only discuss the proof of L^2-norm estimate η_z. Now, we appeal to Aubin-Nitsche duality arguments. Thus, we consider $\Phi\in H_0^1(\Omega)\cap H^2(\Omega)$ as the solution of the elliptic problem:

$$-\mathrm{div}(A\nabla\Phi) = \Psi, \quad \text{in } \Omega, \tag{134}$$

which satisfies the following elliptic regularity result

$$\|\Phi\|_2 \leq C\|\Psi\|. \tag{135}$$

By using (134) and the definition of Π_h, integrating by parts appropriately, and the property (84), we obtain

$$\begin{aligned}
(\eta_z, \Psi) &= (\eta_z, -\mathrm{div}(A\nabla\Phi)) \\
&= (\tilde{z}, -\mathrm{div}(A\nabla\Phi)) + (z_h, \mathrm{div}(A\nabla\Phi)) \\
&= (z_h, \mathrm{div}(\Pi_h(A\nabla\Phi))) + (A\nabla\tilde{z}, \nabla\Phi) \\
&= -(\tilde{\boldsymbol{q}} + A\tilde{\boldsymbol{p}} - A\boldsymbol{p}_d, \nabla\Phi) + (z_h, \mathrm{div}(\Pi_h(A\nabla\Phi))) \\
&= -(\eta_{\boldsymbol{q}}, \nabla\Phi) - (A\eta_{\boldsymbol{p}}, \nabla\Phi) + (z_h, \mathrm{div}(\Pi_h(A\nabla\Phi))) \\
&\quad - (\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d, A\nabla\Phi).
\end{aligned}$$

Using (120), integrating by parts and

$$(\nabla_h w_h, (I - \Pi_h)(A\nabla\Phi)) = 0, \tag{136}$$

we now arrive at

$$\begin{aligned}
(\eta_z, \Psi) &= (\mathrm{div}\eta_{\boldsymbol{q}}, \Phi - P_h\Phi) + (\eta_y, P_h\Phi) - (A\eta_{\boldsymbol{p}}, \nabla\Phi) \\
&\quad - (\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h, (I - \Pi_h)(A\nabla\Phi)) \\
&= -(z_{h,tt} + \mathrm{div}\boldsymbol{q}_h - y_h + y_d, \Phi - P_h\Phi) + (\eta_y, \Phi) - (A\eta_{\boldsymbol{p}}, \nabla\Phi) \\
&\quad - (\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h, (I - \Pi_h)(A\nabla\Phi)) \\
&\leq C\Big(\|h^{1+\min\{1,k\}}(z_{h,tt} + \mathrm{div}\boldsymbol{q}_h - y_h + y_d)\|\|\Phi\|_2 + \|\eta_y\|\|\Phi\| \\
&\quad + \|A\eta_{\boldsymbol{p}}\|\|\nabla\Phi\| + \|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h)\|\|A\nabla\Phi\|_1\Big) \\
&\leq C\Big(\|h^{1+\min\{1,k\}}(z_{h,tt} + \mathrm{div}\boldsymbol{q}_h - y_h + y_d)\| + \|\eta_y\| + \|\eta_{\boldsymbol{p}}\| \\
&\quad + \|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h)\|\Big)\|\Phi\|_2.
\end{aligned} \tag{137}$$

Using elliptic regularity (135) in (137), we obtain

$$\begin{aligned}
\frac{(\eta_z, \Psi)}{\|\Psi\|} &\leq C\Big(\|h^{1+\min\{1,k\}}(z_{h,tt} + \mathrm{div}\boldsymbol{q}_h - y_h + y_d)\| + \|\eta_y\| + \|\eta_{\boldsymbol{p}}\| \\
&\quad + \min_{w_h \in W_h} \|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h)\|\Big).
\end{aligned} \tag{138}$$

Now, taking supremum over Ψ, we obtain estimate (130). □

Remark 3.2. In (130), we can replace $\min_{w_h \in W_h} \|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d - \nabla_h w_h)\|$ by $\|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d + \nabla_h z_h)\|$ or by $\|h(\alpha\boldsymbol{q}_h + \boldsymbol{p}_h - \boldsymbol{p}_d + \nabla_h I_h(z_h))\|$, where $I_h(z_h)$ is an improved version of z_h, which is obtained by post processing z_h. Similar places can be found in (122)-(125) and (131)-(132).

Collecting Lemmas 3.1-3.6, we finally derive the following main results:

Theorem 3.1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solutions of* (12)-(20) *and* (26)-(34)*, respectively. Then the following a posteriori estimates hold for $t \in [0, T]$:*

$$\begin{aligned}
&\|u - u_h\|_{L^2(J;L^2(\Omega))} \\
&\leq C\Big(\eta_1 + \|\eta_z(T)\| + \|\eta_{z,t}\|_{L^2(J;L^2(\Omega))} + \|\eta_z\|_{L^2(J;L^2(\Omega))} + \|\eta_{\boldsymbol{p}}(0)\| \\
&\quad + \|y_0 - y_0^h\| + \|y_1 - y_1^h\| + \|\eta_y(0)\| + \|\eta_{y,t}\|_{L^2(J;L^2(\Omega))} \\
&\quad + \|A\nabla y_0 + \boldsymbol{p}_h(0)\| + \|\eta_{y,t}(0)\| + \|\eta_{y,tt}\|_{L^2(J;L^2(\Omega))}\Big), \qquad (139)
\end{aligned}$$

$$\begin{aligned}
&\|y - y_h\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|u - u_h\|_{L^2(J;L^2(\Omega))} + \|\eta_y\|_{L^\infty(J;L^2(\Omega))}), \qquad (140)
\end{aligned}$$

$$\begin{aligned}
&\|\boldsymbol{p} - \boldsymbol{p}_h\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|u - u_h\|_{L^2(J;L^2(\Omega))} + \|\eta_{\boldsymbol{p}}\|_{L^\infty(J;L^2(\Omega))}), \qquad (141)
\end{aligned}$$

$$\begin{aligned}
&\|z - z_h\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C(\|u - u_h\|_{L^2(J;L^2(\Omega))} + \|\eta_z\|_{L^\infty(J;L^2(\Omega))}), \qquad (142)
\end{aligned}$$

where η_1 is defined in Lemma 3.2 and the estimates for η_y, $\eta_{y,t}$, $\eta_{y,tt}$, $\eta_{\boldsymbol{p}}$, η_z and $\eta_{z,t}$ are given in Lemma 3.6.

Theorem 3.2. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solutions of* (12)-(20) *and* (26)-(34)*, respectively. Then there is a constant $C > 0$ independent of h such that*

$$\|u - u_h\|_{L^\infty(J;L^2(\Omega))} \leq C\|z - z_h\|_{L^\infty(J;L^2(\Omega))}, \qquad (143)$$

$$\begin{aligned}
&\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^\infty(J;L^2(\Omega))} \\
&\leq C\Big(\|z - z_h\|_{L^\infty(J;L^2(\Omega))} + \|\eta_{z,tt}\|_{L^2(J;L^2(\Omega))} + \|\eta_{z,t}(T)\| \\
&\quad + \|A\boldsymbol{p}_h(T) - A\boldsymbol{p}_d(T) + \boldsymbol{q}_h(T)\| + \|\eta_{z,t}\|_{L^2(J;L^2(\Omega))} \\
&\quad + \|\eta_{\boldsymbol{p}}(T)\| + \|\xi_{\boldsymbol{q}}\|_{L^\infty(J;L^2(\Omega))} + \|\mathrm{div}\eta_{\boldsymbol{p}}(0)\| \\
&\quad + \|\mathrm{div}(A\nabla y_0) + \mathrm{div}\boldsymbol{p}_h(0)\| + \|A\nabla y_1 + \boldsymbol{p}_{h,t}(0)\| \\
&\quad + \|\eta_{\boldsymbol{p},t}(0)\| + \|\eta_{y,tt}(0)\| + \|\eta_{y,ttt}\|_{L^2(J;L^2(\Omega))}\Big), \qquad (144)
\end{aligned}$$

where η_1 is defined in Lemma 3.2 and the estimates for η_y, $\eta_{y,t}$, $\eta_{y,tt}$, $\eta_{y,ttt}$, $\eta_{\boldsymbol{p}}$, $\eta_{\boldsymbol{p},t}$, $\mathrm{div}\eta_{\boldsymbol{p}}$, η_z, $\eta_{z,t}$, $\eta_{z,tt}$ and $\eta_{\boldsymbol{q}}$ are given in Lemma 3.6.

Proof. From Lemma 2.1 and (35), we have

$$\|u - u_h\|_{L^\infty(J;L^2(\Omega))} \leq C\|z - z_h\|_{L^\infty(J;L^2(\Omega))}, \qquad (145)$$

$$\|(u - u_h)_t\|_{L^2(J;L^2(\Omega))} \leq C\|(z - z_h)_t\|_{L^2(J;L^2(\Omega))}. \qquad (146)$$

Differentiating the equations (63) and (64) with respect to t, we have

$$(\alpha r_{\boldsymbol{p},tt},\boldsymbol{v})-(r_{y,tt},\mathrm{div}\boldsymbol{v})=0,\quad\forall\,\boldsymbol{v}\in\boldsymbol{V},\tag{147}$$

$$(r_{y,ttt},w)+(\mathrm{div}r_{\boldsymbol{p},t},w)=((u-u_h)_t,w),\quad\forall\,w\in W.\tag{148}$$

Since $r_{y,t}(0)=0$, let $t=0$ and $\boldsymbol{v}=r_{\boldsymbol{p},t}(0)$ in (69), we find that $r_{\boldsymbol{p},t}(0)=0$. Moreover, we have $\mathrm{div}r_{\boldsymbol{p}}(0)=0$. Set $t=0$ and $w=r_{y,tt}(0)$ in (64), we derive

$$\|r_{y,tt}(0)\|\le\|(u-u_h)(0)\|\le C\|z-z_h\|_{L^\infty(J;L^2(\Omega))}.\tag{149}$$

Now, choose $w=r_{y,tt}$ in (148) and $\boldsymbol{v}=r_{\boldsymbol{p},t}$ in (147), respectively. It is easy to see that

$$\begin{aligned}&\|r_{\boldsymbol{p},t}\|^2_{L^\infty(J;L^2(\Omega))}+\|r_{y,tt}\|^2_{L^\infty(J;L^2(\Omega))}\\&\le C(\|z-z_h\|^2_{L^\infty(J;L^2(\Omega))}+\|\xi_{z,t}\|^2_{L^2(J;L^2(\Omega))}\\&\quad+\|\eta_{z,t}\|^2_{L^2(J;L^2(\Omega))})+\delta\|r_{z,t}\|^2_{L^2(J;L^2(\Omega))},\end{aligned}\tag{150}$$

where δ is an arbitrary small positive constant.

Finally, we differentiate the equation (65) with respect to t, we get

$$(\alpha r_{\boldsymbol{q},t},\boldsymbol{v})-(r_{z,t},\mathrm{div}\boldsymbol{v})=-(r_{\boldsymbol{p},t},\boldsymbol{v}),\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}.\tag{151}$$

Since $r_z(T)=0$, let $t=T$ and $\boldsymbol{v}=r_{\boldsymbol{q}}(T)$ in (65), we find that

$$\|\alpha^{\frac12}r_{\boldsymbol{q}}(T)\|\le C\|r_{\boldsymbol{p}}(T)\|.\tag{152}$$

Selecting $\boldsymbol{v}=-r_{\boldsymbol{q}}$ and $w=-r_{z,t}$ as the test functions and add the two relations of (151) and (66), we can obtain that

$$-\frac12\frac{d}{dt}(\|\alpha^{\frac12}r_{\boldsymbol{q}}\|^2+\|r_{z,t}\|^2)=(r_{\boldsymbol{p},t},r_{\boldsymbol{q}})-(r_y,r_{z,t}).\tag{153}$$

Integrating (153) from t to T, using (149), (150), (152), ϵ-Cauchy inequality and applying Gronwall's lemma, we can easily obtain the following error estimate

$$\begin{aligned}&\|r_{\boldsymbol{q}}\|^2_{L^\infty(J;L^2(\Omega))}+\|r_{z,t}\|^2_{L^\infty(J;L^2(\Omega))}\\&\le\delta\|r_{z,t}\|^2_{L^2(J;L^2(\Omega))}+C\bigg(\|z-z_h\|^2_{L^\infty(J;L^2(\Omega))}+\|r_{\boldsymbol{p}}(T)\|^2\\&\quad+\|\xi_{z,t}\|^2_{L^2(J;L^2(\Omega))}+\|\eta_{z,t}\|^2_{L^2(J;L^2(\Omega))}+\|r_y\|^2_{L^2(J;L^2(\Omega))}\bigg).\end{aligned}\tag{154}$$

For sufficiently small δ, substituting the estimates for $\xi_{z,t}$ and r_y in (154), we can derive (144). □

4. Conclusion and future works

In this paper, we derive a posteriori error estimates for the semidiscrete mixed finite element solutions of quadratic optimal control problems governed by hyperbolic equations. Our posteriori error estimates for the linear hyperbolic optimal control problems by mixed finite element methods seem to be new. In the next work, we shall discuss a posteriori analysis for a completely discrete mixed approximation and design the adaptive mixed finite element algorithms. Furthermore, we shall consider a posteriori error estimates of mixed finite element methods for more complicated optimal control problems governed by wave equations.

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