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ASYMPTOTIC EXPANSION OF THE BERGMAN KERNEL FOR TUBE DOMAIN OF INFINITE TYPE

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Abstract. The asymptotic expansions of the Bergman kernels on the diagonals near the boundary points of exponentially-flat infinite type for pseudoconvex tube domain in $\mathbb{C}^2$ are obtained.

1. Introduction

The studies of boundary behavior of Bergman kernel have been important topic for a long time related to the analysis of holomorphic functions for pseudoconvex domains. The boundary geometries of the domains determine the growth rates of the Bergman kernels near the given boundary points. For strongly pseudoconvex domain in $\mathbb{C}^n$, the boundary behavior of Bergman kernel is well understood by the works of Diederich [5], Hörmander [8] and Fefferman [7]. In particular, Fefferman obtained the asymptotic expansion formula for the Bergman kernel on the diagonal near C^∞ smooth strongly pseudoconvex boundary point. For the weakly pseudoconvex domain of finite type in the sense of D'Angelo, there have been considerable and significant achievements. To cite only a few, we list here only [2, 3, 4, 6, 14]. In particular, Kamimoto obtained asymptotic expansions for the domains which have circular symmetries [10] and for finite type tube domains [9, 11].

However, there are very few results for infinite type boundary points. Kim and Lee [12] analyzed Bergman kernel near the exponentially-flat infinite type boundary point for model domain using scaling method. Bharali [1] obtained upper and lower bound estimates of the Bergman kernel near the mildly infinite type boundary point for model domain.

In this paper we obtain an asymptotic expansion of the Bergman kernel function $B(z, \bar{z})$ for any point z near an exponentially-flat infinite type boundary point z_0 of a pseudoconvex tube domain in $\mathbb{C}^2$.

We use Koranyi-Vinberg integral representation of the Bergman kernel for tube domain. Since the singularity of the Bergman kernel near given boundary point is determined by local geometry of the boundary, analysis is focused on

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localized representation of original Koranyi-Vinberg representation. As in the case of finite type, singularity can be stratified by a real blowing up in the case of infinite type.

Since the principal part of defining function for finite type case gives quasi-homogeneous model, the analysis of the Bergman kernel near finite type boundary point focuses on how to handle a perturbation of model case. On the other hand, we don't have such a model for infinite type case. For this reason, first of all, a real blowing up of the infinite type point is constructed using defining function itself. In addition, a proper flat condition for the defining function is needed to analyze the principal part of singularity expansion. Our condition controls the growth of derivatives of all order of the defining function at the infinite type boundary point such that the principal part of expansion formula extends to the horizontal boundary in stratification. Since $f(t) = \exp(\frac{1}{t^{2m}})$ satisfies that condition, we call our condition *exponentially-flat*. Under this growth condition, it will be shown that all the coefficients of expansion in vertical direction vanish.

2. Statement of main theorem

Let Ω be a domain in $\mathbb{C}^2$. Denote by $A^2(\Omega)$ the closed subspace of $L^2(\Omega)$ consisting of holomorphic functions. Take a complete orthonormal basis $\{\varphi_j\}_j$ of $A^2(\Omega)$. The Bergman kernel $B(z) = B(z, \bar{z})$ of Ω (on the diagonal) is defined by $B(z) = \sum_j |\varphi_j(z)|^2$.

Given a domain ω in $\mathbb{R}^2$, the tube domain over the base ω is defined by

$$\Omega = \mathbb{R}^2 + i\omega = \{z = x + iy \in \mathbb{C}^2 : x \in \mathbb{R}^2, y \in \omega\}.$$

Here we set $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$. It is known that Ω is pseudoconvex if and only if ω is convex. We assume that ω is convex with C^∞-smooth boundary. Suppose the boundary point $z_0 = x_0 + iy_0$ is of *infinite type*. That is, there exists a coordinate system (y, V_{y_0}) such that

$$\omega \cap V_{y_0} = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 > f(y_1)\}, \tag{2.1}$$

where f is nonnegative, convex and C^∞-smooth, satisfies $f(0) = 0$ and vanishes to infinite order at $y_1 = 0$, that is, $\lim_{t\to 0} f(t)/|t|^N = 0$ for each $N \in \mathbb{Z}_+$.

In this article, we will use the notation $A(t) \sim B(t)$ in place of $\lim_{t\to 0+} |\frac{A(t)}{B(t)}|$ is finite and nonzero constant. For positive valued function $A(t)$ and $B(t)$, we will also use the notation $A(t) \lesssim B(t)$ if there exists $C(t)$ which is positive constant or satisfies $\lim_{t\to 0+} C(t) > 0$ such that $A(t) \le C(t)B(t)$ near $t = 0$.

We say that the boundary point $z_0 \in \partial\Omega$ is *exponentially-flat*, if there exists a coordinate system such as (2.1) such that

$$\frac{f^{(k)}(t)}{f(t)} \sim t^{-(2m+1)k}. \tag{2.2}$$

In particular, $f(t) = \exp(-\frac{1}{t^{2m}})$ satisfies exponentially-flat condition.

We say that the boundary point z_0 is *quasi-symmetric* if the defining function f additionally satisfies following: if there exist positive constants d and ϵ such that $f(-t) = f(dt)$ for $0 < t < \epsilon$.

Consider a real blowing up of the origin for quasi-symmetric function f as follows: $\pi : (-1/d, 1) \times (0, \epsilon) \to \omega \cap \{(y_1, y_2) : y_2 < L\}$ for some positive constant L such that $\pi(\tau, \rho) = (y_1, y_2)$, where $y_1 = \tau\rho, y_2 = f(\rho)$. Then the main theorem of this article is as follows:

Theorem 2.1. *Let $z_0 = x_0 + iy_0$ be a non Levi-flat, exponentially-flat infinite type and quasi-symmetric boundary point of Ω. There exists a neighborhood V of y_0 and ϵ such that for $(\tau, \rho) \in (-1/d, 1) \times (0, \epsilon) \subset \pi^{-1}(V \cap \omega)$ and x near x_0*

$$B(x + i\pi(\tau, \rho)) = \rho^{-2} f(\rho)^{-2} B_0(\tau, \rho), \qquad (\textit{mod smooth kernel}) \tag{2.3}$$

where $B_0(\tau, \rho)$ is a C^∞ smooth function on $(-1/d, 1)\times(0, \epsilon)$ satisfies followings:

1. *$B_0(\tau, \rho)$ extends C^∞ smoothly to $(-1/d, 1) \times [0, \epsilon)$.*
2. *$B_0(\tau, \rho)$ has the following asymptotic expansion with respect to ρ : for each $N \geq 1$, there exists $R_N(\tau, \rho) \in C((-1/d, 1) \times [0, \epsilon))$ such that*

$$B_0(\tau, \rho) = b(\tau) + R_N(\tau, \rho)\rho^N$$

on $(-1/d, 1) \times (0, \epsilon)$ where $b(\tau)$ is in $C^\infty(-1/d, 1)$ and

$$|R_N(\tau, \rho)| \lesssim \rho^{-(2m+2)N-3} f(\rho) \qquad \textit{as } \rho \to 0+ .$$

Remark 1. The principal term $b(\tau)$ tells us that tangential limits of Bergman kernel depend on orbits approaching to the origin. Each vertical line $\{\tau\} \times (0, \rho_0)$ is mapped by π to $\{y_2 = f(y_1/\tau) : 0 < y_1 < \tau\rho_0\}$, which is a tangential orbit to the origin. Then

$$\begin{aligned} b(\tau) &= \lim_{\rho\to 0+} B_0(\tau, \rho) \\ &= \lim_{\rho\to 0+} \frac{1}{2\pi^2} \int_0^\infty e^{-2s} \left[\int_{-a\rho/f(\rho)}^{a\rho/f(\rho)} e^{-2\tau s v} \frac{1}{\int_{-\epsilon/\rho}^{\epsilon/\rho} e^{-2s[vw + f(\rho w)/f(\rho)]\, dw}} \right] s^2 ds \end{aligned}$$

for $-1/d < \tau < 1$.

Remark 2. Our theorem still holds for infinite type points which become flat less rapidly than exponentially-flat case. The defining function

$$f(t) = \exp\left(-\frac{1}{t^{2m}} \log \frac{1}{|t|}\right)$$

becomes flat less rapidly than exponentially-flat case. It holds that for any positive constant α there exists a neighborhood U_α of 0 such that $|\frac{f^{(k)}(t)}{f(t)}| \lesssim t^{-(2m+1)k-\alpha}$ whenever $t \in U_\alpha$. All the estimates still work under this growth condition with minor change.

3. Integral formula for tube domain

For $y = (y_1, y_2), u = (u_1, u_2)$ in $\mathbb{R}^2$, we set $\langle y, u\rangle = y_1u_1 + y_2u_2$. It is shown in [13], [15] that the Bergman kernel of a tube domain $\Omega = \mathbb{R}^2 + i\omega$ is expressed as

$$B(z) = B(x+iy) = \frac{1}{4\pi^2}\int_{\Lambda^*} e^{-2\langle y,u\rangle}\frac{1}{\varphi(u)}du, \tag{3.1}$$

where

$$\varphi(u) = \int_\omega e^{-2\langle u,w\rangle}dw$$

and $\Lambda^* = \{u \in \mathbb{R}^2 : \varphi(u) < \infty\}$. We need to express Λ^* in a different form to compute the Bergman kernel. For a convex set ω we define its *recession cone*

$$\Lambda_\omega = \{y \in \mathbb{R}^2 : v + ty \in \omega \text{ for all } v \in \omega, t \geq 0\}.$$

Then for unbounded ω, we have

$$\Lambda^* = \{u \in \mathbb{R}^2 : \langle u, y\rangle > 0 \text{ for } y \in \Lambda_\omega\}.$$

For $R > 0$, set $\tilde{B}_R = \{y_1 \in \mathbb{R} : f(y_1) < R\}$, $B_R = \{y_1/R : y_1 \in \tilde{B}_R\}$ and $B = \cap_{R>0}B_R$. Then

$$\Lambda_\omega = \{(s\hat{y}_1, s) \in \mathbb{R}^2 : s \geq 0, \hat{y}_1 \in B\}.$$

Thus we have

$$\Lambda^* = \{(t\hat{u}_1, t) \in \mathbb{R}^2 : t > 0, \hat{u}_1 \in B^*\},$$

where

$$B^* = \{\hat{u}_1 \in \mathbb{R} : \hat{u}_1\hat{y}_1 + 1 > 0 \text{ for } \hat{y}_1 \in B\}.$$

For our case $B^* = \mathbb{R}$, thus

$$\begin{aligned} B(x+iy) &= \frac{1}{2\pi^2}\int_0^\infty e^{-2y_2t}F(y_1,t)\,t^2dt, \\ F(y_1,t) &= \int_{\mathbb{R}} e^{-2ty_1\hat{u}_1}\frac{1}{D(t,\hat{u}_1)}d\hat{u}_1, \\ D(t,\hat{u}_1) &= \int_{\mathbb{R}} e^{-2t[f(w)+\hat{u}_1w]}dw. \end{aligned} \tag{3.2}$$

4. Localization

Localization for integral representation of the Bergman kernel allows us to analyze its singularity using only local geometry around infinite type boundary point. Localization of Bergman kernel for tube domain were introduced in [9], [11]. We follow the arguments in [11]. Let U be an open interval containing the origin. Then we define

$$B_U(z) = \frac{1}{4\pi^2}\int_{\Lambda^*(U)} e^{-2\langle y,u\rangle}\frac{1}{\varphi(u)}du,$$

where

$$\Lambda^*(U) := \{(t\hat{u}_1, t) \in \mathbb{R}^2 : t > 0,\ \hat{u}_1 \in U\}$$

and

$$B_{U,\epsilon}(z) = \frac{1}{4\pi^2} \int_{\Lambda^*(U)} e^{-2\langle y,u\rangle} \frac{1}{\varphi_\epsilon(u)} du,$$

where

$$\varphi_\epsilon(u) = \int_{\omega\cap[(-\epsilon,\epsilon)\times\mathbb{R}+]} e^{-2\langle u,w\rangle} dw.$$

The following localization arguments hold if the defining function f of ω satisfies $f(t) = o(|t|)$ as $t \to 0$.

Proposition 4.1 (Kamimoto)**.** *For any open interval U containing the origin, $B(z) - B_U(z)$ is real analytic near the origin.*

Proposition 4.2 (Kamimoto)**.** *For any $\epsilon > 0$, there exists U such that $B_U(z) - B_{U,\epsilon}(z)$ is real analytic near the origin.*

Thus, for our purposes, it suffices to obtain the asymptotic expansion of $B_{U,\epsilon}(z)$.

5. Proof of main theorem

The localized representation of the Bergman kernel is as follows:

$$B_{a,\epsilon}(y_1, y_2) = \frac{1}{2\pi^2} \int_0^\infty e^{-2y_2 t} F_{a,\epsilon}(y_1; t) t^2\, dt,$$

$$F_{a,\epsilon}(y_1; t) = \int_{-a}^{a} e^{-2ty_1 u} \frac{1}{D_\epsilon(t,u)} du,$$

$$D_\epsilon(t,u) = \int_{-\epsilon}^{\epsilon} e^{-2t[uw+f(w)]} dw,$$

where a depends on the choice of ϵ according to Proposition 4.2. By the blow-up coordinate system $\pi(\tau,\rho) = (\tau\rho, f(\rho))$, $(\tau,\rho) \in (-1/d, 1) \times (0,\epsilon)$ the representation in (3.2) changes into

$$B_{a,\epsilon}(\Phi(\tau,\rho)) = \rho^{-2} f(\rho)^{-2} K_{a,\epsilon}(\tau,\rho), \tag{5.1}$$

$$K_{a,\epsilon}(\tau,\rho) = \frac{1}{2\pi^2} \int_0^\infty e^{-2s} s^2 F_{a,\epsilon}(\tau,\rho; s) ds,$$

$$F_{a,\epsilon}(\tau,\rho; s) = \int_{-\mu}^{\mu} e^{-2\tau s v} \frac{1}{D_\epsilon(\rho; s, v)} dv,$$

$$D_\epsilon(\rho; s, v) = \int_{-\epsilon/\rho}^{\epsilon/\rho} e^{-2s[vw + f(\rho w)/f(\rho)]} dw,$$

where $\mu = a\rho/f(\rho)$.

We decompose the integral $K_{a,\epsilon}$ of (5.1) as follows:

$$\begin{aligned} K_{a,\epsilon}(\tau,\rho) &= \frac{1}{2\pi^2} \int_{f(\rho)}^\infty e^{-2s} F_{a,\epsilon}(\tau,\rho; s) s^2\, ds + \frac{1}{2\pi^2} \int_0^{f(\rho)} e^{-2s} F_{a,\epsilon}(\tau,\rho; s) s^2\, ds \\ &= K^{(1)}_{a,\epsilon}(\tau,\rho) + K^{(2)}_{a,\epsilon}(\tau,\rho). \end{aligned}$$

We will show that boundary values of derivatives of $K^{(1)}_{a,\epsilon}(\tau,\rho)$ at $\rho=0$ equal 0 in Proposition 5.1 and $\rho^{-2}f(\rho)^{-2}K^{(2)}_{a,\epsilon}(\tau,\rho)$ is real analytic in ρ and $f(\rho)$ in Proposition 5.2. It will prove the main theorem.

5.1. Growth estimate of derivative of $K^{(1)}_{a,\epsilon}$

In this section we will use the notation ∂^n_ρ in place of $\frac{\partial^n}{\partial\rho^n}$ whenever we need to simplify expression.

Proposition 5.1. *Suppose that f satisfies exponentially flat condition* (2.2) *at the origin. Then*

$$|\partial^{n'}_\tau\partial^n_\rho K^{(1)}_{a,\epsilon}(\tau,\rho)| \lesssim \rho^{-(2m+2)n-3}f(\rho) \quad \text{as } \rho\to 0+$$

for $-1/d<\tau<1$ and $n\ge 1, n'\ge 0$.

Proof. We have

$$\begin{aligned} &\partial^{n'}_\tau\partial^n_\rho K^{(1)}_{a,\epsilon}(\tau,\rho)\\ &= -\sum_{k=0}^{n-1}\partial^{n-1-k}_\rho(f'(\rho)f(\rho)^2e^{-2f(\rho)}\,\partial^{n'}_\tau\partial^k_\rho F_{a,\epsilon}(\tau,\rho;s)|_{s=f(\rho)})\\ &\quad+\frac{1}{2\pi^2}\int_{f(\rho)}^\infty e^{-2s}\,\partial^{n'}_\tau\partial^n_\rho F_{a,\epsilon}(\tau,\rho;s)\,s^2\,ds \end{aligned} \tag{5.2}$$

for $n\ge 1$. The first part of (5.2) is estimated as follows. Set

$$F^{(k)}_{a,\epsilon}(\tau,\rho;s)=\int_{-\mu}^{\mu}e^{-2\tau sv}\,\partial^k_\rho\frac{1}{D_\epsilon(\rho;s,v)}dv, \tag{5.3}$$

$$\begin{aligned} Q^+_j(\tau,\rho;s)&=\mu' e^{-2\tau s\mu}\,\partial^j_\rho\frac{1}{D_\epsilon(\rho;s,v)}|_{v=\mu},\\ Q^-_j(\tau,\rho;s)&=\mu' e^{2\tau s\mu}\,\partial^j_\rho\frac{1}{D_\epsilon(\rho;s,v)}|_{v=-\mu}. \end{aligned} \tag{5.4}$$

Then we have

$$\partial^i_s\partial^{n'}_\tau\partial^k_\rho F_{a,\epsilon}(\tau,\rho;s)=\sum_{j=0}^{k-1}\partial^i_s\partial^{n'}_\tau\partial^{k-1-j}_\rho(Q^+_j+Q^-_j)\;+\partial^i_s\partial^{n'}_\tau F^{(k)}_{a,\epsilon},\qquad k\ge 1. \tag{5.5}$$

By Lemma 6.6, we have

$$\partial^i_s\partial^{n'}_\tau\partial^{k-1-j}_\rho Q^\pm_j(\tau,\rho;s)|_{s=f(\rho)}\sim\rho^{-(2m+1)k+n'+1}f(\rho)^{-i-1}.$$

By Lemma 6.5, we have

$$\begin{aligned} &\partial^i_s\partial^{n'}_\tau F^{(k)}_{a,\epsilon}(\tau,\rho;s)|_{s=f(\rho)}\\ &=\sum_{i_1+i_2=i}C_{i_1,i_2}\,\tau^{i_1}(-f(\rho))^{n'}\int_{-\mu}^{\mu}e^{-2\tau sv}v^{n'+i_1}\,\partial^{i_2}_s\partial^k_\rho\frac{1}{D_\epsilon}dv|_{s=f(\rho)}\\ &\sim\rho^{-(2m+1)k+n'+2}f(\rho)^{-i-1}. \end{aligned}$$

They imply

$$(\partial_s^i \partial_\tau^{n'} \partial_\rho^k F_{a,\epsilon})|_{s=f(\rho)} \sim \rho^{-(2m+1)k+n'+1} f(\rho)^{-i-1}. \tag{5.6}$$

Using (5.6) and derivative condition on f we have

$$\begin{aligned}
&\partial_\rho^j((\partial_\tau^{n'} \partial_\rho^k F_{a,\epsilon})|_{s=f(\rho)}) \\
&= (\partial_\tau^{n'} \partial_\rho^{k+j} F_{a,\epsilon})|_{s=f(\rho)} \\
&\quad + \sum_{j'=0}^{j-1} \sum_{j''=1}^{j-j'} (\partial_s^{j''} \partial_\tau^{n'} \partial_\rho^{k+j'} F_{a,\epsilon})|_{s=f(\rho)} \sum_{\substack{\alpha_1+\cdots+\alpha_{j''}=j-j' \\ \alpha_1,\ldots,\alpha_{j''}>0}} f^{(\alpha_1)} \cdots f^{(\alpha_{j''})} \\
&\sim \rho^{-(2m+1)(k+j)+n'+1} f(\rho)^{-1}. \qquad (5.7)
\end{aligned}$$

By

$$\begin{aligned}
&\partial_\rho^j \left(f'(\rho) f(\rho)^2 e^{-2f(\rho)}\right) \\
&= e^{-f(\rho)} \sum_{j'=0}^{j} \sum_{\alpha_1+\cdots+\alpha_{j'+3}=j+1} C_{\alpha_1,\ldots,\alpha_{j'+3}} f^{(\alpha_1)}(\rho) \cdots f^{(\alpha_{j'+3})}(\rho) \\
&\sim \rho^{-(2m+1)(j+1)} f(\rho)^3
\end{aligned}$$

and (5.7) we have

(5.8)

$$\sum_{k=0}^{n-1} \partial_\rho^{n-1-k} (f'(\rho) f(\rho)^2 e^{-f(\rho)} \, \partial_\tau^{n'} \partial_\rho^k F_{a,\epsilon}(\tau,\rho;s)|_{s=f(\rho)}) \sim \rho^{-(2m+1)n+n'+1} f(\rho)^2.$$

The second part of (5.2) is separated by (5.5) as

$$\begin{aligned}
&\int_{f(\rho)}^\infty e^{-2s} \, \partial_\tau^{n'} \partial_\rho^n F_{a,\epsilon}(\tau,\rho;s) \, s^2 ds \\
&= \sum_{j=0}^{n-1} \int_{f(\rho)}^\infty e^{-2s} \, \partial_\tau^{n'} \partial_\rho^{n-1-j} (Q_j^+ + Q_j^-)(\tau,\rho;s) \, s^2 ds \qquad (5.9) \\
&\quad + \int_{f(\rho)}^\infty e^{-2s} \, \partial_\tau^{n'} F_{a,\epsilon}^{(n)}(\tau,\rho;s) \, s^2 ds.
\end{aligned}$$

The first part of (5.9) is estimated by following lemma.

Lemma 5.1. *Under the assumptions of Proposition 5.1,*

$$\left| \int_{f(\rho)}^\infty e^{-2s} \, \partial_\tau^{n'} \partial_\rho^{n-1-j} \, Q_j^\pm(\tau,\rho;s) \, s^2 \, ds \right| \lesssim \rho^{-(2m+2)(n-1)-2} f(\rho)^2$$

for $j = 0, 1, \ldots, n-1$.

Proof. Estimation of integration of derivatives of (5.4) requires a revised equation of (6.16) which is not evaluated at $s = f(\rho)$. Applying Lemma 6.7 to the first line of (6.16), we have

$$\partial_\rho^q \left(\partial_\rho^j \frac{1}{D_\epsilon}|_{v=\mu} \right) \sim \rho^{-(2m+1)(q+j)} \frac{1}{D_\epsilon}|_{v=\mu} \left(\sum_{j'=1}^{q} \sum_{j''=1}^{j'} \sum_{k=0}^{q+j-j'} C_{j',j'',k}^{(q,j)}(\rho;s) s^{j''+k} f^{-j''-k} + \sum_{k=0}^{q+j} C_k^{(q,j)}(\rho;s) s^k f^{-k} \right),$$

where $C_{j',j'',k}^{(q,j)}(\rho;s) = O(1)$ and $C_k^{(q,j)}(\rho;s) = O(1)$ as $\rho \to 0+$. Applying it and (6.15) into (6.14), we have

$$\begin{aligned} &\partial_\tau^{n'} \partial_\rho^{n-1-j} Q_j^\pm(\tau,\rho;s) \\ \sim &\sum_{q_1+q_2=n-1-j} C_{q_1,q_2}\, \rho^{-(2m+1)(n-1)+n'+1} f^{-n'-1} s^{n'} e^{\mp\mu\tau s} \frac{1}{D_\epsilon}|_{v=\pm\mu} \\ &\sum_{r=0}^{q_1} (-\tau)^r \rho^r \left(\sum_{j'=1}^{q_2} \sum_{j''=1}^{j'} \sum_{k=0}^{q_2+j-j'} C_{j',j'',k}^{(q_2,j)}(\rho;s)\, s^{j''+k+r} f^{-j''-k-r} + \sum_{k=0}^{q_2+j} C_k^{(q_2,j)}(\rho;s)\, s^{k+r} f^{-k-r} \right). \end{aligned}$$

Set

$$f_0(t) = \max\{f(t), f(-t)\} \qquad \text{for } 0 < t < \epsilon.$$

It holds that

$$\begin{aligned} D_\epsilon(\rho;s,v)|_{v=\mu} &\geq \int_{-\epsilon}^{\epsilon} e^{-2s(\mu w + f(\rho w)/f(\rho))} dw \\ &\gtrsim e^{-2sf_0(\epsilon\rho)/f(\rho)} \frac{\sinh(2\epsilon\mu s)}{\mu s} \qquad (5.10) \\ &\gtrsim e^{2(-f_0(\epsilon\rho)/f(\rho)+\epsilon\mu)s} \frac{(1-e^{-4a\epsilon\rho})}{\rho} f(\rho) s^{-1} \qquad \text{for } s \geq f(\rho). \end{aligned}$$

It implies that

$$\begin{aligned} &\left| \int_{f(\rho)}^{\infty} e^{-2s}\, \partial_\tau^{n'} \partial_\rho^{n-1-j} Q_j^\pm\, s^2\, ds \right| \\ \lesssim &\sum_{q_1+q_2=n-1-j} C_{q_1,q_2}\, \rho^{-(2m+1)(n-1)+n'+1} f(\rho)^{-n'-1} \sum_{r=0}^{q_1} |\tau|^r \rho^r \\ &\int_{f(\rho)}^{\infty} e^{-\lambda s} \left(\sum_{j'=1}^{q_2} \sum_{j''=1}^{j'} \sum_{k=0}^{q_2+j-j'} f(\rho)^{-j''-k-r} s^{j''+k+r+n'+2} + \sum_{k=0}^{q_2+j} f(\rho)^{-k-r} s^{k+r+n'+2} \right) ds \\ \sim &\rho^{-(2m+2)(n-1)-2} f(\rho)^2, \end{aligned}$$

where $\lambda = 1 - \frac{f_0(\epsilon\rho)}{f(\rho)} + \mu(|\tau| + \epsilon)$. It proves the lemma. □

The boundary limit of the second integral of (5.9) is estimated as follows.

Lemma 5.2. *Under the assumptions of Proposition 5.1,*

$$\left|\int_{f(\rho)}^{\infty} e^{-2s}\, \partial_\tau^{n'} F_{a,\epsilon}^{(n)}(\tau,\rho;s)\, s^2\, ds\right| \lesssim \rho^{-(2m+2)n-2} f(\rho)^2$$

for $-1/d < \tau < 1$, $n \geq 1$.

Proof. By Lemma 6.4 there exist bounded functions $E_l^{(p)}(\rho;s,v) \in C([0,\rho_0) \times \mathbb{R}_+ \times \mathbb{R})$ such that

$$\begin{aligned}
&\partial_\tau^{n'} F_{a,\epsilon}^{(n)}(\tau,\rho;s)\\
&= \int_{-\mu}^{\mu} e^{-2\tau s v}\,(-sv)^{n'}\partial_\rho^n \frac{1}{D_\epsilon(\rho;s,v)}\, dv\\
&\sim \sum_{p=0}^{n}\sum_{l=0}^{p} \rho^{-n-l-N_l} f(\rho)^{-p+l}\, s^{p+n'} \int_{-\mu}^{\mu} e^{-2\tau s v}\, E_l^{(p)}(\rho;s,v)\, \frac{1}{D_\epsilon(\rho;s,v)} v^{l+n'}\, dv,
\end{aligned}$$

where $N_p = 0$ and $N_l = 2m(n-l)$ for $l < p$. Using (5.10) we have

$$\begin{aligned}
&\left|\int_{-\mu}^{\mu} e^{-2\tau s v}\, E_l^{(p)}(\rho;s,v)\, \frac{1}{D_\epsilon(\rho;s,v)} v^{l+n'}\, dv\right|\\
&\quad\leq \int_0^{\mu} (e^{-2(\epsilon+|\tau|)sv} + e^{-2(\epsilon-|\tau|)sv}) \frac{2sve^{2sf_0(\rho\epsilon)/f(\rho)}}{1-e^{-2\epsilon s v}} v^{l+n'} dv.
\end{aligned}$$

It holds that

$$\begin{aligned}
&\int_0^{\mu} e^{-2(\epsilon\pm|\tau|)sv} \frac{2sv}{1-e^{-2\epsilon s v}} v^{l+n'} dv\\
&\leq \frac{e^{2a\epsilon\rho}}{\epsilon}\int_0^{\frac{a\rho}{2s}} e^{-(2\epsilon\pm|\tau|)sv} v^{l+n'} dv + C(\rho)\rho^{-1} s \int_{\frac{a\rho}{2s}}^{\mu} e^{-2(\epsilon\pm|\tau|)sv} v^{l+n'+1} dv\\
&\lesssim \frac{(1/s+\rho^{-1}\mu)}{(\epsilon\pm|\tau|)} \mu^{l+n'} e^{-(\epsilon\pm|\tau|)\mu s}\sinh((\epsilon\pm|\tau|)\mu s),
\end{aligned}$$

where $C(\rho) \to C > 0$ as $\rho \to 0+$. Thus

$$\begin{aligned}
&\int_{f(\rho)}^{\infty} e^{-2s(1-f_0(\epsilon\rho)/f(\rho))} s^{p+n'+2}\left[\int_0^{\mu} e^{-2(\epsilon\pm|\tau|)sv}\frac{2sv}{1-e^{-2\epsilon s v}} v^{l+n'} dv\right] ds\\
&\lesssim \frac{\sinh\left(\frac{\epsilon\pm|\tau|}{2}\right)}{\epsilon\pm|\tau|} \rho^{-p+l-3} f(\rho)^{p-l+2}
\end{aligned}$$

for $|\tau| < \epsilon$. It implies that for $|\tau| < \epsilon$

(5.11)

$$\left|\int_{f(\rho)}^{\infty} e^{-2s} F_n(\tau,\rho;s)\, s^2\, ds\right| \lesssim \sum_{p=0}^{n}\sum_{l=0}^{p} \rho^{-n-p-N_l-3} f(\rho)^2 \sim \rho^{-(2m+2)n-3} f(\rho)^2.$$

For $|\tau| \geq \epsilon$, by $D_\epsilon(\rho; s, v) \gtrsim e^{-s f_0(\epsilon\rho)/f(\rho)}$

$$\begin{aligned}&\int_{f(\rho)}^{\infty} e^{-2s} s^{p+n'+2} \left[\int_{-\mu}^{\mu} e^{-2|\tau|sv} \frac{1}{D_\epsilon(\rho; s, v)} v^{l+n'}\, dv\right] ds \\ &\leq \int_{f(\rho)}^{\infty} e^{-2(1-\frac{f_0(\epsilon\rho)}{f(\rho)}-\mu|\tau|)s} s^{p+n'+2} \mu^{l+n'+1}\, ds \\ &\sim \rho^{-p+l-2} f(\rho)^{p-l+2}.\end{aligned}$$

Thus for $|\tau| \geq \epsilon$,

$$\begin{aligned}&\left|\int_{f(\rho)}^{\infty} e^{-2s}\, \partial_\tau^{n'} F_n(\tau, \rho; s)\, s^2\, ds\right| \\ &\lesssim \sum_{p=0}^{n} \sum_{l=0}^{p} \rho^{-n-p-N_l-2} f(\rho)^2 \sim \rho^{-(2m+2)n-2} f(\rho)^2.\end{aligned} \tag{5.12}$$

It proves the lemma. □

Now by (5.8), Lemma 5.1 and Lemma 5.2, we obtain Proposition 5.1. □

5.2. Real analyticity of K_2

Proposition 5.2. *$\rho^{-2} f(\rho)^{-2} K_2(\tau, \rho)$ is real analytic in ρ and $f(\rho)$.*

Proof. Set $F_j(\tau, \rho) = \frac{1}{j!} \partial_s^j F_{a,\epsilon}(\tau, \rho; s)|_{s=0}$. Then

$$K_2(\tau, \rho) = \sum_{j=0}^{N-1} F_j(\tau, \rho) \int_0^{f(\rho)} e^{-2s} s^{j+2}\, ds + \int_0^{f(\rho)} e^{-2s}\, F_N(\tau, \rho; s)\, s^{N+2} ds.$$

We have

$$\begin{aligned}F_j(\tau, \rho) &= \frac{1}{j!} \sum_{j_1+j_2=j} \binom{j}{j_1} (-\tau)^{j_1} \sum_{k=0}^{j_2} C_{j_2,k}\, \rho^{-k+1} f(\rho)^{-j_2+k} \int_{-\mu}^{\mu} v^{j_1+k} dv \\ &= \rho^2 f(\rho)^{-j-1} \sum_{j'=0}^{j} {C'}_{j,j'} (\tau\rho)^{j'}\end{aligned}$$

and

$$\begin{aligned}&\sup_{0\leq s\leq f(\rho)} |F_N(\tau, \rho; s)| \\ &\leq \frac{1}{N!} \sum_{j_1+j_2=N} \binom{N}{j_1} |\tau|^{j_1} \sum_{k=0}^{j_2} \rho^{-k} f(\rho)^{-j_2+k} \sup_{0\leq s\leq f(\rho)} \int_{-\mu}^{\mu} e^{-\tau sv} |v|^{j_1+k}\, |\tilde{D}_k^{(j_2)}(\rho; s, v)|\, \frac{1}{D_\epsilon}\, dv \\ &\leq C_\epsilon\, e^{|\tau|\rho} \rho^2 f(\rho)^{-N-1} \sum_{j'=0}^{N} {C''}_{N,j'}\, (|\tau|\rho)^{j'},\end{aligned}$$

where $\tilde{D}_k^{(j_2)}$ is bounded for $v \in \mathbb{R}$. In particular, ${C''}_{N,j} \leq M^N$ for $0 \leq j \leq N$ for some positive number M. It completes the proof. □

6. Proof of lemmas

In this section we will give proofs of lemmas which we use in the proof of main proposition.

6.1. Estimates of derivatives of D_ϵ and $\frac{1}{D_\epsilon}$

Here we present estimates of growths of derivatives of D_ϵ, which we need to estimate $\partial_\rho^n \partial_\tau^{n'} \partial_\rho^n K_{a,\epsilon}^{(1)}$ through (5.1) and (5.2). The firsthand estimate is on growth of derivatives of $\frac{1}{D_\epsilon(\rho;s,v)}$ and its evaluation at $v=\mu$ or $s=f(\rho)$. We have

$$D_\epsilon(\rho;s,v) = \int_{-\epsilon/\rho}^{\epsilon/\rho} e^{-2s(vw+\frac{f(\rho w)}{f(\rho)})}\,dw = \rho^{-1}\int_{-\epsilon}^{\epsilon} e^{-2s(\frac{vu}{\rho}+\frac{f(u)}{f(\rho)})}du.$$

Set $\varphi(u,v;\rho) = \frac{vu}{\rho} + \frac{f(u)}{f(\rho)}$ for convenience.

Lemma 6.1. *For $q \geq 1$,*

$$\begin{aligned}
&\partial_v^p \partial_\rho^q \partial_s^r D_\epsilon(\rho;s,v)\\
&\sim \rho^{-p-q-1} \sum_{r'=\max(r-p,0)}^{r} s^{p-r+r'} \int_{-\epsilon}^{\epsilon} e^{-s\varphi}\,\varphi^{r'} u^p\,du\\
&\quad + \sum_{q'=1}^{q} \rho^{-p-q+q'-1} \sum_{j=1}^{q'} \sum_{r'=\max(r-p-j,0)}^{r} \sum_{|\beta|=q'} s^{p+j-r+r'} \int_{-\epsilon}^{\epsilon} e^{-s\varphi}\varphi^{r'}\,\partial_\rho^\beta \varphi\, u^p\,du,
\end{aligned}$$

where $|\beta| = \beta_1 + \cdots + \beta_j$ and $\partial_\rho^\beta \varphi = \partial_\rho^{\beta_1}\varphi \cdots \partial_\rho^{\beta_j}\varphi$.

Proof. We have

$$\begin{aligned}
&\partial_v^p \partial_\rho^q \partial_s^r D_\epsilon\\
&= \partial_\rho^q \partial_s^r \left[(-s)^p \rho^{-p-1} \int_{-\epsilon}^{\epsilon} e^{-s\varphi} u^p du\right]\\
&= C_{p,q}\,\rho^{-p-q-1} \int_{-\epsilon}^{\epsilon} \partial_s^r((-s)^p e^{-s\varphi}) u^p du\\
&\quad + \sum_{q'=1}^{q} \sum_{j=1}^{q'} \sum_{|\beta|=q'} C_{p,q,q',\beta}\,\rho^{-p-q+q'-1} \int_{-\epsilon}^{\epsilon} \partial_s^r((-s)^{p+j} e^{-s\varphi})\,\partial_\rho^\beta \varphi\, u^p\,du.
\end{aligned}$$

Using

$$\partial_s^r(s^p e^{-s\varphi}) = \sum_{r'=\max(r-p,0)}^{r} C_{r,r'} s^{p-r+r'} \varphi^{r'} e^{-s\varphi}$$

we obtain the lemma. □

Lemma 6.2. *Suppose that f satisfies exponentially-flat condition* (2.2) *at the origin. Then*

$$\partial_v^p \partial_\rho^q \partial_s^r \frac{1}{D_\epsilon}|_{v=\mu, s=f(\rho)} \sim \rho^{-p-(2m+1)q-1} f(\rho)^{p-r} \qquad \text{as } \rho \to 0+ .$$

Proof. For $\beta_1 + \cdots + \beta_j = q'$,

$$\partial_\rho^{\beta_1}\varphi \cdots \partial_\rho^{\beta_j}\varphi|_{v=\mu} \sim \rho^{-(2m+1)q'} f(\rho)^{-j} \left(f(u)^j + \Psi_{q',j}(\rho,u)\right), \tag{6.1}$$

where $\Psi_{q',j}(\rho,u) = O(\rho)$ as $\rho \to 0+$. By Lemma 6.1

$$\begin{aligned}
&\partial_v^p \partial_\rho^q \partial_s^r D_\epsilon|_{v=\mu, s=f(\rho)} \\
&\sim \rho^{-p-q-1} f(\rho)^{p-r} \int_{-\epsilon}^{\epsilon} e^{-s\varphi} \sum_{l=0}^{\min(p,r)} (au + f(u))^l u^p du \\
&\quad + \sum_{q'=1}^{q} \rho^{-p-q-2mq'-1} f(\rho)^{p-r} \sum_{j=1}^{q'} \int_{-\epsilon}^{\epsilon} e^{-s\varphi}\, u^p \left(f(u)^j + \Psi_{q'',j}(\rho,u)\right) \sum_{l=0}^{\min(p,r)} (au+f(u))^l \, du \\
&\sim \rho^{-p-(2m+1)q-1}\, f(\rho)^{p-r}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\partial_v^p \partial_\rho^q \partial_s^r \frac{1}{D_\epsilon}|_{v=\mu, s=f(\rho)} \\
&= \sum_{l=1}^{\max(p,q,r)} \sum_{|\alpha|=p, |\beta|=q, |\gamma|=r} C_{\alpha,\beta,\gamma} \frac{\partial_v^{\alpha_1}\partial_\rho^{\beta_1}\partial_s^{\gamma_1} D_\epsilon}{D_\epsilon} \cdots \frac{\partial_v^{\alpha_l}\partial_\rho^{\beta_l}\partial_s^{\gamma_l} D_\epsilon}{D_\epsilon} \frac{1}{D_\epsilon} \\
&\sim \rho^{-p-(2m+1)q} f(\rho)^{p-r}.
\end{aligned}$$

We obtain the lemma. □

For such growth estimates of evaluations of the derivatives of D_ϵ as Lemma 6.2, (6.1) is enough. But if we want to estimate integral of derivatives of $\frac{1}{D_\epsilon}$, we need more thorough estimate than (6.1). Set $D_{p,q}(\rho; s, v) = \int_{-\epsilon}^{\epsilon} e^{-s\varphi} u^p f(u)^q du$.

Lemma 6.3. *Suppose that f satisfies exponentially flat condition* (2.2) *at the origin. Then*

$$\begin{aligned}
&\sum_{\beta_1+\cdots+\beta_j=q} \int_{-\epsilon}^{\epsilon} e^{-s\varphi}\ \partial_\rho^{\beta_1}\varphi \cdots \partial_\rho^{\beta_j}\varphi \, du \\
&\sim \sum_{l=0}^{j} v^l \rho^{-q-l-2m(q-l)(1-\delta_{j-l,0})} f(\rho)^{-j+l}\ D_{l,j-l}(\rho; s, v).
\end{aligned}$$

Proof. We have

$$\partial_\rho^{\beta_1}\varphi \cdots \partial_\rho^{\beta_j}\varphi = \sum_{l=0}^{j} v^{j-l} u^{j-l} f(u)^l \rho^{-j+l} H_{(\beta,l)}(\rho),$$

where

$$H_{(\beta,l)}(\rho) = \sum_{(\beta'_1,\dots,\beta'_j)} C_{\beta'_1,\dots,\beta'_j} \left(\frac{1}{f(\rho)}\right)^{(\beta'_1)} \cdots \left(\frac{1}{f(\rho)}\right)^{(\beta'_l)} \rho^{-(\beta'_{l+1}+\cdots+\beta'_j)},$$

where $(\beta'_1,\dots,\beta'_j)$ is a rearrangement of $(\beta_1,\dots,\beta_j)$ and summation is taken over all possible such rearrangements. Then since $\left(\frac{1}{f(\rho)}\right)^{(p)} \sim \rho^{-(2m+1)p} f(\rho)^{-1}$,

$$\begin{aligned} H_{(\beta,l)} &\sim f(\rho)^{-l} \sum_{(\beta'_1,\dots,\beta'_j)} C_{\beta'_1,\dots,\beta'_j} \rho^{-(2m+1)(\beta'_1+\cdots+\beta'_l)-(\beta'_{l+1}+\cdots+\beta'_j)} \\ &\sim f(\rho)^{-l} \sum_{(\beta'_1,\dots,\beta'_j)} C_{\beta'_1,\dots,\beta'_j} \rho^{-2m(\beta'_1+\cdots+\beta'_l)-q} \\ &\sim \begin{cases} \rho^{-q} & l=0 \\ \rho^{-q-2m(q-j+l)} f(\rho)^{-l} & l \geq 1. \end{cases} \end{aligned}$$

It proves the lemma. □

Lemma 6.4. *Under the same assumption of Lemma* 6.3*, for* $n \geq 1$*, there exists* $E_l^{(p)}(\rho; s, v) \in C([0,\rho_0) \times \mathbb{R}_+ \times \mathbb{R})$ *satisfying* $E_l^{(p)}(\rho; s, v) \lesssim 1$ *such that*

$$\partial_\rho^n \frac{1}{D_\epsilon(\rho; s, v)} \sim \sum_{p=0}^{n} \sum_{l=0}^{p} \rho^{-n-l-N_l} f(\rho)^{-p+l} s^p \frac{E_l^p(\rho; s, v)}{D_\epsilon(\rho; s, v)} v^l,$$

where $N_p = 0$ *and* $N_l = 2m(n-l)$ *for* $l < p$.

Proof. Set $\hat{D}_{p,q} = D_{p,q}/D_\epsilon$. Then by Lemmas 6.1 and 6.3, we have

$$\begin{aligned} \frac{\partial_\rho^q D_\epsilon}{D_\epsilon} &= \rho^{-q} + \sum_{\substack{q'+q''=q \\ q'' \geq 1}} \rho^{-q'-1} \sum_{j=1}^{q''} (-s)^j \sum_{\beta_1+\cdots+\beta_j = q''} \frac{1}{D_\epsilon} \int_{-\epsilon}^{\epsilon} e^{-s\varphi}\, \partial_\rho^{\beta_1}\varphi \cdots \partial_\rho^{\beta_j}\varphi \, du \\ &\sim \rho^{-q} \, (1 + \sum_{j=1}^{q} \left(\sum_{l=0}^{j} B_{j,l}^{(q)}(\rho; s, v)\, v^l \right) (-s)^j \,), \end{aligned}$$

where

$$B_{j,l}^{(q)}(\rho; s, v) = \begin{cases} \rho^{-j} \hat{D}_{j,0} & l = j \\ \rho^{-2mq+(2m-1)l} \, f^{-j+l} \, \hat{D}_{l,j-l} & 0 \leq l \leq j-1. \end{cases}$$

We have

$$\begin{aligned} \partial_\rho^n \frac{1}{D_\epsilon} &= \sum_{k=1}^{n} \sum_{\substack{\alpha_1+\cdots+\alpha_k=n \\ \alpha_1,\dots,\alpha_k>0}} \frac{\partial_\rho^{\alpha_1} D_\epsilon}{D_\epsilon} \cdots \frac{\partial_\rho^{\alpha_k} D_\epsilon}{D_\epsilon} \frac{1}{D_\epsilon} \\ &\sim \rho^{-n} \sum_{p=0}^{n} (-s)^p \sum_{l=0}^{p} \hat{B}_l^{(p)}(\rho; s, v)\, v^l \frac{1}{D_\epsilon}, \qquad (6.2) \end{aligned}$$

where

$$\hat{B}_l^{(p)}(\rho; s, v) = \sum_{k=1}^{n} \sum_{\substack{\alpha_1+\cdots+\alpha_k=n \\ \alpha_1,\ldots,\alpha_k>0}} \sum_{j_1+\cdots+j_k=p} \sum_{\substack{l_1+\cdots+l_k=l \\ l_1\le j_1,\ldots,l_k\le j_k}} B_{j_1,l_1}^{(\alpha_1)} \cdots B_{j_k,l_k}^{(\alpha_k)}.$$

Define

$$E_l^{(p)}(\rho; s, v) = \rho^{l+N_l} f(\rho)^{p-l} \hat{B}_l^{(p)}(\rho; s, v), \tag{6.3}$$

where $N_p = 0$ and $N_l = m(n-l)$ for $l < p$. Then $E_l^{(p)} \lesssim 1$. It is proved as follows. For $l = p$, $\hat{B}_l^{(p)}$ is a sum of such terms as $B_{j_1,j_1}^{(\alpha_1)} \cdots B_{j_k,j_k}^{(\alpha_k)} = \rho^{-p}\hat{D}_{j_1,0} \cdots \hat{D}_{j_k,0}$. For $l < p$, $\hat{B}_l^{(p)}$ is a sum of such terms as

$$\begin{aligned} &B_{j_1,j_1}^{(\alpha_1)} \cdots B_{j_{k'},j_{k'}}^{(\alpha_k)} B_{j_{k'+1},l_{k'+1}}^{(\alpha_k)} \cdots B_{j_k,l_k}^{(\alpha_k)} \\ &= \rho^{-(j_1+\cdots+j_{k'})-m(\alpha_{k'+1}+\cdots+\alpha_k)+(m-1)(l_{k'+1}+\cdots+l_k)} \\ &\quad \hat{D}_{j_1,0} \cdots \hat{D}_{j_{k'},0} \cdots \hat{D}_{l_{k'+1},j_{k'+1}-l_{k'+1}} \cdots \hat{D}_{l_k,j_k-l_k}, \end{aligned}$$

where k' satisfies $0 < k' < k$ and $l_1 = j_1, \ldots, l_{k'} = j_{k'}, l_{k'+1} < j_{k'+1}, \ldots, l_k < j_k$. Then the exponent of ρ^{-1} is $l + m\{(\alpha_{k'+1} - l_{k'+1}) + \cdots + (\alpha_k - l_k)\}$. Since $(j_{k'+1} - l_{k'+1}) + \cdots + (j_k - l_k) = p - l$ and $(\alpha_{k'+1} - j_{k'+1}) + \cdots + (\alpha_k - j_k) \le n - p$, the exponent of $\rho^{-1} \le l - m(n-p+p-l) = l - m(n-l)$. The exponent of $f(\rho)^{-1}$ is $(j_{k'+1} - l_{k'+1}) + \cdots + (j_k - l_k) = p - l$. It completes the proof. □

6.2. Integrals of derivatives of $\frac{1}{D_\epsilon}$

In this section we will estimate integrals in v of derivatives of $\frac{1}{D_\epsilon(\rho;s,v)}$. It require more thorough analysis than Lemma 6.2.

Lemma 6.5. *For $q \ge 1$*

$$\int_{-\mu}^{\mu} e^{-\tau s v}\, v^n\, \partial_\rho^q \partial_s^r \frac{1}{D_\epsilon(\rho; s, v)}\, dv|_{s=f(\rho)} \sim \rho^{-(2m+1)q+n+2} f(\rho)^{-r-n-1}$$

as $\rho \to 0+$.

Proof. From Lemmas 6.1 and 6.3

$$\begin{aligned} &\frac{\partial_\rho^q \partial_s^r D_\epsilon}{D_\epsilon}|_{s=f(\rho)} \\ &\sim \rho^{-q} \sum_{r'=0}^{r} v^{r'} \rho^{-r'} f(\rho)^{r'-r} \hat{D}_{r',r-r'} \\ &\quad + \rho^{-q} \sum_{q'=1}^{q} \sum_{j=1}^{q'} \sum_{l=0}^{j} \sum_{r'=0}^{r} v^{l+r'} \rho^{-(l+r')-2m(q'-l)(1-\delta_{lj})} f(\rho)^{l+r'-r} \hat{D}_{(j,l),(r,r')} \end{aligned} \tag{6.4}$$

$$\sim \rho^{-q} \sum_{q'=0}^{q} \sum_{l=0}^{q'} \sum_{r'=0}^{r} v^{l+r'} \rho^{-(l+r')-2m(q'-l)} f(\rho)^{l+r'-r} \hat{D}^{(q')}_{l,(r,r')},$$

where

$$\hat{D}_{(j,l),(r,r')} = \begin{cases} \sum_{r''=0}^{r-r'} \hat{D}_{l+r',j-l+r''} & j \geq r \text{ or } j < r,\ r' > r-j \\ \sum_{r''=r-j-r'}^{r-r'} \hat{D}_{l+r',j-l+r''} & j < r,\ 0 \leq r' \leq r-j \end{cases}$$

and

$$\hat{D}^{(q')}_{l,(r,r')} = \begin{cases} \sum_{j=l+1}^{q'} \hat{D}_{(j,l),(r,r')} & l < q' \\ \sum_{j=1}^{q'} \hat{D}_{(j,0),(r,r')} & l = q' > 0 \\ \hat{D}_{(0,0),(r,r')} & l = q' = 0. \end{cases}$$

Set

$$G_k^{(q,r)}(\rho, f(\rho); v) = \sum_{\substack{l+r'=k \\ 0\leq l\leq q,\ 0\leq r'\leq r}} \sum_{q'=0}^{q-l} \rho^{-2mq'} \hat{D}^{(l+q')}_{l,(r,r')}, \qquad 0 \leq k \leq q+r. \tag{6.5}$$

Then (6.4) has an expansion in v as follows:

$$\frac{\partial_\rho^q \partial_s^r D_\epsilon}{D_\epsilon}|_{s=f(\rho)} \sim \sum_{k=0}^{q+r} v^k \rho^{-q-k} f(\rho)^{k-r} G_k^{(q,r)}(\rho, f(\rho); v). \tag{6.6}$$

From (6.6) we have

$$\begin{aligned} \partial_\rho^q \partial_s^r \frac{1}{D_\epsilon}|_{s=f(\rho)} &\sim \sum_{i=1}^{\max(q,r)} \sum_{\substack{q_1+\cdots+q_i=q \\ r_1+\cdots+r_i=r}} \frac{\partial_\rho^{q_1}\partial_s^{r_1} D_\epsilon}{D_\epsilon} \cdots \frac{\partial_\rho^{q_i}\partial_s^{r_i} D_\epsilon}{D_\epsilon} \frac{1}{D_\epsilon} \\ &\sim \sum_{k=0}^{q+r} v^k \rho^{-q-k} f(\rho)^{k-r} A_k^{(q,r)}(\rho, f(\rho); v) \frac{1}{D_\epsilon}, \end{aligned} \tag{6.7}$$

where
(6.8)

$$A_k^{(q,r)}(\rho, f(\rho); v) = \sum_{i=1}^{\max(q,r)} \sum_{\substack{q_1+\cdots+q_i=q \\ r_1+\cdots+r_i=r}} \sum_{\substack{k_1+\cdots+k_i=k \\ 0\leq k_1\leq q_1+r_1,\ldots,0\leq k_i\leq q_i+r_i}} G_{k_1}^{(q_1,r_1)} \cdots G_{k_i}^{(q_i,r_i)}.$$

Since $\hat{D}_{l,l'} \lesssim 1$ and $D_\epsilon \gtrsim \rho^{-1} e^{-sf(\epsilon)/f(\rho)}$, we have
(6.9)

$$\left| A_k^{(q,r)}(\rho, f(\rho); v) \frac{1}{D_\epsilon} \right| |_{s=f(\rho)} \lesssim \begin{cases} \rho^{-2mq+1}, & k \leq r \\ \rho^{-2m(q+r-k)+1}, & k > r \end{cases} \qquad \text{as } \rho \to 0+.$$

Since

$$\int_{-\epsilon}^{\epsilon} e^{-s\varphi} du \leq \frac{2\rho}{sv} \sinh(\epsilon sv/\rho) \lesssim e^{\epsilon sv/\rho}$$

and

$$|D_{l,l'}(\rho;s,v)| \gtrsim \begin{cases} \frac{1}{\rho}\, e^{-sf(\epsilon)/f(\rho)}, & l:\ \text{even} \\ \frac{1}{\rho^2}\, s|v|\, e^{-sf(d\epsilon)/f(\rho)}, & l:\ \text{odd} \end{cases}$$

we have

$$\hat{D}_{l,l'}|_{s=f(\rho)} \gtrsim \begin{cases} e^{-\epsilon\frac{f(\rho)}{\rho}v}, & l:\ \text{even} \\ \frac{f(\rho)}{\rho}|v|\, e^{-\epsilon\frac{f(\rho)}{\rho}v}, & l':\ \text{odd}. \end{cases} \tag{6.10}$$

By (6.5), (6.8) and (6.10), there exist nonzero integers N_i, N_i' such that

$$\begin{aligned} &\left| A_k^{(q,r)}(\rho, f(\rho); v)\frac{1}{D_\epsilon}\right| |_{s=f(\rho)} \\ &\gtrsim \begin{cases} \sum_{i=1}^{\max(q,r)} \rho^{-2mq+1}\left(\frac{f(\rho)}{\rho}|v|\right)^{N_i} e^{-\epsilon(i+1)\frac{f(\rho)}{\rho}v}, & k\le r \\ \sum_{i=1}^{\max(q,r)} \rho^{-2m(q+r-k)+1}\left(\frac{f(\rho)}{\rho}|v|\right)^{N_i'} e^{-\epsilon(i+1)\frac{f(\rho)}{\rho}v}, & k> r \end{cases} \end{aligned} \tag{6.11}$$

as $\rho \to 0+$. Since

$$\int_{-\mu}^{\mu} e^{-\tau f(\rho)v} v^l dv = \begin{cases} 2\rho^{l+1} f(\rho)^{-l-1}\sum_{k=0}^{\infty} \frac{\tau^{2k}\rho^{2k}}{(2k)!(2k+l+1)}, & l:\ \text{even} \\ -2\rho^{l+2} f(\rho)^{-l-1}\sum_{k=0}^{\infty} \frac{\tau^{2k+1}\rho^{2k}}{(2k+1)!(2k+l+2)}, & l:\ \text{odd} \end{cases} \tag{6.12}$$

we have

(6.13)

$$\int_{-\mu}^{\mu} e^{-\tau s v} v^l A_k^{(q,r)}(\rho, f(\rho); v)\frac{1}{D_\epsilon} dv|_{s=f(\rho)} \sim \begin{cases} \rho^{-2mq+l+\delta_l} f(\rho)^{-l-1}, & k\le r \\ \rho^{-2m(q+r-k))+l+\delta_l} f(\rho)^{-l-1}, & k>r \end{cases}$$

where $\delta_l = 2$ if l is even and $\delta_l = 3$ if l is odd. By applying (6.13) to (6.7), we obtain the lemma. □

6.3. Estimation of derivatives of $F_{a,\epsilon}(\tau, \rho; s)$

In the proof of Proposition 5.1, the main estimate of $\partial_\tau^{n'}\partial_\rho^{n} K_{a,\epsilon}^{(1)}(\tau,\rho)$ is reduced to estimate of derivatives of $F_{a,\epsilon}(\tau,\rho;s)$, which is decomposed into two parts by (5.5). Recall that the first part is given by the sum of

$$\partial_\rho^{k-1-j}\partial_\tau^{n'} Q_j^+(\tau,\rho;s) = \partial_\rho^{k-1-j}\partial_\tau^{n'}\left(\mu' e^{-\mu\tau s}\partial_\rho^j \frac{1}{D_\epsilon(\rho;s,v)}|_{v=\mu}\right).$$

We will estimate of the growths of them evaluated at $s = f(\rho)$ and of its integrals in s.

Lemma 6.6. *Suppose that f satisfies exponentially flat condition* (2.2) *at the origin. Then*

$$\partial_s^p \partial_\rho^q \partial_\tau^{n'} Q_j^{\pm}(\tau,\rho;s)|_{s=f(\rho)} \sim \rho^{-(2m+1)(q+j+1)+n'+1} f(\rho)^{-p-1}$$

as $\rho \to 0+$.

Proof. We have

(6.14)

$$\partial_s^p \partial_\rho^q \partial_\tau^{n'} Q_j^\pm$$
$$\sim \sum_{\substack{p_0+p_1+p_2=p \\ q_1+q_2=q}} C_{\substack{p_0,p_1,p_2 \\ q_1,q_2}} (-s)^{n'-p_0} \partial_s^{p_1} \partial_\rho^{q_1} (\mu' \mu^{n'} e^{\mp\mu\tau s}) \partial_s^{p_2} \partial_\rho^{q_2} \left(\partial_\rho^j \frac{1}{D_\epsilon}|_{v=\pm\mu} \right).$$

Since

(6.15)

$$\partial_s^{p_1} \partial_\rho^{q_1} (\mu' \mu^{n'} e^{-\mu\tau s})$$
$$= (-\tau)^{p_1} \left((\mu' \mu^{n'+p_1})^{(q_1)} + \sum_{l=1}^{q_1} (\mu' \mu^{n'+p_1})^{(q_1-l)} \sum_{r=1}^{l} \sum_{|\beta|=l} (-\tau s)^r \mu^{(\beta_1)} \cdots \mu^{(\beta_r)} \right) e^{-\mu\tau s}$$
$$\sim \sum_{r=0}^{q_1} (-\tau)^{p_1+r} s^r \rho^{-(2m+1)(q_1+1)+n'+p_1+r+1} f(\rho)^{-n'-p_1-r-1} e^{-\mu\tau s}$$

for $q_1 > 0$ and by Lemma 6.2

$$\text{(6.16)} \qquad \partial_s^{p_2} \partial_\rho^{q_2} \left(\partial_\rho^j \frac{1}{D_\epsilon}|_{v=\mu} \right)|_{s=f(\rho)}$$
$$= \sum_{j'=1}^{q_2} \sum_{j''=1}^{j'} (\partial_s^{p_2} \partial_v^{j''} \partial_\rho^{q_2+j-j'} \frac{1}{D_\epsilon})|_{v=\mu, s=f(\rho)} \sum_{|\alpha|=j'} C_\alpha \mu^{(\alpha_1)} \cdots \mu^{(\alpha_{j''})}$$
$$+ (\partial_s^{p_2} \partial_\rho^{q_2+j} \frac{1}{D_\epsilon})|_{v=\mu, s=f(\rho)}$$
$$\sim \rho^{-(2m+1)(q_2+j)} f(\rho)^{-p_2}$$

for $q_2 > 0$ we have

$$\text{(6.17)} \qquad \partial_s^p \partial_\rho^q \partial_\tau^{n'} Q_j^\pm(\tau, \rho; s)|_{s=f(\rho)} \sim \rho^{-(2m+1)(q+j+1)+1} f(\rho)^{-p-1}.$$

It is desired estimate. □

When we integrate derivatives of $Q_j^\pm(\tau, \rho; s)$ in s, a replacement of (6.16), which is not evaluated at $s = f(\rho)$, is needed.

Lemma 6.7. *There exist continuous functions $L_{(p,q),k}(\rho; s)$ on $(0, \rho_0) \times \mathbb{R}_+$, which are bounded as $\rho \to 0+$, such that*

$$\partial_v^p \partial_\rho^q \frac{1}{D_\epsilon(\rho; s, v)}|_{v=\mu} \sim \rho^{-p-(2m+1)q} \sum_{k=0}^{q} s^{p+k} f(\rho)^{-k} L_{(p,q),k}(\rho; s) \frac{1}{D_\epsilon}|_{v=\mu}.$$

Proof. By Lemma 6.1 and Lemma 6.3 we have

$$\text{(6.18)} \qquad \frac{\partial_v^p \partial_\rho^q D_\epsilon}{D_\epsilon}|_{v=\mu} \sim \rho^{-p-q} \sum_{j=0}^{q} s^{p+j} f(\rho)^{-j} L_{p,j}(\rho; s),$$

where

$$L_{p,j}(\rho;s)=\begin{cases}\hat{D}_{p,0}|_{v=\mu} & j=0\\ \hat{D}_{p+j,0}|_{v=\mu}+\sum_{l=0}^{j-1}\rho^{-2m(q-l)}\hat{D}_{p+l,j-l}|_{v=\mu} & j\geq 1.\end{cases}\tag{6.19}$$

Then

$$\begin{aligned}\partial_v^p\partial_\rho^q\frac{1}{D_\epsilon}|_{v=\mu} &= \sum_{i=1}^{\max(p,q)}\sum_{\substack{p_1+\cdots+p_i=p\\ q_1+\cdots+q_i=q}}\frac{\partial_v^{p_1}\partial_\rho^{q_1}D_\epsilon}{D_\epsilon}\cdots\frac{\partial_v^{p_i}\partial_\rho^{q_i}D_\epsilon}{D_\epsilon}\frac{1}{D_\epsilon}|_{v=\mu}\\ &\sim \sum_{k=0}^{q}\rho^{-p-(2m+1)q}f(\rho)^{-k}\,s^{p+k}\,L_{(p,q),k}(\rho;s),\end{aligned}\tag{6.20}$$

where

$$\begin{aligned}&L_{(p,q),k}(\rho;s)\\ &=\rho^{2mq}\sum_{i=1}^{\max(p,q)}\sum_{\substack{p_1+\cdots+p_i=p\\ q_1+\cdots+q_i=q}}\sum_{\substack{j_1+\cdots+j_i=k\\ 0\leq j_1\leq q_1,\ldots,0\leq j_i\leq q_i}}L_{p_1,j_1}\cdots L_{p_i,j_i}\\ &\sim\left(\sum_{i=1}^{\max(p,q)}\sum_{p_1+\cdots+p_i=p}\sum_{\substack{j_1+\cdots+j_i=k\\ 0\leq j_1\leq q_1,\ldots,0\leq j_i\leq q_i}}\hat{D}_{p_1+j_1,0}\cdots\hat{D}_{p_i+j_i,0}+O(\rho)\right)\end{aligned}$$

as $\rho\to 0+$ and $|\hat{D}_{p_1+j_1,0}\cdots\hat{D}_{p_i+j_i,0}|\lesssim 1$. It completes the proof. $\square$

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