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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR A LOGARITHMIC WAVE EQUATION ARISING FROM Q-BALL DYNAMICS

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Abstract. In this paper we investigate an initial boundary value problem of a logarithmic wave equation. We establish the global existence of weak solutions to the problem by using Galerkin method, logarithmic Sobolev inequality and compactness theorem.

1. Introduction

In this paper we study the global existence of weak solutions for the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u - u\log|u|^2 + u_t + |u|^2 u = 0, \ (x,t) \in \Omega \times (0,T), \\ u = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, u is a complex scalar field. This model arises from the study of Q-ball dynamics in theoretical physics (see [18]). This type of problems have many applications in many branches of physics such as nuclear physics, optics and geophysics [5, 9, 19]. The model (1.1) is closely related to the following equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \Delta u + u - \varepsilon u\log|u|^2 = 0, \ (x,t) \in \mathcal{O} \times (0,T), \\ u = 0, \ (x,t) \in \partial\mathcal{O} \times (0,T), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), \ x \in \mathcal{O}, \end{cases} \tag{1.2}$$

where $\mathcal{O}$ is a finite interval $[a, b]$, the parameter ε measures the force of the nonlinear interaction and the nonlinear effects in quantum mechanics are very

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small. The problem (1.2) is a relativistic version of logarithmic quantum mechanics introduced in [3,4] and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [15,16,21,22].

It is easy to see that the problem (1.2) can be see as one dimensional case of (1.1) without the first order derivative term and cubic term. In [12], by using compactness method Gorka obtained the global existence of weak solutions to the problem (1.2). In [2] Bartkowski and Gorka studied the corresponding Cauchy problem for (1.2) $\mathcal{O} = \mathbb{R}$ without boundary conditions. The global existence of weak solutions, classical solutions and the traveling wave were obtained.

The model (1.1) is introduced in [18] for studying the dynamics of Q-ball in theoretical physics. The logarithmic nonlinearity is of much interest in physics, since it appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics [1,10,20].

The main difference between our work and [12] lies in: our problem is in three dimensional case and involves another nonlinear term $|u|^2u$; there is no restrictions on the coefficient of the logarithmic nonlinear term $u\log|u|^2$.

Recently in [18] a numerical study of the model (1.1) is given. However, there is no theoretical analysis for the problem. The purpose of this paper is to give a mathematical analysis for the problem (1.1). We mainly establish the global existence of weak solutions to the problem (1.1). This can be realized in a few steps. Firstly we write the problem in a weak version. Secondly we construct approximate solutions by the Galerkin method. Finally we prove the convergence of the sequence of the approximate solutions. To get a priori estimates of the approximate solutions, we employ the Gross logarithmic Sobolev inequality and logarithmic Gronwall inequality, which are fundamental here.

We also mention some related mathematical work involving the logarithmic term in the literature. In [7] Thierry and Alain establish the existence and uniqueness of a solution for the corresponding Cauchy problem (1.1) in $\mathbb{R}^3$ without the first order term and the cubic nonlinear term. There have been some works on the logarithmic Schrödinger equation (for example, see [6,8,13, 14]).

The rest of our paper is organized as follows. In Section 2 we state the main result. Section 3 is devoted to the proof of the main result.

2. Main result

We use the following notations throughout this paper: denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\langle\cdot,\cdot\rangle$ the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|_p$ the L^p norm. We also use C to denote a universal positive constant may take different values in different places. We introduce the definition of weak solutions for the problem (1.1).

Definition 2.1. A function u defined on $[0,T]$ is called a weak solution of the problem (1.1) if $u \in C_w([0,T];H_0^1(\Omega))$, $u' \in C_w([0,T];L^2(\Omega)), u(0) =$

$u^0, u'(0) = u^1$ and u satisfies

$$\langle u''(t), \phi\rangle + (\nabla u, \nabla\phi) + (u, \phi) + (u', \phi) - (u \log |u|^2, \phi) + (|u|^2 u, \phi) = 0 \tag{2.1}$$

for a.e. $t \in [0, T]$ and all test functions $\phi \in H_0^1(\Omega)$.

Our main results read as follows.

Theorem 2.1. *Assume that $u^0(x) \in H_0^1(\Omega)$, $u^1(x) \in L^2(\Omega)$. Then, the problem* (1.1) *admits global weak solution defined on* $[0, T]$ *for any* $T > 0$.

3. Proof of Theorem 2.1

In this section we carry out the proof of Theorem 2.1. The proof is based on Galerkin method. To proceed the proof we need the Gross logarithmic Sobolev inequality and the logarithmic Gronwall inequality. For the convenience of the reader we state the results here.

Lemma 3.1. *Assume $v \in H_0^1(\Omega)$ and Ω is a bounded smooth domain in $\mathbb{R}^3$. Then, for any $a > 0$, it holds that*

$$\int_\Omega |v|^2 \log |v| \mathrm{d}x \le \left(\frac{3}{4} \log \frac{4a}{\mathrm{e}}\right) \|v\|_2^2 + \frac{a}{4}\|\nabla v\|_2^2 + \|v\|_2^2 \log \|v\|_2.$$

Proof. See [17]. □

Lemma 3.2. *Assume that $w(t)$ is nonnegative, $w(t) \in L^\infty(0, T)$, $w(0) \ge 0$, and it satisfies*

$$w(t) \le w(0) + a \int_0^t w(s) \log[a + w(s)]\mathrm{d}s,\ t \in [0, T],$$

where $a > 1$ is a positive constant. Then we have

$$w(t) \le (a + w(0))^{\mathrm{e}^{at}}, \quad t \in [0, T].$$

Proof. See [7, 11]. □

We use the standard Galerkin method to construct approximate solutions. Let $\{w_j\}_{j=1}^\infty$ be the eigenfunctions of the operator $A = -\triangle$ with zero Dirichlet boundary condition and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well-known that $\{w_j\}_{j=1}^\infty$ forms an orthonormal basis for $L^2(\Omega)$ as well as for $H_0^1(\Omega)$. Moreover, the linear span of $\{w_j\}_{j=1}^\infty$ is dense in $L^q(\Omega)$ for any $1 \le q \le 6$. Let $\mathcal{P}_k$ be the orthogonal projection of $L^2(\Omega)$ onto V_k = the linear span of $\{w_1, \ldots, w_k\}$, $k \ge 1$. Let $u_k(t) = \sum_{j=1}^k g_{k,j}(t) w_j$ be an approximate solution to (1.1) in V_k. Then $u_k(t)$ verifies the following system of ODEs:

$$\begin{aligned}&\langle u_k''(t), w_j\rangle + (\nabla u_k(t), \nabla w_j) + (u_k, w_j) + (u_k'(t), w_j) \\ &\quad - (u_k \log |u_k|^2, w_j) + (|u_k|^2 u_k, w_j) = 0,\end{aligned} \tag{3.1}$$

$$u_k(0) = \mathcal{P}_k u^0,\ u_k'(0) = \mathcal{P}_k u^1, \tag{3.2}$$

for $j = 1, \dots, k$. More specifically,

$$u_k(0) = \sum_{j=1}^{k} u_{k,j}(0) w_j,\ u_k'(0) = \sum_{j=1}^{k} u_{k,j}'(0) w_j,$$

where

$$u_{k,j}(0) = (u^0, w_j),\ u_{k,j}'(0) = (u^1, w_j),\ \ j = 1, \dots, k.$$

Obviously, $u_k(0) \to u^0$ strongly in $H_0^1(\Omega)$, $u_k'(0) \to u^1$ strongly in $L^2(\Omega)$ as $k \to \infty$. By using the Cauchy-Peano theorem, we know that the system (3.1)-(3.2) admits a solution $g_{k,j}(t) \in C^2[0, T_k]$ for every $k \geq 1$ and some $T_k > 0$. Then we can obtain an approximate solution $u_k(t)$ of the problem (1.1) over $[0, T_k]$.

Now we try to get the a priori estimate for the approximate solutions $u_k(t)$ of the problem (1.1).

Multiplying (3.1) by $g_{k,j}'(t)$ and summing with respect to j from 1 to k, we have

(3.3)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 - \int_\Omega |u_k(t)|^2 \log|u_k(t)|\mathrm{d}x + \frac{1}{4}\|u_k(t)\|_4^4\right] + \|u_k'(t)\|_2^2 = 0.$$

Integrating (3.3) over $(0, t), 0 < t \leq T_k$, we get

(3.4)
$$\begin{aligned}
&\frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 - \int_\Omega |u_k(t)|^2 \log|u_k(t)|\mathrm{d}x \\
&\quad + \frac{1}{4}\|u_k(t)\|_4^4 + \int_0^t \|u_k'(s)\|_2^2 \mathrm{d}s \\
&= \frac{1}{2}\|u_k'(0)\|_2^2 + \frac{1}{2}\|\nabla u_k(0)\|_2^2 + \|u_k(0)\|_2^2 - \int_\Omega |u_k(0)|^2 \log|u_k(0)|\mathrm{d}x + \frac{1}{4}\|u_k(0)\|_4^4 \\
&\leq C_0 - \int_\Omega |u_k(0)|^2 \log|u_k(0)|\mathrm{d}x,
\end{aligned}$$

where $C_0 = C(\|u^0\|_{H^1(\Omega)}, \|u^1\|_{L^2(\Omega)})$ is a positive constant. To deal with the last term in (3.4), we use the elementary inequality

$$|t^2 \log t| \leq C(1 + t^3), \quad \forall\, t > 0, \tag{3.5}$$

where $C > 0$ is a positive constant. Then by (3.5) we have

$$\begin{aligned}
\left| -\int_\Omega |u_k(0)|^2 \log|u_k(0)|\mathrm{d}x \right| &\leq C|\Omega| + C\int_\Omega |u_k(0)|^3 \mathrm{d}x \\
&\leq C(1 + \|u_k(0)\|_{H_0^1(\Omega)}^3) \\
&\leq C(1 + \|u^0\|_{H_0^1(\Omega)}^3).
\end{aligned} \tag{3.6}$$

Hence combining (3.4) and (3.6) gives

$$\frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 + \frac{1}{4}\|u_k(t)\|_4^4 + \int_0^t \|u_k'(s)\|_2^2 \mathrm{d}s$$

$$\leq C + \int_\Omega |u_k(t)|^2 \log |u_k(t)| \mathrm{d}x. \tag{3.7}$$

Now we use Gross Sobolev inequality in Lemma 3.1 to estimate the last term on the righthandside of (3.7) as follows

$$\int_\Omega |u_k(t)|^2 \log |u_k(t)| \mathrm{d}x$$

$$\leq \left(\frac{3}{4}\log\frac{4a}{\mathrm{e}}\right)\|u_k(t)\|_2^2 + \frac{a}{4}\|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 \log \|v\|_2. \tag{3.8}$$

Inserting (3.8) into (3.7), we have

$$\begin{aligned} &\frac{1}{2}\|u_k'(t)\|_2^2 + \left(\frac{1}{2} - \frac{a}{4}\right)\|\nabla u_k(t)\|_2^2 + \left(1 - \frac{3}{4}\log\frac{4a}{\mathrm{e}}\right)\|u_k(t)\|_2^2 \\ &\quad + \frac{1}{4}\|u_k(t)\|_4^4 + \int_0^t \|u_k'(s)\|_2^2 \mathrm{d}s \\ &\leq C + \|u_k(t)\|_2^2 \log \|u_k(t)\|_2. \end{aligned} \tag{3.9}$$

By taking $a = \frac{1}{4}$ in (3.9) we obtain

$$\begin{aligned} &\|u_k'(t)\|_2^2 + \|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 + \|u_k(t)\|_4^4 \\ &\leq C(1 + \|u_k(t)\|_2^2 \log \|u_k(t)\|_2). \end{aligned} \tag{3.10}$$

Noting that

$$u_k(t) = u_k(0) + \int_0^t u_k'(s)\mathrm{d}s,$$

we have

$$\begin{aligned} \|u_k(t)\|_2^2 &\leq 2\|u_k(0)\|_2^2 + 2T\int_0^t \|u_k'(s)\|_2^2 \mathrm{d}s \\ &\leq 2\|u_k(0)\|_2^2 + \max\{1, 2T\}\frac{1+C}{C}\int_0^t \|u_k'(s)\|_2^2 \mathrm{d}s. \end{aligned} \tag{3.11}$$

Then it follows from (3.10) and (3.11) that

$$\|u_k(t)\|_2^2 \leq A + B\int_0^t \|u_k(s)\|_2^2 \log \|u_k(s)\|_2 \mathrm{d}s, \tag{3.12}$$

where

$$A = 2\|u_k(0)\|_2^2 + \max\{1, 2T\}(1+C)T,\ B = \max\{1, 2T\}(1+C).$$

Noting $B \geq 1$, then by the logarithmic Gronwall inequality in Lemma 3.2, we get

$$\|u_k(t)\|_2^2 \leq (A+B)^{e^{Bt}} \leq C_T. \tag{3.13}$$

Therefore from (3.10) and (3.13) we conclude that

$$\|u_k'(t)\|_2^2 + \|\nabla u_k(t)\|_2^2 + \|u_k(t)\|_2^2 + \|u_k(t)\|_4^4 \leq C_T. \tag{3.14}$$

The estimate (3.14) implies that $T_k = T$ and

$$u_k \quad \text{is uniformly bounded in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{3.15}$$

$$u_k' \quad \text{is uniformly bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.16}$$

By a standard discussion, we obtain

$$u_k'' \quad \text{is uniformly bounded in} \quad L^\infty(0,T;H^{-1}(\Omega)). \tag{3.17}$$

Hence we can infer from (3.14)-(3.16) that there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$u_k \to u \quad \text{weakly star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{3.18}$$

$$u_k' \to u' \quad \text{weakly star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.19}$$

$$u_k'' \to u'' \quad \text{weakly star in} \quad L^\infty(0,T;H^{-1}(\Omega)). \tag{3.20}$$

Then using (3.18)-(3.20) and Aubin-Lions lemma we have

$$u_k \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{3.21}$$

which implies

$$u_k \to u \quad \text{a.e. in } (0,T)\times\Omega. \tag{3.22}$$

It follows from (3.22) that

$$u_k \log|u_k|^2 \to u\log|u|^2 \quad \text{a.e. in } (0,T)\times\Omega, \tag{3.23}$$

$$|u_k|^2 u_k \to |u|^2 u \quad \text{a.e. in } (0,T)\times\Omega. \tag{3.24}$$

Using (3.5) again we can estimate the logarithmic nonlinear term as

$$\begin{aligned}\int_\Omega |u_k \log|u_k|^2|^2 \mathrm{d}x &= 4\int_\Omega |u_k|^2(\log|u_k|)^2\mathrm{d}x\\ &\leq C|\Omega| + C\int_\Omega |u_k|^6\mathrm{d}x\\ &\leq C(\|u_k\|^6_{H_0^1(\Omega)}+1)\\ &\leq C.\end{aligned}$$

That is to say,

$$u_k\log|u_k|^2 \quad \text{is uniformly bounded in} \quad L^\infty(0,T;L^2(\Omega)).$$

Then there exists some function $\chi \in L^\infty(0,T;L^2(\Omega))$ such that

$$u_k\log|u_k|^2 \to \chi \quad \text{weakly star} \quad L^\infty(0,T;L^2(\Omega)).$$

In view of (3.23) and Lemma 3.1.5 in [23], we have

$$\chi = u\log|u|^2,$$

which implies

$$u_k\log|u_k|^2 \to u\log|u|^2 \quad \text{weakly star} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.25}$$

Next we show the convergence of nonlinear term $|u_k|^2u_k$. By Sobolev inequality, we have

$$\int_\Omega \big||u_k|^2u_k\big|^2\mathrm{d}x = \int_\Omega |u_k|^6\mathrm{d}x \le C\|u_k\|^6_{H^1_0(\Omega)} \le C.$$

Then there exists some function $\tilde\chi \in L^\infty(0,T;L^2(\Omega))$ such that

$$|u_k|^2u_k \to \tilde\chi \quad \text{weakly star} \quad L^\infty(0,T;L^2(\Omega)).$$

Using (3.24) and Lemma 3.1.5 in [23], we get

$$\tilde\chi = |u|^2u,$$

which concludes

$$|u_k|^2u_k \to |u|^2u \quad \text{weakly star} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.26}$$

Now, using the convergence (3.18)-(3.20), (3.25) and (3.26) we can pass to the limit in (3.1) to obtain

$$\langle u'', w_j\rangle + (\nabla u, \nabla w_j) + (u, w_j) + (u', w_j) - (u\log|u|^2, w_j) + (|u|^2u, w_j) = 0,\ \forall j.$$

Since the system $\{w_j\}_{i=1}^\infty$ is dense in $H^1_0(\Omega)$, we have

$$\langle u'', v\rangle + (\nabla u, \nabla v) + (u, v) + (u', v) - (u\log|u|^2, v) + (|u|^2u, v) = 0$$

for any $v \in H^1_0(\Omega)$. That is to say u satisfies the equation (1.1) in the weak sense.

In what follows, we check that u satisfies the initial condition. From (3.18) and (3.19), we have

$$u_k(0) \to u(0) \quad \text{weakly in } L^2(\Omega).$$

Since we have chosen $u_k(0)$ such that

$$u_k(0) \to u^0 \quad \text{strongly in } H^1_0(\Omega).$$

Therefore, we have

$$u(0) = u^0. \tag{3.27}$$

It can be inferred from (3.21) that

$$\langle u_k'', w_j\rangle \to \langle u'', w_j\rangle \text{ in } L^\infty(0,T),$$

which implies that

$$\langle u_k'(0), w_j\rangle \to \langle u'(0), w_j\rangle.$$

Noting that

$$u_k'(0) \to u^1 \quad \text{strongly in } L^2(\Omega),$$

then we have

$$u'(0) = u^1. \tag{3.28}$$

As a consequence of (3.27) and (3.28), the initial condition is satisfied. Therefore, the global existence of weak solutions to the problem (1.1) is established. Then the proof of Theorem 2.1 is complete.

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