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CONGRUENCES OF THE WEIERSTRASS $\wp(x)$ AND $\wp''(x)(x=\frac{1}{2},\frac{\tau}{2},\frac{\tau+1}{2})$-FUNCTIONS ON DIVISORS

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Abstract. In this paper, we find the coefficients for the Weierstrass $\wp(x)$ and $\wp''(x)(x=\frac{1}{2},\frac{\tau}{2},\frac{\tau+1}{2})$-functions in terms of the arithmetic identities appearing in divisor functions which are proved by Ramanujan ([23]). Finally, we reprove congruences for the functions $\mu(n)$ and $\nu(n)$ in Hahn's article [11, Theorems 6.1 and 6.2].

1. Introduction

In a series of articles [18] Liouville stated many identities for general functions satisfying certain parity conditions. When specialized these yield results of number-theoretic interest. The Liouville identities are equivalent to identities among elliptic functions. In this article we considered the Weierstrass $\wp$-functions and identities of the basic hypergeometric series. Let $\sigma_s(N)$ denote the sum of sth power of the positive divisors of N, and let $\sigma_s(0)=\frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann Zeta-function. Ramanujan ([23]) wrote several formulas for

$$\sigma_r(0)\sigma_s(N)+\sigma_r(1)\sigma_s(N-1)+\cdots+\sigma_r(N)\sigma_s(0). \tag{1}$$

Some of these convolution sums involving divisor functions had been considered earlier by Glaisher [8], [9], MacMahon [20, pp. 303–341], Melfi [21], Huard, Ou, Spearman and Williams [12], etc.

For $N,m,r,s,d\in\mathbb{Z}$ with $d,s>0$ and $r\geq 0$, we define some necessary divisor functions and infinite products for later use, which appear in many areas of number theory:

$$\sigma_{s,r}(N;m)=\sum_{\substack{d|N\\ d\equiv r \bmod m}} d^s,$$

$$\sigma(N):=\sigma_1(N)=\sum_{d|N} d,$$

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$$S_1 := \sum_{N \text{ odd}} \sigma_{1,1}(N;2)q^N,$$
$$S_2 := \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N;2)q^N,$$
$$(a;q)_\infty := (a)_\infty := \prod_{n\geq 0}(1-aq^n).$$

In Section 2, we state the coefficients for $\wp(\frac{\tau}{2})$, $\wp(\frac{\tau+1}{2})$ and $\wp(\frac{1}{2})$ dealing with the summation of odd divisors and even divisors. They permit us to obtain $\Delta(\tau)$ from $g_2(\tau)$ and $g_3(\tau)$ and to get the differences between roots.

In Section 3, it will be shown that the derivatives of the Weierstrass $\wp$-functions have the infinite q-series. We can retrieve the actual values of the coefficients belonging to q-series. Also, we introduce the tables about the coefficients of $\wp''(x)(x = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2})$-functions and also $(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2$, $(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2$ and $(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2$ for $0 \leq N \leq 51$. If r is a non-zero integer, we define the function $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^r.$$

Note that $p_{-1}(n) = p(n)$, the ordinary partition function. A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For example, if 1 is allowed to have 2 colors, say r(red), and g(green), then all colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, $1_r + 1_g$. Setting $p_{e,r}(n)$ and $p_{o,r}(n)$ denote the number of r-colored partitions into an even (respectively, odd) number of distinct parts, it is easy to see that

$$p_r(n) = p_{e,r}(n) - p_{o,r}(n),$$

when r is a positive integer. In [11, Theorems 6.1 and 6.2], Hahn considered congruences for the function $\mu(n)$ and $\nu(n)$ which was defined by $\sum_{n=0}^{\infty}\mu(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$ and $\sum_{n=0}^{\infty}\nu(n)q^n := \prod_{n=1}^{\infty}(1-q^{2n})^8(1+q^n)^8$.

Using the function $\wp''(\frac{\tau}{2},\tau)$ and $\wp''(\frac{\tau}{2},\tau) \times (\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2$, we prove

$$\nu(n-1) \equiv \tilde{\sigma_3}(n) \pmod 3 \quad \text{and} \quad \mu(3n-1) \equiv 0 \pmod 3$$

(Please see Remarks 3.9 and 3.12).

2. Divisor functions

N. J. Fine's list of identities of the basic hypergeometric series type appeared in [7]. While studying these identities, we found that some identities appeared more than once on the list, usually in similar form (see [3], [4]). In this section, we state two identities that appeared in [7, pp. 78–79]:

$$\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} = 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N\\ \omega \text{ odd}}} \omega, \tag{2}$$

$$\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \sum_{N \text{ odd}} \sigma(N)q^N. \tag{3}$$

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{H}$ the complex upper half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\},$$

and the Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \omega^{-2k}.$$

We used the notations $\wp(z)$ and G_{2k} instead of $\wp(z;\Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ has been fixed. Now, Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

Proposition 2.1 ([15, p. 251])**.** *Let* $e_1 = \wp(\frac{\tau}{2})$, $e_2 = \wp(\frac{1}{2})$ *and* $e_3 = \wp(\frac{\tau+1}{2})$, *where* $P_0 = \prod_{n=1}^{\infty}(1-q^{2n})$, $P_1 = \prod_{n=1}^{\infty}(1-q^{2n-1})$, $P_2 = \prod_{n=1}^{\infty}(1+q^{2n})$ *and* $P_3 = \prod_{n=1}^{\infty}(1+q^{2n-1})$. *Then,*

(a) $e_2 - e_1 = \pi^2 {P_0}^4 {P_3}^8$.
(b) $e_2 - e_3 = \pi^2 {P_0}^4 {P_1}^8$.
(c) $e_3 - e_1 = 2^4\pi^2 q {P_0}^4 {P_2}^8$.

Let us recall

$$\prod_{n=1}^{\infty}(1-q^{2n-1}) = \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{2n})},$$

$$\prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)}\frac{(1-q^{2n})}{(1-q^{4n})}, \tag{4}$$

$$\prod_{n=1}^{\infty}(1+q^{2n}) = \prod_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^{2n})}.$$

Equations (2), (3) and (4) suggest that

(5)

$$\begin{aligned}
\wp(\frac{\tau}{2}) &= -\frac{\pi^2}{3}\left(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} + 16\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}\right) \\
&= -\frac{\pi^2}{3}\Big(1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N\Big) \\
&= -\frac{\pi^2}{3}\Big(1 + 8\sum_{N \text{ odd}} q^N\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N\Big) \\
&= -\frac{\pi^2}{3}\Big(1 + 24\sum_{N=1}^{\infty} q^N\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega\Big) \\
&= -\frac{\pi^2}{3}\Big(1 + 24\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N\Big) \\
&= -\frac{\pi^2}{3}(1 + 24S_1 + 24S_2).
\end{aligned}$$

Similarly, the relations (2) and (3) yield the following arithmetic results [13], [14]:

(6)
$$\begin{aligned}
\wp(\frac{\tau+1}{2}) &= -\frac{\pi^2}{3}\left(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 32\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}\right) \\
&= -\frac{\pi^2}{3}(1 - 24S_1 + 24S_2),
\end{aligned}$$

(7)
$$\begin{aligned}
\wp(\frac{1}{2}) &= \frac{2\pi^2}{3}\left(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}\right) \\
&= \frac{2\pi^2}{3}(1 + 24S_2),
\end{aligned}$$

(8)
$$g_2(\tau) = \frac{4\pi^4}{9}[3(1+24S_2)^2 + 24^2S_1^2],$$

and

(9)
$$g_3(\tau) = \frac{8\pi^6}{27}[(1+24S_2)^3 - 24^2S_1^2(1+24S_2)].$$

We consider the formula for the modular discriminant $\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24} = g_2(\tau)^3 - 27g_3(\tau)^2$, where the Dedekind η-function is given by the infinite product $\eta(\tau) = q^{\frac{1}{12}}(q^2;q^2)_\infty$ ([26]). From (8) and (9) we see that

$$\begin{aligned}\Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2\\ &= \Big\{\frac{4\pi^4}{9}[3(1+24S_2)^2 + 24^2S_1^2]\Big\}^3\\ &\quad - 27\Big\{\frac{8\pi^6}{27}[(1+24S_2)^3 - 24^2S_1^2(1+24S_2)]\Big\}^2\\ &= 4096\pi^{12}S_1^2(-1+8S_1-24S_2)^2(1+8S_1+24S_2)^2.\end{aligned} \tag{10}$$

Calculating the differences between roots, writing them in terms of infinite sums, and using (5), (6) and (7), we get

(11)

$$\begin{aligned}\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}) &:= \pi^2\sum_{n=0} a_nq^n\\ &= \frac{2\pi^2}{3}(1+24S_2) + \frac{\pi^2}{3}(1+24S_1+24S_2)\\ &= \pi^2\Big(1 + 8\sum_{N\text{ odd}}\sigma_{1,1}(N;2)q^N + 24\sum_{N\geq 2\text{ even}}\sigma_{1,1}(N;2)q^N\Big),\end{aligned}$$

(12)

$$\begin{aligned}\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}) &:= \pi^2\sum_{n=0} b_nq^n\\ &= \frac{2\pi^2}{3}(1+24S_2) + \frac{\pi^2}{3}(1-24S_1+24S_2)\\ &= \pi^2\Big(1 - 8\sum_{N\text{ odd}}\sigma_{1,1}(N;2)q^N + 24\sum_{N\geq 2\text{ even}}\sigma_{1,1}(N;2)q^N\Big),\end{aligned}$$

and

$$\begin{aligned}\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}) &:= \pi^2\sum_{n=0} c_nq^n\\ &= -\frac{\pi^2}{3}(1-24S_1+24S_2) + \frac{\pi^2}{3}(1+24S_1+24S_2)\\ &= \pi^2\sum_{N\text{ odd}} 16\sigma_{1,1}(N;2)q^N.\end{aligned} \tag{13}$$

If we define

$$H(q) := \frac{1}{16}\left(\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^8 - 1\right) := \sum_{n=0} h(n)q^n,$$

then by

$$\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^8 = 1 + 16\sum_{N\geq 1} q^N\sum_{d|N}(-1)^d d^3 \tag{14}$$

as in [7, p. 77], we find that

$$H(q) = \sum_{N\geq 1} (\sigma_{3,0}(N;2) - \sigma_{3,1}(N;2))q^N. \tag{15}$$

Thus, we can obtain the table below for the coefficients appearing in the infinite sum with positive integers for $0 \leq N \leq 51$.

N	a_N	b_N	c_N	h_N	N	a_N	b_N	c_N	h_N
0	1	1	0	0	26	336	336	0	15386
1	8	−8	16	−1	27	320	−320	640	−20440
2	24	24	0	7	28	192	192	0	24424
3	32	−32	64	−28	29	240	−240	480	−24390
4	24	24	0	71	30	576	576	0	24696
5	48	−48	96	−126	31	256	−256	512	−29792
6	96	96	0	196	32	24	24	0	37447
7	64	−64	128	−344	33	384	−384	768	−37296
8	24	24	0	583	34	432	432	0	34398
9	104	−104	208	−757	35	384	−384	768	−43344
10	144	144	0	882	36	312	312	0	53747
11	96	−96	192	−1332	37	304	−304	608	−50654
12	96	96	0	1988	38	480	480	0	48020
13	112	−112	224	−2198	39	448	−448	896	−61544
14	192	192	0	2408	40	144	144	0	73458
15	192	−192	384	−3528	41	336	−336	672	−68922
16	24	24	0	4679	42	768	768	0	67424
17	144	−144	288	−4914	43	352	−352	704	−79508
18	312	312	0	5299	44	288	288	0	94572
19	160	−160	320	−6860	45	624	−624	1248	−95382
20	144	144	0	8946	46	576	576	0	85176
21	256	−256	512	−9632	47	384	−384	768	−103824
22	288	288	0	9324	48	96	96	0	1810412
23	192	−192	384	−12168	49	456	−456	912	−117993
24	96	96	0	16324	50	744	744	0	110257
25	248	−248	496	−15751	51	576	−576	1152	−137592

Therefore, we summarize the above results obtained from (11) to (14) as follows.

Proposition 2.2. *Let n be non-negative integers and let $\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}) =: \pi^2 \sum_{n=0} a_n q^n$, $\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}) := \pi^2 \sum_{n=0} b_n q^n$, $\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}) := \pi^2 \sum_{n=0} c_n q^n$ and $H(q) := \frac{1}{16}\left(\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^8 - 1\right) := \sum_{n=0} h(n)q^n$. The following assertions hold*:

(a) $a_n - b_n = c_n$.

(b) $a_{2n-1} = -b_{2n-1}$ *and* $a_{2n} = b_{2n}$ *and* $c_{2n} = 0$.
(c) $h(0) = 0$ *and* $h(n) = \sigma_{3,1}(n;2) - \sigma_{3,0}(n;2)$ $(n > 0)$.

By (11), (12), (13), (15) and Proposition 2.2, we see the following result.

Corollary 2.3. *Let* p *be any odd positive integer and* $n > 0$.
(a) $a_{2^n} = 24$ *and* $a_{2^n p} = 24(p+1)$.
(b) $a_p = 8(p+1)$ *and* $c_p = 16(p+1)$.

Ramanujan's theta functions $\varphi(q)$, $\psi(q)$ and $f(-q)$ [1, Entry 22, p. 36] are defined, for $|q| < 1$, by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{16}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n\frac{(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{17}$$

and

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n\frac{(3n+1)}{2}} = (q;q)_\infty. \tag{18}$$

Here, the product representations arise from the Jacobi triple product identity.

From (16), (17), (18), Proposition 2.1 and Proposition 2.2, we can deduce the following.

Remark 2.4. (a) $e_2 - e_1 = \pi^2\varphi^4(q) = \pi^2 \sum_{n=0} a_n q^n$.
(b) $e_2 - e_3 = \pi^2\varphi^4(-q) = \pi^2 \sum_{n=0} b_n q^n$.

Let us introduce the triangular numbers. It is immediate from the definitions of $\psi(q)$ and $\varphi(q)$ in (16) and (17), respectively, that if

$$\varphi^s(q) := \sum_{n=0}^{\infty} r_s(n) q^n \tag{19}$$

and

$$\phi^s(q) := \sum_{n=0}^{\infty} \delta_s(n) q^n, \tag{20}$$

then $r_s(n)$ and $\delta_s(n)$ are the number of representations of n as a sum of s square and s triangular numbers, respectively. Clearly, $r_s(0) = \delta_s(0) = 1$. Here, for each nonnegative integer n, the triangular number T_n is defined by

$$T_n := \frac{n(n+1)}{2}.$$

Proposition 2.5. *In* [11, Theorem 5.1(5.3)], *for each positive integer n, we have*

$$r_4(n) = 16\hat{\sigma}(\frac{n}{2}) + 8\hat{\sigma}(n),$$

with $\hat{\sigma}_s(n) = \sum_{d|n}(-1)^{\frac{n}{d}-1}d^s$.

Combining (11) with Remark 2.4(a) and applying (16) to (15), we can get the following.

Corollary 2.6. *If $r_4(n)$, $r_8(n)$ are defined by* (19)*, then*

(a)

$$r_4(n) = \begin{cases} 8\sigma_{1,1}(n;2) & n \text{ odd} \\ 24\sigma_{1,1}(n;2) & n \text{ even.} \end{cases}$$

(b)

$$r_8(n) = \begin{cases} 16\sigma_{3,1}(n;2) & n \text{ odd} \\ 16(\sigma_{3,0}(n;2) - \sigma_{3,1}(n;2)) & n \text{ even} \end{cases}$$

which is also described in [11, p. 17].

3. The coefficients of $\wp''$-functions

Formula (1) is appeared in several articles ([2], [6, p. 338], [9], [10], [16, p. 678], [17, p. 106], [19, p. 81], [22, p. 146], [23], [24, p. 115], and [27]). Using (5), (6), (7), and (8), we can obtain

$$\begin{aligned} 5\sigma_3(M) = \sigma_{1,1}(M;2) &+ 12\sum_{k=1}^{M-1}\sigma_{1,1}(k;2)\sigma_{1,1}(M-k;2) \\ &+ 4\sum_{k=1}^{M}\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2M-2k+1;2) \end{aligned} \tag{21}$$

(see [14]).

Glaisher [5, p. 300] and Ramanujan [23] proved that

$$\sigma(1)\sigma(2n-1)+\sigma(3)\sigma(2n-3)+\cdots+\sigma(2n-1)\sigma(1) = \frac{1}{8}[\sigma_3(2n)-\sigma_3(n)]. \tag{22}$$

From (21) and (22), we find that

$$\sum_{k=1}^{M-1}\sigma_{1,1}(k;2)\sigma_{1,1}(M-k;2) = \frac{1}{24}[11\sigma_3(M) - \sigma_3(2M) - 2\sigma_{1,1}(M;2)] \tag{23}$$

(see [14]).

In this section, we will find the coefficients of the Weierstrass $\wp''(z)$-functions using (23). Recall ([25], p. 63) that we have expressed $\wp''(\frac{1}{2},\tau)$, $\wp''(\frac{\tau}{2},\tau)$ and $\wp''(\frac{\tau+1}{2},\tau)$.

We see from (5), (6) and (7) that

$$\begin{aligned}\wp''(\frac{\tau}{2},\tau) &= 2(e_1-e_2)(e_1-e_3)\\ &= 2\Big[-\frac{\pi^2}{3}(1+24S_1+24S_2)-\frac{2\pi^2}{3}(1+24S_2)\Big]\\ &\quad\times\Big[-\frac{\pi^2}{3}(1+24S_1+24S_2)+\frac{\pi^2}{3}(1-24S_1+24S_2)\Big]\\ &= 32\pi^4 S_1(1+8S_1+24S_2)\\ &= 32\pi^4\sum_{M=1}^{\infty}\sigma_{1,1}(2M-1;2)q^{2M-1}\\ &\quad+256\pi^4\sum_{M,N=1}^{\infty}\sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N-1;2)q^{2(M+N-1)}\\ &\quad+768\pi^4\sum_{M,N=1}^{\infty}\sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N;2)q^{2(M+N)-1}.\end{aligned}\tag{24}$$

Then, using (23), we replace n by $n=2L-1$ to obtain

$$\begin{aligned}&\sum_{k=1}^{L-1}\sigma_{1,1}(2k-1)\sigma_{1,1}(2L-1-(2k-1))\\ &=\frac{1}{2}\sum_{k=1}^{2L-2}\sigma_{1,1}(k)\sigma_{1,1}(2L-1-k)\\ &=\frac{1}{48}(11\sigma_3(2L-1)-\sigma_3(4L-2)-2\sigma_{1,1}(2L-1;2))\\ &=\frac{1}{24}(\sigma_3(2L-1)-\sigma(2L-1)).\end{aligned}\tag{25}$$

We observe that σ_3 is multiplicative, that is,

$$11\sigma_3(2L+1)-\sigma_3(4L+2)=11\sigma_3(2L+1)-\sigma_3(2)\sigma_3(2L+1)=2\sigma_3(2L+1).\tag{26}$$

Comparing (23), (22), (26) with (24), we have

$$\wp''(\frac{\tau}{2},\tau)=32\pi^4\Big\{\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1}+\sum_{K=1}^{\infty}[\sigma_3(2K)-\sigma_3(K)]q^{2K}\Big\}.\tag{27}$$

From (2), (3) and (27) it is easy to see that

$$\begin{aligned}\wp''(\frac{\tau}{2},\tau) &= 32\pi^4 q\frac{(q^2;q^2)_\infty^{16}}{(q)_\infty^{8}}\\ &= 32\pi^4\Big(\sum_{n=1}^{\infty}\sigma_3(n)q^n-\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\Big).\end{aligned}\tag{28}$$

Summarizing the above results, obtained from (23) to (28) are as follow.

Theorem 3.1. *Let* $\wp''(\frac{\tau}{2},\tau) := \pi^4 \sum_{n=0} d_n q^n$ *and let* $2n = 2^l Q$ *be an integer with* Q *odd.*

(a) $\wp''(\frac{\tau}{2},\tau) = 32\pi^4 q \frac{(q^2;q^2)_\infty^{16}}{(q)_\infty^8}$.
(b) $d_0 = 0$.
(c) $d_{2n-1} = 32\sigma_3(2n-1)$.
(d) $d_{2^l Q} = 32 \cdot 8^l \sigma_3(Q)$.

And let us evaluate $\wp''(\frac{1}{2},\tau)$.

$$\begin{aligned}
\wp''(\frac{1}{2},\tau) &= 2(e_2-e_1)(e_2-e_3) \\
&= 2\Big[\frac{2\pi^2}{3}(1+24S_2)+\frac{\pi^2}{3}(1+24S_1+24S_2)\Big] \\
&\quad \times \Big[\frac{2\pi^2}{3}(1+24S_2)+\frac{\pi^2}{3}(1-24S_1+24S_2)\Big] \\
&= 2\pi^4\Big[-64\Big(\sum_{M=1}^{\infty}\sigma_{1,1}(2M-1;2)q^{2M-1}\Big)\Big(\sum_{N=1}^{\infty}\sigma_{1,1}(2N-1;2)q^{2N-1}\Big) \\
&\quad + \Big(1+24\sum_{L=1}^{\infty}\sigma_{1,1}(2L;2)q^{2L}\Big)\Big(1+24\sum_{K=1}^{\infty}\sigma_{1,1}(2K;2)q^{2K}\Big)\Big] \\
&= 2\pi^4 + 96\pi^4\sum_{L=1}^{\infty}\sigma_{1,1}(2L;2)q^{2L} \\
&\quad - 128\pi^4\sum_{M,N=1}^{\infty}\sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N-1;2)q^{2(M+N-1)} \\
&\quad + 1152\pi^4\sum_{K,L=1}^{\infty}\sigma_{1,1}(L;2)\sigma_{1,1}(K;2)q^{2(L+K)}.
\end{aligned}$$

Similarly, we can use the Glaisher's proof (22) for the term

$$\sum_{M,N=1}^{\infty}\sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N-1;2)$$

and (23) with $M = N+1$ for the term

$$\sum_{K,L=1}^{\infty}\sigma_{1,1}(L;2)\sigma_{1,1}(K;2).$$

At last, we get

$$\wp''(\frac{1}{2},\tau) = 2\pi^4 + \pi^4\sum_{N=1}^{\infty}[544\sigma_3(N) - 64\sigma_3(2N)]q^{2N}. \tag{29}$$

Theorem 3.2. *Let* $\wp''(\frac{1}{2},\tau) := \pi^4 \sum_{n=0} e_n q^n$ *and let* $2n = 2^l Q$ *be an integer with* Q *odd.*

(a) $\wp''(\frac{1}{2}, \tau) = 2\pi^4 \frac{(q^2;q^2)_\infty^{16}}{(q^4;q^4)_\infty^8}$.
(b) $e_0 = 2$.
(c) $e_{2^l Q} = [480\sigma_3(2^{l-1}) - 64 \cdot 8^l]\sigma_3(Q)$.

Since $\wp''(\frac{\tau+1}{2}, \tau)$ has as the same factor as $\wp''(\frac{\tau}{2}, \tau)$ except for the sign, we can track $\wp''(\frac{\tau}{2}, \tau)$ to get the formulae (30). By (12) and (13) we see that

$$\begin{aligned}
\wp''(\frac{\tau+1}{2}, \tau) &= 2(e_3 - e_1)(e_3 - e_2) \\
&= 2\Big[-\frac{\pi^2}{3}(1 - 24S_1 + 24S_2) + \frac{\pi^2}{3}(1 + 24S_1 + 24S_2)\Big] \\
&\quad \times \Big[-\frac{\pi^2}{3}(1 - 24S_1 + 24S_2) - \frac{2\pi^2}{3}(1 + 24S_2)\Big] \\
&= -32\pi^4 \sum_{M=1}^{\infty} \sigma_{1,1}(2M-1;2)q^{2M-1} \\
&\quad + 256\pi^4 \sum_{M,N=1}^{\infty} \sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N-1;2)q^{2(M+N-1)} \\
&\quad - 768\pi^4 \sum_{M,N=1}^{\infty} \sigma_{1,1}(2M-1;2)\sigma_{1,1}(2N;2)q^{2(M+N)-1}.
\end{aligned}$$

Next, by (22) and (23) we have that

(30)
$$\wp''(\frac{\tau+1}{2}, \tau) = -32\pi^4 \Big\{ \sum_{L=1}^{\infty} \sigma_3(2L-1)q^{2L-1} - \sum_{K=1}^{\infty} [\sigma_3(2K) - \sigma_3(K)]q^{2K} \Big\}.$$

Thus, we derive from (2), (3) and (30) that

$$\begin{aligned}
\wp''(\frac{\tau+1}{2}, \tau) &= -32\pi^4 q \frac{(q^4;q^4)_\infty^8 (q)_\infty^8}{(q^2;q^2)_\infty^8} \\
&= 32\pi^4 \Big(\sum_{n=1}^{\infty} (-1)^n \sigma_3(n) q^n - \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \Big).
\end{aligned}$$

Theorem 3.3. *Let $\wp''(\frac{\tau+1}{2}, \tau) := \pi^4 \sum_{n=0} f_n q^n$ and let $2n = 2^l Q$ be an integer with Q odd.*

(a) $\wp''(\frac{\tau+1}{2}, \tau) = -32\pi^4 q \frac{(q^4;q^4)_\infty^8 (q)_\infty^8}{(q^2;q^2)_\infty^8}$.
(b) $f_0 = 0$.
(c) $f_{2n-1} = -32\sigma_3(2n-1)$.
(d) $f_{2^l Q} = 32 \cdot 8^l \sigma_3(Q)$.

For $\wp''(\frac{1}{2}, \tau)$ in (29), the following corollary is satisfied.

Corollary 3.4. *Let N be any non-negative integer.*

(a) *If $N \equiv 2 \pmod 4$, then $e_N < 0$ and $e_N \equiv 0 \pmod{32}$.*

(b) *If* $N \equiv 0 \pmod 4$*, then* $e_N > 0$ *and* $e_N \equiv 0 \pmod{32}$*, where* $N > 1$.

Proof. In (29), if $N = 2k-1$ with $k \in \mathbb{N}$, then the coefficients of $q^{2(2k-1)}$ becomes like this:

$$\begin{aligned} e_{4k-2} &= 544\sigma_3(2k-1) - 64\sigma_3(2(2k-1)) \\ &= 544\sigma_3(2k-1) - 64\sigma_3(2)\sigma_3(2k-1) \\ &= -32\sigma_3(2k-1). \end{aligned}$$

So, e_{4k-2} always has a negative sign and $e_{4k-2} \equiv 0 \pmod{32}$.
But for $N = 2k$, the coefficients of $q^{2(2k)}$ is $544\sigma_3(2k) - 64\sigma_3(2(2k))$. Let $k = 2^{r_1}Q$ with $r_1 \geq 0$ and Q be odd. Then,

$$\begin{aligned} 544\sigma_3(2k) - 64\sigma_3(2(2k)) &= 544\sigma_3(2^{r_1+1}Q) - 64\sigma_3(2^{r_1+2}Q) \\ &= [544\sigma_3(2^{r_1+1}) - 64\sigma_3(2^{r_1+2})]\sigma_3(Q) \\ &= 32[-2\cdot 8^{r_1+2} + 15(8^{r_1+1} + 8^{r_1} + \cdots + 1)]\sigma_3(Q) \\ &= 32 \cdot \frac{8^{r_1+2} - 15}{7}\sigma_3(Q). \end{aligned}$$

Since $r_1 \geq 0$, so $544\sigma_3(2k) - 64\sigma_3(2(2k)) > 0$. And $544\sigma_3(2k) - 64\sigma_3(2(2k)) = 32[-2\cdot 8^{r_1+2} + 15(8^{r_1+1} + 8^{r_1} + \cdots + 1)]\sigma_3(Q)$ shows that $e_{4k} \equiv 0 \pmod{32}$. □

Now, let us investigate the relation of coefficients of $\wp''(\frac{\tau}{2}, \tau)$ and $\wp''(\frac{\tau+1}{2}, \tau)$ in (27) and (30), respectively.

Corollary 3.5. (a) *If* $N = 2k-1$*, then* $d_N = -f_N$*.*
(b) *If* $N = 2k$*, then* $d_N = f_N$*, where* $N > 1$ *and* $k \in \mathbb{N}$*.*

Finally, we can obtain given the table below for the coefficients appearing in the infinite sum with an positive integer $0 \leq N \leq 51$.

Now, we can consider the coefficients for $(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2$, $(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2$ and $(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2$ from (11), (12) and (13), respectively.

Theorem 3.6. *Let* $(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2 := \pi^4 \sum_{n=0} \alpha_n q^n$*,* $(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2 := \pi^4 \sum_{n=0} \beta_n q^n$ *and* $(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2 := \pi^4 \sum_{n=0} \gamma_n q^n$*. Then we get the following.*

(a)

$$\alpha_n = \begin{cases} 1 & n = 0 \\ 16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$

(b)

$$\beta_n = \begin{cases} 1 & n = 0 \\ -16\sigma_3(n) & n \text{ odd} \\ 256\sigma_3(\frac{n}{2}) - 16\sigma_3(n) & n \text{ even.} \end{cases}$$

N	$\frac{1}{32}d_N$	$\frac{1}{32}e_N$	$\frac{1}{32}f_N$	N	$\frac{1}{32}d_N$	$\frac{1}{32}e_N$	$\frac{1}{32}f_N$
0	0	$\frac{1}{16}$	0	26	17584	-2198	17584
1	1	0	-1	27	20440	0	-20440
2	8	-1	8	28	22016	2408	22016
3	28	0	-28	29	24390	0	-24390
4	64	7	64	30	28224	-3528	28224
5	126	0	-126	31	29792	0	-29792
6	224	-28	224	32	32768	4679	32768
7	344	0	-344	33	37296	0	-37296
8	512	71	512	34	39312	-4914	39312
9	757	0	-757	35	43344	0	-43344
10	1008	-126	1008	36	48448	98611	48448
11	1332	0	-1332	37	50654	0	-50654
12	1792	196	1792	38	54880	-6860	54880
13	2198	0	-2198	39	61544	0	-61544
14	2752	-344	2752	40	64512	8946	64512
15	3528	0	-3528	41	68922	0	-68922
16	4096	583	4096	42	77056	-9632	77056
17	4914	0	-4914	43	79508	0	-79508
18	6056	-757	6056	44	85248	9324	85248
19	6860	0	-6860	45	95382	0	-95382
20	8064	882	8064	46	97344	-12168	97344
21	9632	0	-9632	47	103824	0	-103824
22	10656	-1332	10656	48	114688	16324	114688
23	12168	0	-12168	49	117993	0	-117993
24	14336	1988	14336	50	126008	-15751	126008
25	15751	0	-15751	51	137592	0	-137592

(c)

$$\gamma_n = \begin{cases} 32\sigma_3(n) - 32\sigma_3(\frac{n}{2}) & n\ even \\ 0 & otherwise. \end{cases}$$

Proof. (a) It follows from (11) that

$$\begin{aligned} &(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2 \\ &= \Big[\pi^2\Big(1 + 8\sum_{N \text{ odd}} \sigma_{1,1}(N;2)q^N + 24\sum_{N\geq 2 \text{ even}} \sigma_{1,1}(N;2)q^N\Big)\Big]^2 \\ &= \pi^4\Big(1 + 8\sum_{n=1}^{\infty}\sigma_{1,1}(2n-1;2)q^{2n-1} + 24\sum_{k=1}^{\infty}\sigma_{1,1}(2k;2)q^{2k}\Big) \end{aligned}$$

$$\times \Big(1+8\sum_{m=1}^{\infty}\sigma_{1,1}(2m-1;2)q^{2m-1}+24\sum_{l=1}^{\infty}\sigma_{1,1}(2l;2)q^{2l}\Big)$$
$$=\pi^4\Big[1+16\sum_{n=1}^{\infty}\sigma_{1,1}(2n-1;2)q^{2n-1}+48\sum_{k=1}^{\infty}\sigma_{1,1}(2k;2)q^{2k}$$
$$+64\sum_{n,m=1}^{\infty}\sigma_{1,1}(2n-1;2)\sigma_{1,1}(2m-1;2)q^{2(n+m-1)}$$
$$+384\sum_{k,m=1}^{\infty}\sigma_{1,1}(2k;2)\sigma_{1,1}(2m-1;2)q^{2(k+m)-1}$$
$$+576\sum_{k,l=1}^{\infty}\sigma_{1,1}(2k;2)\sigma_{1,1}(2l;2)q^{2(k+l)}\Big].$$

It follows from (22) that

$$\sum_{n,m=1}^{\infty}\sigma_{1,1}(2n-1;2)\sigma_{1,1}(2m-1;2)q^{2(n+m-1)}$$
$$=\sum_{M=1}^{\infty}\sum_{k=1}^{M}\sigma_{1,1}(2k-1;2)\sigma_{1,1}(2(M-k)+1;2)q^{2M}$$
$$=\sum_{M=1}^{\infty}\frac{1}{8}[\sigma_3(2M)-\sigma_3(M)]q^{2M},$$

from (25) that

$$\sum_{k,m=1}^{\infty}\sigma_{1,1}(2k;2)\sigma_{1,1}(2m-1;2)q^{2(k+m)-1}$$
$$=\sum_{M=1}^{\infty}\sum_{n=1}^{M}\sigma_{1,1}(2n;2)\sigma_{1,1}(2(M-n)+1;2)q^{2M+1}$$
$$=\sum_{M=1}^{\infty}\frac{1}{24}[\sigma_3(2M+1)-\sigma_1(2M+1)]q^{2M+1},$$

and from (23) that

$$\sum_{k,l=1}^{\infty}\sigma_{1,1}(2k;2)\sigma_{1,1}(2l;2)q^{2(k+l)}$$
$$=\sum_{k,l=1}^{\infty}\sigma_{1,1}(k;2)\sigma_{1,1}(l;2)q^{2(k+l)}$$
$$=\sum_{M=1}^{\infty}\sum_{n=1}^{M}\sigma_{1,1}(n;2)\sigma_{1,1}(M+1-n;2)q^{2(M+1)}$$

$$= \frac{1}{24} \sum_{M=1}^{\infty} [11\sigma_3(M+1) - \sigma_3(2(M+1)) - 2\sigma_{1,1}(M+1;2)] q^{2(M+1)},$$

and thus we can obtain the desired result.

(b) In a similar manner like (a).

(c) Using (13) and (22), we find, upon direct calculation, that

$$\Big(\wp(\frac{\tau+1}{2}\Big) - \wp(\frac{\tau}{2}))^2 = \Big(\pi^2 \sum_{N \text{ odd}} 16\sigma_{1,1}(N;2) q^N\Big)^2$$
$$= 256\pi^4 \sum_{n,m=1}^{\infty} \sigma_{1,1}(2n-1;2)\sigma_{1,1}(2m-1;2) q^{2(n+m-1)}. \qquad \square$$

Thus, let us make a table about $\frac{1}{16}\alpha_N$, $\frac{1}{16}\beta_N$ and $\frac{1}{16}\gamma_N$ for $0 \le N \le 51$.

N	$\frac{1}{16}\alpha_N$	$\frac{1}{16}\beta_N$	$\frac{1}{16}\gamma_N$	N	$\frac{1}{16}\alpha_N$	$\frac{1}{16}\beta_N$	$\frac{1}{16}\gamma_N$
0	$\frac{1}{16}$	$\frac{1}{16}$	0	26	15386	15386	35168
1	1	−1	0	27	20440	−20440	0
2	7	7	16	28	24424	24424	44032
3	28	−28	0	29	24390	−24390	0
4	71	71	128	30	24696	24696	56448
5	126	−126	0	31	29792	−29792	0
6	196	196	448	32	37447	37447	65536
7	344	−344	0	33	37296	−37296	0
8	583	583	1024	34	34398	34398	78624
9	757	−757	0	35	43344	−43344	0
10	882	882	2016	36	53747	53747	96896
11	1332	−1332	0	37	50654	−50654	0
12	1988	1988	3584	38	48020	48020	109760
13	2198	−2198	0	39	61544	−61544	0
14	2408	2408	5504	40	87858	87858	129024
15	3528	−3528	0	41	68922	−68922	0
16	4679	4679	8192	42	67208	67208	154544
17	4914	−4914	0	43	79508	−79508	0
18	5299	5299	12112	44	93612	93612	172416
19	6860	−6860	0	45	95382	−95382	0
20	7986	7986	18048	46	85176	85176	194688
21	9632	−9632	0	47	103824	−103824	0
22	9324	9324	21312	48	131012	131012	229376
23	12168	−12168	0	49	117993	−117993	0
24	16324	16324	28672	50	110257	110257	252016
25	15751	−15751	0	51	137592	−137592	0

Corollary 3.7. *Let N be any non-negative integer. Then*

(a) $d_{2N-1} = 2\alpha_{2N-1}$.
(b) $f_{2N} = \gamma_{2N}$.

Hahn (see [11]) proved a congruence for the function $\nu(n)$ which is defined by

$$\sum_{n=0}^{\infty} \nu(n)q^n := \prod_{n=1}^{\infty}(1-q^{2n})^8(1+q^n)^8. \tag{31}$$

Thus $\nu(n)$ is the number of partitions of n into 16 colors, 8 appear at most once (say S_1), and 8 are even and appear at most once (say S_2), weighted by the parity of colors from the set S_2.

Proposition 3.8 ([11]). *If* $\nu(n)$ *is defined by* (31), *then*

$$\nu(n-1) \equiv \tilde{\sigma_3}(n) \pmod 3.$$

Remark 3.9. By (27) we see that

$$\wp''(\frac{\tau}{2}, \tau) = 32\pi^4\Big\{\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1} + \sum_{K=1}^{\infty}[\sigma_3(2K)-\sigma_3(K)]q^{2K}\Big\}$$

and

$$\nu(n-1) = \begin{cases} \sigma_3(n) & n \text{ odd}, \\ \sigma_3(n) - \sigma_3(\frac{n}{2}) & \text{otherwise}. \end{cases}$$

If n is odd, then $\nu(n-1) = \sigma_3(n) = \tilde{\sigma_3}(n)$. If $2n$ is even, then

$$\sigma_3(2n) - \sigma_3(n) - \tilde{\sigma_3}(2n) = \sigma_3(2n) - \sigma_3(n) - \sigma_3(2n) + 2^4\sigma_3(n) = 15\sigma_3(n).$$

Therefore, $\sigma_3(2n) - \sigma_3(n) = \nu(2n-1) \equiv \tilde{\sigma_3}(2n) \pmod 3$.
We can reprove Proposition 3.8.

We prove a congruence for the function $\mu(n)$ [11, 6.3] which is defined by

$$\sum_{n=0}^{\infty} \mu(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8. \tag{32}$$

It follows that

$$\mu(n) = \mu_e(n) - \mu_o(n),$$

where $\mu_e(n)$ and $\mu_o(n)$ are the number of 16-colored partitions into an even (respectively, odd) number of distinct parts, where all the parts of the latter eight colors are even.

Next, we can also retrieve $\mu(3n-1) \equiv 0 \pmod 3$ shown in [11].

Theorem 3.10. *Let* $\sum_{n=0}^{\infty}\mu(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$ *in* (32). *Then*

(a) $\mu(2n-1) \equiv \sigma_1(2n-1) \pmod 6$.
(b) $\mu(2n) \equiv (2n+1)[\sigma_1(2n) + \sigma_1(n)] \pmod 6$.

Proof. By (27) and Theorem 3.6(a), we can obtain:

(33)

$$\begin{aligned}
&\wp''(\frac{\tau}{2},\tau)\times(e_2-e_1)^2\\
&=32\pi^4\Big\{\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1}+\sum_{K=1}^{\infty}[\sigma_3(2K)-\sigma_3(K)]q^{2K}\Big\}\\
&\quad\times\pi^4\Big\{1+16\sum_{M=1}^{\infty}\sigma_3(2M-1)q^{2M-1}+16\sum_{M=1}^{\infty}[16\sigma_3(M)-\sigma_3(2M)]q^{2M}\Big\}\\
&=32\pi^8\Big\{\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1}+240\sum_{K,M=1}^{\infty}\sigma_3(2K-1)\sigma_3(M)q^{2(L+M)-1}\\
&\quad+16\sum_{L,M=1}^{\infty}\sigma_3(2L-1)\sigma_3(2M-1)q^{2(L+M-1)}+\sum_{K=1}^{\infty}[\sigma_3(2K)-\sigma_3(K)]q^{2K}\\
&\quad+\sum_{K,M=1}^{\infty}[272\sigma_3(2K)\sigma_3(M)-16\sigma_3(2K)\sigma_3(2M)-256\sigma_3(K)\sigma_3(M)]q^{2(M+K)}\Big\}.
\end{aligned}$$

(a) Let us pay attention to q^{2N-1} in (33):

$$\begin{aligned}
&\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1}+240\sum_{K,M=1}^{\infty}\sigma_3(2K-1)\sigma_3(M)q^{2(L+M)-1}\\
&\equiv\sum_{L=1}^{\infty}\sigma_3(2L-1)q^{2L-1}\pmod 6.
\end{aligned}\tag{34}$$

By the definition of $\mu(n)$, it means that $\mu(2n-1)\equiv\sigma_1(2n-1)\pmod 6$.

(b) Now, let us consider q^{2N} in (33). Since $\sigma_3(M)\equiv\sigma_1(M)\pmod 6$, the term with q^{2N} can be changed like this;

(35)

$$\begin{aligned}
S:=\ &16\sum_{L,M=1}^{\infty}\sigma_3(2L-1)\sigma_3(2M-1)q^{2(L+M-1)}+\sum_{K=1}^{\infty}[\sigma_3(2K)-\sigma_3(K)]q^{2K}\\
&+\sum_{K,M=1}^{\infty}[272\sigma_3(2K)\sigma_3(M)-16\sigma_3(2K)\sigma_3(2M)-256\sigma_3(K)\sigma_3(M)]q^{2(M+K)}\\
\equiv\ &16\sum_{L,M=1}^{\infty}\sigma_1(2L-1)\sigma_1(2M-1)q^{2(L+M-1)}+\sum_{K=1}^{\infty}[\sigma_1(2K)-\sigma_1(K)]q^{2K}\\
&+\sum_{K,M=1}^{\infty}[272\sigma_1(2K)\sigma_1(M)-16\sigma_1(2K)\sigma_1(2M)-256\sigma_1(K)\sigma_1(M)]q^{2(M+K)}\\
&\pmod 6.
\end{aligned}$$

Then, by (22)

$$16\sum_{L,M=1}^{\infty}\sigma_1(2L-1)\sigma_1(2M-1)q^{2(L+M-1)}=\sum_{K=1}^{\infty}2[\sigma_3(2K)-\sigma_3(K)]q^{2K},$$

and by [12, (4.4)],

$$\begin{aligned}&272\sum_{K,M=1}^{\infty}\sigma_1(2K)\sigma_1(M)q^{2(M+K)}\\&=272\sum_{K=1}^{\infty}\sum_{l=1}^{K}\sigma_1(l)\sigma_1(2(K-l+1))q^{2(K+1)}\\&=\frac{34}{3}\sum_{K=1}^{\infty}[2\sigma_3(2(K+1))-(6K+5)\sigma_1(2(K+1))\\&\qquad+8\sigma_3(K+1)-(12K+11)\sigma_1(K+1)]q^{2(K+1)}.\end{aligned}$$

Also by [12, (3.10)]

$$\begin{aligned}&-256\sum_{K,M=1}^{\infty}\sigma_1(K)\sigma_1(M)q^{2(M+K)}\\&=-256\sum_{K=1}^{\infty}\sum_{l=1}^{K}\sigma_1(l)\sigma_1(K+1-l)q^{2(K+1)}\\&=\sum_{K=1}^{\infty}-\frac{64}{3}[5\sigma_3(K+1)-(6K+5)\sigma_1(K+1)]q^{2(K+1)}.\end{aligned}$$

Lastly, we get

$$\begin{aligned}&-16\sum_{K,M=1}^{\infty}\sigma_1(2K)\sigma_1(2M)q^{2(M+K)}\\&=-16\sum_{K=1}^{\infty}\sum_{n=1}^{K}\sigma_1(2n)\sigma_1(2(K-n+1))q^{2(K+1)}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{n=1}^{K}\sigma_1(2n)\sigma_1(2(K-n+1))\\&=\sum_{n=1}^{2K+1}\sigma_1(n)\sigma_1(2(K+1)-n)-\sum_{n=1}^{K+1}\sigma_1(2n-1)\sigma_1(2(K+1)-2n+1)\\&=\frac{1}{12}[5\sigma_3(2(K+1))-(12K+11)\sigma_1(2(K+1))]\\&\quad-\frac{1}{8}[\sigma_3(2(K+1))-\sigma_3(K+1)],\end{aligned}$$

we have

$$\begin{aligned}&-16\sum_{K,M=1}^{\infty}\sigma_1(2K)\sigma_1(2M)q^{2(M+K)}\\ =&\sum_{K=1}^{\infty}\Big\{-\frac{4}{3}[5\sigma_3(2(K+1))-12K\sigma_1(2(K+1))-11\sigma_1(2(K+1))]\\ &\quad+2\sigma_3(2(K+1))-2\sigma_3(K+1)\Big\}q^{2(K+1)}.\end{aligned}$$

Again applying $\sigma_3(M) \equiv \sigma_1(M) \pmod 6$ to the results of the above calculations, we can ultimately get (35) like this:

$$\begin{aligned}S \equiv &\sum_{K=1}^{\infty}[-21\sigma_1(2(K+1))-52K\sigma_1(2(K+1))\\ &\quad-39\sigma_1(K+1)+8K\sigma_1(K+1)]q^{2(K+1)} \pmod 6.\end{aligned}$$

Then, we claim that

$$\mu(2n) \equiv (2n+1)[\sigma_1(2n)+\sigma_1(n)] \pmod 6 \tag{36}$$

for $n = K+1$. □

Corollary 3.11. *If $\mu(n)$ is defined by* (32)*, then*

$$\mu(3n-1) \equiv 0 \pmod 6.$$

Proof. Let us consider the odd and even cases in Theorem 3.10 for $3n-1$. From (34), $\mu(6n+5) \equiv \sigma_3(6n+5) \equiv \sigma_1(6n+5) \pmod 6$. Let $6n+5 = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}q_1^{f_1}q_2^{f_2}\cdots q_s^{f_s}$ with distinct primes $p_1 \equiv p_2 \equiv \cdots \equiv p_r \equiv 1 \pmod 6$ and $q_1 \equiv q_2 \equiv \cdots \equiv q_s \equiv -1 \pmod 6$. Because of $6n+5$, we have $f_1+f_2+\cdots+f_s \equiv 1 \pmod 2$.

Without loss of generality, suppose that $f_1 \equiv 1 \pmod 2$. Then,

$$\begin{aligned}\sigma_1(6n+5) &= \sigma_1(p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}q_1^{f_1}q_2^{f_2}\cdots q_s^{f_s})\\ &= \sigma_1(p_1)^{e_1}\sigma_1(p_2)^{e_2}\cdots\sigma_1(p_r)^{e_r}\sigma_1(q_1)^{f_1}\sigma_1(q_2)^{f_2}\cdots\sigma_1(q_s)^{f_s}\\ &\equiv 0 \pmod 6,\end{aligned}$$

since $1+q_1+{q_1}^2+\cdots+{q_1}^{f_1} \equiv 0 \pmod 6$. Thus, $\mu(6n+5) \equiv 0 \pmod 6$.

On the other hand, from (36) we evaluate that $\mu(6n+2) \equiv (6n+3)[\sigma_1(2(3n+1))+\sigma_1(3n+1)] \pmod 6$.

Let $3n+1 = 2^rQ$ with $r \geq 0$ and odd Q.

Then,

$$\sigma_1(2(3n+1))+\sigma_1(3n+1) = \sigma_1(2^{r+1}Q)+\sigma_1(2^rQ) = 2(3\cdot 2^r-1)\sigma_1(Q).$$

So,

$$\mu(6n+2) \equiv 6(2n+1)(3\cdot 2^r-1)\sigma_1(Q) \equiv 0 \pmod 6.$$

□

Remark 3.12. $\mu(3n-1) \equiv 0 \pmod 6$ shown by us induces that $\mu(3n-1) \equiv 0 \pmod 3$ which is also the Hahn's result in [11, Theorem 6.1].

References

[1] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989.

[2] M. Besgue, *Extrait d'une lettre de M. Besgue à M Liouville*, J. Math. Pures Appl. **7** (1862), 256.

[3] B. Cho, D. Kim, and J. K. Koo, *Divisor functions arising from q-series*, Publ. Math. Debrecen **76** (2010), no. 3-4, 495–508.

[4] ———, *Modular forms arising from divisor functions*, J. Math. Anal. Appl. **356** (2009), no. 2, 537–547.

[5] L. E. Dickson, *History of the Theory of Numbers. Vol. I: Divisibility and Primality*, Chelsea Publishing Co., New York New York, 1966.

[6] ———, *History of the Theory of Numbers. Vol. II: Diophantine analysis*, Chelsea Publishing Co., New York New York, 1966.

[7] N. J. Fine, *Basic hypergeometric series and applications*, American Mathematical Society, Providence, RI, 1988.

[8] J. W. L. Glaisher, *Expressions for the five powers of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **15** (1885), 33–36.

[9] ———, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1–20.

[10] ———, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884).

[11] H. Hahn, *Convolution sums of some functions on divisors*, Rocky Mountain J. Math. **37** (2007), no. 5, 1593–1622.

[12] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Willians, *Elementary evaluation of certain convolution sums involving divisor functions*, Number theory for the millennium, II (Urbana, IL, 2000), 229274, A K Peters, Natick, MA, 2002.

[13] D. Kim and M. Kim, *Divisor functions and Weierstrass functions arising from q series*, Bull. Korean Math. Soc. **49** (2012), no. 4, 693–704.

[14] D. Kim, H. Park, and A. Kim, *Arithmetic identities arising from the basic hypergeometric series and elliptic curve*, Submitted.

[15] S. Lang, *Elliptic Functions*, Addison-Wesly, 1973.

[16] D. H. Lehmer, *Selected papers, Vol. II*, Charles Babbage Research Centre, St. Pierre, Manitoba, 1981.

[17] ———, *Some functions of Ramanujan*, Math. Student **27** (1959), 105–116.

[18] J. Liouville, *Sur quelques formules generales qui peuvent etre utiles dans la theorie des nombres*, Jour. de Math. (2) (1864), 389–400.

[19] J. Lützen, *Joseph Liouville 1809–1882*, Master of Pure and Applied Mathematics, Springer-Verlag, Berlin/Heidelberg, 1998.

[20] P. A. MacMahon, *Divisors of numbers and their continuations in the theory of partitions*, Proc. London Math. Soc. (2) **19** (1920), no. 1, 75–113.

[21] G. Melfi, *On some modular identities*, Number theory (Eger, 1996), 371–382, de Gruyter, Berlin, 1998.

[22] S. Ramanujan, *Collected papers Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, RI, 2000.

[23] ———, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184.

[24] R. A. Rankin, *Elementary proofs of relations between Eisenstein series*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 2, 107–117.

[25] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.
[26] ———, *The Arithmetic of Elliptic Curves*, Springer-Verlag, 1986.
[27] N. P. Skoruppa, *A quick combinatorial proof of Eisenstein series identities*, J. Number Theory **43** (1993), no. 1, 68–73.

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