Bull. Korean Math. Soc. **50** (2013), No. 1, pp. 263–273
http://dx.doi.org/10.4134/BKMS.2013.50.1.263

MEAN-VALUE PROPERTY AND CHARACTERIZATIONS OF SOME ELEMENTARY FUNCTIONS

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Abstract. A mean-value result, saying that the difference quotient of a differentiable function in a real interval is a mean value of its derivatives at the endpoints of the interval, leads to the functional equation

$$\frac{f(x)-F(y)}{x-y} = M\left(g(x), G(y)\right), \qquad x \neq y,$$

where M is a given mean and f, F, g, G are the unknown functions. Solving this equation for the arithmetic, geometric and harmonic means, we obtain, respectively, characterizations of square polynomials, homographic and square-root functions. A new criterion of the monotonicity of a real function is presented.

Introduction

In a recent paper [4] the following counterpart of the Lagrange mean-value theorem has been proved. *If a real function f defined on an interval $I \subset \mathbb{R}$ is differentiable, and f' is one-to-one, then there exists a unique mean function $M : f'(I) \times f'(I) \to f'(I)$ such that*

$$\frac{f(x)-f(y)}{x-y} = M\left(f'(x), f'(y)\right), \qquad x, y \in I,\ x \neq y.$$

One can show [5] that, in this equality, M is a power mean if and only if one of the following cases occurs:

M is the arithmetic mean, that is $M = \mathcal{A}$ where

$$\mathcal{A}(u, v) = \frac{u+v}{2}, \qquad u, v \in \mathbb{R},$$

and the function f is a quadratic polynomial;

M is the geometric, that is $M = \mathcal{G}$ where

$$\mathcal{G}(u, v) = \sqrt{uv}, \qquad u, v > 0,$$

and f is a homographic function;

Received July 28, 2011.
2010 *Mathematics Subject Classification.* Primary 39B22, 26A24, 26A48.
Key words and phrases. mean-value theorem, classical means, monotonic functions, quadratic function, homographic function, square root function, functional equation.

M is the harmonic, that is $M = \mathcal{H}$ where

$$\mathcal{H}(u,v) = \frac{2uv}{u+v}, \qquad u, v > 0,$$

and f is the square root of an affine function.

In the present paper, assuming that $M \in \{\mathcal{A}, \mathcal{G}, \mathcal{H}\}$, we consider the functional equation

$$(*) \qquad \frac{f(x) - F(y)}{x - y} = M\left(g(x), G(y)\right), \qquad x, y \in I, x \neq y,$$

where all functions $f, F, g, G : I \to \mathbb{R}$ are unknown.

In the auxiliary Section 1 we first observe *if the real functions* f, F *defined on a set* $I \subset R$ *of the cardinality grater than* 2 *satisfy the inequality*

$$\frac{f(x) - F(y)}{x - y} \geq 0, \qquad x, y \in I, x \neq y,$$

then both functions are nondecreasing. Moreover, if I *is an interval, the function* f *and* F *coincide at the continuity points of these functions* (Theorem 1). This new criterion of monotonicity appears to be very helpful in proving of the main results.

In Section 2, assuming that $M = \mathcal{A}$, we show that (without any regularity assumptions) the functions f, F, g, G satisfy the equation $(*)$ if and only if the functions f and F are equal to a quadratic polynomials and the functions $g = G$ is its derivative (Theorem 2).

In Section 3 we consider the case $M = \mathcal{G}$. We show that f, F, g, G satisfy the equation $(*)$ if and only if $f = F$ is either affine or homographic, and the functions g and G, up to a multiplicative constant, are equal to the derivative of f (Theorem 3).

In Section 4, assuming that $M = \mathcal{H}$, we prove that f, F, g, G satisfy the equation $(*)$ if and only if $f = F$ is either affine or square root of an affine function, and $g = G$ is the derivative of f (Theorem 4).

The idea of this paper is due to J. Aczél [1], who characterized the quadratic polynomials using only the mean value property of their derivatives (cf. also M. Kuczma [3] and J. Aczél and M. Kuczma [2]).

1. Difference quotient and a criterion of monotonicity

Lemma 1. *Let* $I \subset \mathbb{R}$ *be an arbitrary set such that* $\operatorname{card} I \geq 3$. *If the functions* $f, F : I \to \mathbb{R}$ *satisfy the inequality*

$$(1.1) \qquad \frac{f(x) - F(y)}{x - y} \geq 0, \qquad x, y \in I, x \neq y,$$

then, for all $x_1, x_2, x_3 \in I$ *such that* $x_1 < x_2 < x_3$,

$$(1.2) \qquad f(x_1) \leq F(x_2) \leq f(x_3) \quad \textit{and} \quad F(x_1) \leq f(x_2) \leq F(x_3);$$

in particular, the functions f *and* F *are nondecreasing.*

If inequality (1.1) *is sharp, then inequalities* (1.2) *are sharp and* f, F *are strictly increasing.*

Proof. Let $x_1, x_2, x_3 \in I$ be such that $x_1 < x_2 < x_3$. Taking $x := x_1$ and $y := x_2$ in (1.1) we get $f(x_1) \le F(x_2)$ and taking $x := x_3$ and $y := x_2$ in (1.1) we get $F(x_2) \le f(x_3)$ whence $f(x_1) \le F(x_2) \le f(x_3)$. Similarly, taking first $x := x_2$ and $y := x_1$; then $x := x_3$ and $y := x_2$ we get the remaining inequalities in (1.2). □

To show that the assumption $\operatorname{card} I \ge 3$ is indispensable, consider the following:

Example 1. Take $I := \{1, 2\}$ and $f, F : I \to \mathbb{R}$ defined by $f(1) = 3$, $f(2) = 2$ and $F(1) = 1$, $F(2) = 4$. Then

$$\frac{f(1) - F(2)}{1 - 2} = \frac{f(2) - F(1)}{2 - 1} = 1 > 0,$$

and the function f is not decreasing.

Theorem 1. *Let* $I \subset \mathbb{R}$ *be an interval. The functions* $f, F : I \to \mathbb{R}$ *satisfy the inequality* (1.2) :

$$\frac{f(x) - F(y)}{x - y} \ge 0, \qquad x, y \in I, x \ne y,$$

if and only if f *and* F *are nondecreasing and* $f(x) = F(x)$ *at every point of the continuity of one of these function.*

Proof. In view of Lemma 1, the functions f and F are nondecreasing. Denote by C_f the set of all continuity points of f. If $x \in C_f$ then, for all $s, t \in I$ such that $s < x < t$ in view of the first of inequalities of (1.2),

$$f(s) \le F(x) \le f(t).$$

Letting here s and t tend to x, and using the continuity of f, we hence get $f(x) = F(x)$. If $x \in C_F$ we argue similarly. The converse implication is obvious. □

Remark 1. Theorem 1 generalizes the classical criterion of the monotonicity of a differentiable function: the nonnegativity of derivative.

2. Mean-value property for the arithmetic mean

Theorem 2. *Let* $I \subset \mathbb{R}$ *be a set such that* $\operatorname{card} I > 3$. *The functions* $f, F, g, G : I \to \mathbb{R}$, *and satisfy the functional equation*

$$\frac{f(x) - F(y)}{x - y} = \mathcal{A}(g(x), G(y)), \qquad x, y \in I,\ x \ne y \tag{2.1}$$

if and only if

$$f(x) = F(x) = \frac{a}{2}x^2 + bx + d, \qquad g(x) = G(x) = ax + b, \qquad x \in I$$

for some $a, b, c, d \in \mathbb{R}$.

Proof. Assume that the functions f, F, g, G satisfy the equation (2.1). Then by the definition of $\mathcal{A}$,

$$f(x) - F(y) = \frac{g(x) + G(y)}{2}(x - y), \qquad x, y \in I,\ x \neq y. \tag{2.2}$$

Replacing here y by z we get

$$f(x) - F(z) = \frac{g(x) + G(z)}{2}(x - z), \qquad x, z \in I,\ x \neq z.$$

Subtracting the respective sides of these two equations we obtain

$$2[F(z) - F(y)] = [G(y) - G(z)]x + (z - y)g(x) + [zG(z) - yG(y)],$$

whence

$$g(x) = \frac{G(z) - G(y)}{z - y}x + \frac{2[F(z) - F(y)] - [zG(z) - yG(y)]}{z - y} \tag{2.3}$$

for all $x, y, z \in I$, $y \neq x \neq z$. Since in this formula $y, z \in I$, $y \neq z$, can be here chosen arbitrarily, it follows that

$$g(x) = ax + b, \qquad x \in I \tag{2.4}$$

for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 0$. Moreover the equation (2.3) implies that

$$\frac{G(z) - G(y)}{z - y} = a, \qquad y, z \in I,\ y \neq z, \tag{2.5}$$

and

$$2\frac{F(z) - F(y)}{z - y} - \frac{zG(z) - yG(y)}{z - y} = b, \qquad y, z \in I,\ y \neq z. \tag{2.6}$$

Equality (2.5) implies that, for some real c,

$$G(x) = ax + c, \qquad x \in I. \tag{2.7}$$

Since

$$\frac{zG(z) - yG(y)}{z - y} = \frac{G(z) - G(y)}{z - y}z + G(y),$$

from (2.6) and (2.7) we get

$$2\frac{F(z) - F(y)}{z - y} = az + b, \qquad y, z \in I,\ y \neq z,$$

whence, obviously,

$$F(x) = \frac{a}{2}x^2 + bx + d, \qquad x \in I \tag{2.8}$$

for some real d. Setting the functions g, G and F given by (2.4), (2.7) and (2.8) into the equation (2.2) we get

$$f(x) = \frac{a}{2}x^2 + \frac{b + c}{2}x + d + \frac{b - c}{2}y, \qquad x, y \in I,\ x \neq y.$$

Since the function on the right-hand side cannot depend on y, it follows that

$$c = b \tag{2.9}$$

and, consequently,

$$f(x) = \frac{a}{2}x^2 + bx + d, \qquad x \in I. \tag{2.10}$$

Moreover, from (2.7) and (2.9) we get

$$G(x) = ax + b, \qquad x \in I. \tag{2.11}$$

Since the functions f given by (2.10), F given by (2.8), g given by (2.4) and G given by (2.11) satisfy the equation (2.1), the proof is completed. □

3. Mean-value property for the geometric mean

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval. The functions $f, F : I \to \mathbb{R}$, and $g, G : I \to (0, \infty)$ satisfy the functional equation*

$$\frac{f(x) - F(y)}{x - y} = \mathcal{G}\left(g(x), G(y)\right), \qquad x, y \in I, \ x \neq y, \tag{3.1}$$

if and only if, one of the following cases occurs:

(i) *there are $p, q, r \in \mathbb{R}$, $p > 0$, $q > 0$, such that*

$$f(x) = F(x) = pqx + r, \quad g(x) = p^2, \quad G(x) = q^2, \qquad x \in I;$$

(ii) *there are $p, q, r, s \in \mathbb{R}$, $p > 0$, $q > 0$, such that*

$$f(x) = F(x) = s - \frac{pq}{x + r}, \quad g(x) = \frac{p^2}{(x + r)^2}, \quad G(x) = \frac{q^2}{(x + r)^2}, \qquad x \in I.$$

Proof. If the functions f, F, g, G satisfy the equation (3.1), then by the definition of $\mathcal{G}$,

$$f(x) - F(y) = (x - y)\sqrt{g(x)G(y)}, \qquad x, y \in I, \ x \neq y. \tag{3.2}$$

Replacing here y by z we get

$$f(x) - F(z) = (x - z)\sqrt{g(x)G(z)}, \qquad x, z \in I, \ x \neq z.$$

As the right-hand side of the equation (3.1) is positive, by Lemma 1 the function F (as well as f) is one-to-one. Subtracting the respective sides of these two equations we obtain

$$F(z) - F(y) = \left[\left(\sqrt{G(y)} - \sqrt{G(z)}\right)x + \left(z\sqrt{G(z)} - y\sqrt{G(y)}\right)\right]\sqrt{g(x)},$$

whence

$$\sqrt{g(x)} = \frac{1}{\frac{\sqrt{G(y)} - \sqrt{G(z)}}{F(z) - F(y)}x + \frac{z\sqrt{G(z)} - y\sqrt{G(y)}}{F(z) - F(y)}}, \qquad x, y, z \in I, y \neq x \neq z \tag{3.3}$$

for all $x, y, z \in I$, $y \neq x \neq z$. Since $y, z \in I$, $y \neq z$, can be here arbitrarily fixed, it follows that

$$g(x) = \frac{1}{(ax+b)^2}, \qquad x \in I \tag{3.4}$$

for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 > 0$. Now the equation (3.3) implies that

$$\frac{\sqrt{G(y)} - \sqrt{G(z)}}{F(z) - F(y)} = a, \qquad y, z \in I,\ y \neq z, \tag{3.5}$$

and

$$\frac{z\sqrt{G(z)} - y\sqrt{G(y)}}{F(z) - F(y)} = b, \qquad y, z \in I,\ y \neq z. \tag{3.6}$$

From (3.5) we have

$$\sqrt{G(y)} + aF(y) = \sqrt{G(z)} + aF(z), \qquad y, z \in I,\ y \neq z,$$

which implies that, for some $k \in \mathbb{R}$,

$$\sqrt{G(x)} + aF(x) = k, \qquad x \in I.$$

Similarly, from (3.6), we get

$$x\sqrt{G(x)} - bF(x) = m, \qquad x \in I.$$

The last two equations imply that, for some $c > 0$,

$$G(x) = \frac{c^2}{(ax+b)^2}, \qquad x \in I, \tag{3.7}$$

and

$$F(x) = \frac{nx+d}{ax+b}, \qquad x \in I \tag{3.8}$$

for some $n, d \in \mathbb{R}$. From (3.2) and (3.4) we obtain that

$$f(x) = \frac{kx+m}{ax+b}, \qquad x \in I \tag{3.9}$$

for some $k, m \in \mathbb{R}$. Setting the functions (3.4), (3.7), (3.8) and (3.9) into the equation (3.2) we get

$$a(k-n)xy - (ad - bk + c)x + (am - bn + c)y + b(m-d) = 0$$

for all $x, y \in I$, $x \neq y$. It follows that

$$a(k-n) = 0, \quad ad - bk + c = 0, \quad am - bn + c = 0, \quad b(m-d) = 0. \tag{3.10}$$

If $a = 0$, then $b \neq 0$, and this system of equations simplifies to

$$bk - c = 0, \quad bn - c = 0, \quad m - d = 0.$$

Hence

$$k = n = \frac{c}{b}, \quad m = d,$$

whence

$$f(x) = F(x) = \frac{c}{b^2}x + \frac{d}{b}, \qquad x \in I.$$

In this case, by (3.4) and (3.7), the functions g and G are constant; more precisely, we have

$$g(x) = \frac{1}{b^2}, \qquad G(x) = \left(\frac{c}{b}\right)^2, \qquad x \in I.$$

Since $\frac{c}{b^2} = \frac{1}{b} \cdot \frac{c}{b}$, setting

$$p := \frac{1}{b^2}, \quad q := \frac{c}{b}, \quad r := \frac{d}{b},$$

we obtain

$$f(x) = F(x) = pqx + r, \quad g(x) = p^2, \quad G(x) = q^2, \qquad x \in I.$$

If $b = 0$, then $a \neq 0$, and the system (3.10) becomes

$$k - r = 0, \quad ad + c = 0, \quad am + c = 0.$$

Hence

$$r = k, \quad m = d = -\frac{c}{a},$$

whence

$$f(x) = F(x) = \frac{k}{a} - \frac{c}{a^2}\frac{1}{x}, \qquad x \in I.$$

Replacing in his formula k by ad we get

$$f(x) = F(x) = d - \frac{c}{a^2}\frac{1}{x}, \qquad x \in I.$$

In this case, by (3.4) and (3.7), we have

$$g(x) = \frac{1}{a^2x^2}, \qquad G(x) = \frac{c^2}{a^2x^2}, \qquad x \in I.$$

Setting

$$p := \frac{1}{a}, \quad q := \frac{c}{a}, \quad s := \frac{k}{a},$$

we hence get

$$f(x) = F(x) = s - \frac{pq}{x}, \quad g(x) = \frac{p^2}{x^2}, \quad G(x) = \frac{q^2}{x^2}, \qquad x \in I.$$

If $ab \neq 0$, the from (3.10) we obtain

$$n = k = \frac{ad + c}{b}, \qquad m = d,$$

whence

$$f(x) = F(x) = \frac{c}{b}\frac{x}{ax + b} + \frac{d}{b} = d + \frac{c}{ab} - \frac{c}{a^2}\frac{1}{x + \frac{b}{a}}, \qquad x \in I.$$

From (3.4) and (3.7) we have

$$g(x) = \frac{1}{a^2} \frac{1}{\left(x + \frac{b}{a}\right)^2}, \qquad G(x) = \frac{c^2}{a^2} \frac{1}{\left(x + \frac{b}{a}\right)^2}, \qquad x \in I.$$

Therefore, setting

$$p := \frac{1}{a}, \quad q := \frac{c}{a}, \quad r := \frac{b}{a}, \quad s := d + \frac{c}{ab},$$

we hence obtain

$$f(x) = F(x) = s - \frac{pq}{x+r}, \qquad g(x) = \frac{p^2}{(x+r)^2}, \quad G(x) = \frac{q^2}{(x+r)^2}, \qquad x \in I.$$

Note that taking here $r = 0$ we get the formulas obtained in the previous case. Since the geometric mean is a positive function, (3.1) and Lemma 1 imply that f is strictly increasing. It follows that p and q has to be of the same sign. Of course, we can assume that both are positive. Since in each of these three cases, the functions f, F, g, G satisfy the equation (3.1), the proof is completed. □

4. Mean-value property for the harmonic mean

Theorem 4. *Let $I \subset \mathbb{R}$ be an arbitrary set such that* $\operatorname{card} I > 3$. *The functions $f, F : I \to \mathbb{R}$, and $g, G : I \to (0, \infty)$ satisfy the functional equation*

$$\frac{f(x) - F(y)}{x - y} = \mathcal{H}\left(g(x), G(y)\right), \qquad x, y \in I, \ x \neq y, \tag{4.1}$$

if and only if, one of the following cases occurs:

(i) *there are $p, q \in \mathbb{R}$, $p > 0$, such that*

$$f(x) = F(x) = px + q, \qquad g(x) = G(x) = p, \qquad x \in I;$$

(ii) *there are $p, q, r \in \mathbb{R}$, $p > 0$, such that*

$$f(x) = F(x) = 2p\sqrt{x+q} + r, \qquad g(x) = G(x) = \frac{p}{\sqrt{x+q}}, \qquad x \in I;$$

(iii) *there are $p, q, r \in \mathbb{R}$, $p > 0$, such that*

$$f(x) = F(x) = 2p\sqrt{q-x} + r, \qquad g(x) = G(x) = \frac{p}{\sqrt{q-x}} \qquad x \in I.$$

Proof. Assume that the functions $f, F, g, G : I \to \mathbb{R}$ satisfy the equation (4.1). Since the right-hand side of the equation (4.1) is positive, Theorem 1 implies that the functions f and F are strictly increasing and

$$F(x) = f(x), \qquad x \in J := C_f \cup C_F, \tag{4.2}$$

where C_f denotes the set of all continuity points of f. Thus, from (4.1) we get

$$\frac{f(x) - f(y)}{x - y} = \mathcal{H}\left(g(x), G(y)\right), \qquad x, y \in I, \ x \neq y,$$

whence, by the definition of $\mathcal{H}$, we get

$$\frac{x-y}{f(x)-f(y)} = \frac{\frac{1}{g(x)} + \frac{1}{G(y)}}{2}, \qquad x, y \in I,\ x \neq y.$$

Take arbitrary $u, v \in f(J)$. Setting $x := f^{-1}(u)$, $y := f^{-1}(v)$ in this equation we get

$$\frac{f^{-1}(u) - f^{-1}(v)}{u-v} = \frac{\frac{1}{g\circ f^{-1}(u)} + \frac{1}{G\circ f^{-1}(v)}}{2}, \qquad u, v \in f(J),\ u \neq v.$$

Since, obviously, $\operatorname{card} f(J) = \operatorname{card} J > 3$, applying Theorem 2 with I replaced by $f(J)$; f and F replaced by f^{-1}, g replaced by $\frac{1}{g\circ f^{-1}}$ and G replaced by $\frac{1}{G\circ f^{-1}}$ we conclude that there are $a, b, c, d \in \mathbb{R}$ such that, for some $a, b, c, d \in \mathbb{R}$.,

$$f^{-1}(u) = \frac{a}{2}u^2 + bu + d, \qquad u \in f(J), \tag{4.3}$$

and

$$\frac{1}{g\circ f^{-1}(u)} = \frac{1}{G\circ f^{-1}(u)} = au + b, \qquad u \in f(J). \tag{4.4}$$

If $a \neq 0$ we hence we get that, for every $x \in J$, either

$$f(x) = -\frac{b}{a} - \frac{1}{a}\sqrt{2ax + (b^2 - 2ad)} \ \text{ and } \ g(x) = G(x) = -\frac{1}{\sqrt{2ax + (b^2 - 2ad)}}$$

or

$$f(x) = -\frac{b}{a} + \frac{1}{a}\sqrt{2ax + (b^2 - 2ad)} \ \text{ and } \ g(x) = G(x) = \frac{1}{\sqrt{2ax + (b^2 - 2ad)}}.$$

Note that the first of these formulas must be omitted, as g (and G) has positive values. Since, by Lemma 1, f is increasing in the interval I, the set J is dense in the interval I, and

$$f(x) = -\frac{b}{a} + \frac{1}{a}\sqrt{2ax + (b^2 - 2ad)}, \qquad x \in J,$$

it follows that $J = I$. Hence, by (4.2),

$$f(x) = F(x) = -\frac{b}{a} + \frac{1}{a}\sqrt{2ax + (b^2 - 2ad)}, \qquad x \in I,$$

and, by (4.4)

$$g(x) = G(x) = \frac{1}{\sqrt{2ax + (b^2 - 2ad)}}, \qquad x \in I.$$

Assume that $a > 0$. Setting here

$$p := \frac{1}{\sqrt{2a}}, \qquad q := \frac{b^2 - 2ad}{2a}, \qquad r := -\frac{b}{a},$$

we obtain

$$f(x) = F(x) = 2p\sqrt{x+q} + r, \qquad g(x) = G(x) = \frac{p}{\sqrt{x+q}} \qquad x \in I,$$

where $p > 0$ and $q, r \in \mathbb{R}$ are arbitrary.

If $a < 0$, setting

$$p := \frac{1}{\sqrt{-2a}}, \qquad q := \frac{2ad - b^2}{2a}, \qquad r := -\frac{b}{a},$$

we obtain

$$f(x) = F(x) = 2p\sqrt{q - x} + r, \qquad g(x) = G(x) = \frac{p}{\sqrt{q - x}} \qquad x \in I,$$

where $p > 0$ and $q, r \in \mathbb{R}$ are arbitrary.

Now assume that $a = 0$. In view of (4.3),

$$f^{-1}(u) = bu + d, \qquad u \in f\,(J)\,.$$

Since f is strictly increasing, it follows that $b > 0$ and

$$f(x) = F(x) = \frac{1}{b}x - \frac{d}{b}, \qquad x \in I,$$

and, by (4.4),

$$g(x) = G\,(x) = \frac{1}{b}, \qquad x \in I.$$

Setting $p := \frac{1}{b}$ and $q := -\frac{d}{b}$ we obtain

$$f(x) = F(x) = px + q, \qquad g(x) = G\,(x) = p, \qquad x \in I,$$

where $p > 0$ and $q \in \mathbb{R}$ are arbitrary. □

5. Remark concerning on a relevant functional equation with five unknown functions

From Theorem 2 we obtain the following:

Corollary 1. *Let X be a set such that* $\operatorname{card} X > 3$ *and let $f, F, g, G, h : X \to \mathbb{R}$. Suppose that h is one-to-one. The functions f, F, g, G, h satisfy the functional equation*

$$\frac{f(x) - F(y)}{h(x) - h(y)} = \mathcal{A}\left(g(x), G(y)\right), \qquad x, y \in X,\ x \neq y$$

if and only if

$$f(x) = F(x) = \frac{a}{2}h\,(x)^2 + bh\,(x) + d, \qquad g(x) = G(x) = ah\,(x) + b, \qquad x \in I$$

for some $a, b, c, d \in \mathbb{R}$.

To prove it is enough to apply Theorem 1 to the functional equation

$$\frac{f \circ h^{-1}(u) - F \circ h^{-1}(u)}{u - v} = \mathcal{A}\left(g \circ h^{-1}(u), G \circ h^{-1}(v)\right), \quad u, v \in h\,(X)\,,\ u \neq v.$$

In a similar way one could formulate respective generalizations of Theorems 3 and 4.

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