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A FUBINI THEOREM FOR GENERALIZED ANALYTIC FEYNMAN INTEGRAL ON FUNCTION SPACE

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Abstract. In this paper we establish a Fubini theorem for generalized analytic Feynman integral and L_1 generalized analytic Fourier-Feynman transform for the functional of the form

$$F(x) = f(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_m, x\rangle),$$

where $\{\alpha_1, \ldots, \alpha_m\}$ is an orthonormal set of functions from $L^2_{a,b}[0,T]$. We then obtain several generalized analytic Feynman integration formulas involving generalized analytic Fourier-Feynman transforms.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. The concept of L_1 analytic Fourier-Feynman transform(FFT) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced an L_2 analytic FFT on Wiener space. In [9], Johnson and Skoug developed an L_p analytic FFT theory for $1 \le p \le 2$ which extended the results in [1, 2] and gave various relationships between the L_1 and the L_2 theories. In [7, 8], Huffman, Skoug and Storvick established Fubini theorems for various analytic Wiener and Feynman integrals. In [5], Chang and Lee extend the results of [7, 8] to a very general function space $C_{a,b}[0,T]$ and Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$.

The function space $C_{a,b}[0,T]$ induced by a generalized Brownian motion was introduced by J. Yeh in [12] and was used extensively by Chang and Chung [3]. In this paper, we establish a Fubini theorem for generalized analytic Feynman integral and L_1 generalized analytic Fourier-Feynman transform. We then establish the generalized analytic Feynman integration formulas involving L_1 generalized analytic Fourier-Feynman transforms.

The stochastic process used in this paper as well as in [3, 4, 5, 12], is non-stationary in time, is subject to a drift $a(t)$, and can be used to explain the

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position of the Ornstein-Uhlenbeck process in an external force field [11]. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega) = 0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \ldots, Y(t_n,\omega))$ is normally distributed with the density function

$$K(\vec{t},\vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\Big)^{-1/2} \cdot \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\bigg\},$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [13, pp. 18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [13, p. 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

Given two complex-valued measurable functions F and G on $C_{a,b}[0,T]$, F is said to be equal to G for scale almost everywhere(s-a.e.) if for each $\rho > 0$, $\mu(\{x \in C_{a,b}[0,T] : F(\rho x) \neq G(\rho x)\}) = 0$ [6, 10]. We write that $F \approx G$ if F=G for s-a.e..

Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \bigg\{v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty\bigg\},$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t)+|a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b} = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v,\phi_j)_{a,b}\phi_j(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists.

Now, we state the definition of the generalized analytic Feynman integral.

Definition 2.1. Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a measurable functional such that for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)d\mu(x)$$

exists. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that

$$\int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)d\mu(x) = \lim_{\lambda\to -iq} \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x),$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

The following notations are used throughout this paper:

$$A_{\alpha_j} \equiv \int_0^T \alpha_j(t)da(t) \quad \text{and} \quad B_{\alpha_j} \equiv \int_0^T \alpha_j^2(t)db(t)$$

for $\alpha_j \in L_{a,b}^2[0,T]$, $j = 1, \ldots, m$.

3. A Fubini theorem for generalized analytic Feynman integral

In this section we consider the functional of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_m, x\rangle), \tag{3.1}$$

where $\{\alpha_1, \ldots, \alpha_m\}$ is an orthonormal set of functions from $L_{a,b}^2[0,T]$. We then establish a Fubini theorem for generalized analytic Feynman integral. Finally, we obtain several generalized analytic Feynman integration formulas.

In [5], the authors investigated several Fubini theorems involving the generalized Feynman integral and the generalized Fourier-Feynman transforms of functionals in the Banach algebra $\mathcal{S}(L_{a,b}^2[0,T])$. The Banach algebra $\mathcal{S}(L_{a,b}^2[0,T])$ consists of the functionals which are stochastic Fourier transforms of complex-measures on $L_{a,b}^2[0,T]$. The results in [5] are characterized by the complex-measures corresponding to the functionals in $\mathcal{S}(L_{a,b}^2[0,T])$ (for instance, see [5, equation (3.12)]).

In this paper, we consider the cylinder functional F given by equation (3.1). In fact, the class of all cylinder functionals forms a dense subset in $L^2(C_{a,b}[0,T])$. In order to establish our Fubini theorem for the generalized analytic Feynman integral of the functional F given by (3.1), we use a weight function (3.3) which is a cylinder functional.

Throughout the rest of this paper, we will use the following conventions: for given $q_1, \ldots, q_n \in \mathbb{R} - \{0\}$, let

$$Q_n = \frac{q_1 \cdots q_n}{\sum_{j=1}^n \frac{q_1 \cdots q_n}{q_j}} \quad \text{where} \quad \sum_{j=1}^k \frac{q_1 \cdots q_k}{q_j} \neq 0 \quad \text{for all} \quad k = 2, \ldots, n, \tag{3.2}$$

and for simplicity, let

$$E_{q_1,\ldots,q_n}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, \cdot\rangle) = \exp\bigg\{ \bigg(-\frac{A_{\alpha_j}}{B_{\alpha_j}} + \frac{A_{\alpha_j}\sum_{j=1}^n(-iq_j)^{-1/2}}{B_{\alpha_j}(\sum_{j=1}^n(-iq_j)^{-1})^{1/2}} \bigg)(-iQ_n)^{\frac{1}{2}}\langle \alpha_j, \cdot\rangle - \frac{A_{\alpha_j}^2 \sum_{1\le l<k\le n}(-iq_l)^{-1/2}(-iq_k)^{-1/2}}{B_{\alpha_j}\sum_{j=1}^n(-iq_j)^{-1}} \bigg\} \tag{3.3}$$

for $j = 1, \ldots, m$.

We start this section by stating the following theorem which is a simple modification of [4, Theorem 4.1].

Theorem 3.1. *Let $q \in \mathbb{R} - \{0\}$ be given. Let F be given by equation* (3.1) *with*

$$\int_{\mathbb{R}^m} |f(\vec{u})| \exp\left\{ \left(\frac{1+|q|}{2}\right)^{\frac{1}{2}} \sum_{j=1}^{m} \frac{|A_{\alpha_j} u_j|}{B_{\alpha_j}} \right\} d\vec{u} < +\infty. \tag{3.4}$$

Then the generalized analytic Feynman integral with parameter q exists and is given by the formula

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x) d\mu(x) \\ &= \left(\prod_{j=1}^{m} \frac{-iq}{2\pi B_{\alpha_j}}\right)^{\frac{1}{2}} \int_{\mathbb{R}^m} f(\vec{u}) \exp\left\{ -\sum_{j=1}^{m} \frac{[\sqrt{-iq}u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}} \right\} d\vec{u}. \end{aligned} \tag{3.5}$$

In our next theorem, we establish a Fubini theorem for the generalized analytic Feynman integrals.

Theorem 3.2. *Let $q_1, q_2 \in \mathbb{R} - \{0\}$. Let F be given by equation* (3.1) *whose associated function f satisfies the condition* (3.4) *with $|q|$ replaced with* $\max\{|q_1|, |q_2|, |Q_2|\}$, *where Q_2 is given by equation* (3.2). *Then*

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} F(x+y) d\mu(x) d\mu(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(x+y) d\mu(y) d\mu(x). \end{aligned} \tag{3.6}$$

Also, all expressions in (3.6) *are given by the expression*

$$\begin{aligned} &\left(\prod_{j=1}^{m} \frac{-iQ_2}{2\pi B_{\alpha_j}}\right)^{\frac{1}{2}} \int_{\mathbb{R}^m} f(\vec{w}) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(w_j) \\ &\qquad \cdot \exp\left\{ -\sum_{j=1}^{m} \frac{[\sqrt{-iQ_2}w_j - A_{\alpha_j}]^2}{2B_{\alpha_j}} \right\} d\vec{w}. \end{aligned} \tag{3.7}$$

Proof. First, by using equations (3.1) and (3.5) with q replaced with q_2, we obtain that for all $\lambda_1 > 0$

(3.8)

$$\begin{aligned} &\int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} F(x + \lambda_1^{-1/2} y) d\mu(x) d\mu(y) \\ &= \int_{C_{a,b}[0,T]} \left[\left(\prod_{j=1}^{m} \frac{-iq_2}{2\pi B_{\alpha_j}}\right)^{\frac{1}{2}} \int_{\mathbb{R}^m} f(u_1 + \lambda_1^{-1/2}\langle \alpha_1, y\rangle, \dots, u_m + \lambda_1^{-1/2}\langle \alpha_m, y\rangle) \right. \end{aligned}$$

$$\cdot \exp\bigg\{-\sum_{j=1}^m \frac{[\sqrt{-iq_2}u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\}d\vec{u}\bigg] d\mu(y)$$
$$= \bigg(\prod_{j=1}^m \frac{-iq_2}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\bigg(\prod_{j=1}^m \frac{\lambda_1}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} f(\vec{u}+\vec{v})$$
$$\cdot \exp\bigg\{-\sum_{j=1}^m \frac{[\sqrt{-iq_2}u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}} - \sum_{j=1}^m \frac{[\sqrt{\lambda_1}v_j - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\}d\vec{u}d\vec{v}.$$

Let $u_j + v_j = w_j$ and $v_j = s_j$ for $j = 1, \dots, m$. Then we obtain that for all $\lambda_1 > 0$,

$$\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_2}} F(x+\lambda_1^{-1/2}y)d\mu(x)d\mu(y)$$
$$= \bigg(\prod_{j=1}^m \frac{-iq_2}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\bigg(\prod_{j=1}^m \frac{\lambda_1}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} f(\vec{w})$$
$$\cdot \exp\bigg\{-\sum_{j=1}^m \frac{[\sqrt{-iq_2}(w_j - s_j) - A_{\alpha_j}]^2}{2B_{\alpha_j}} - \sum_{j=1}^m \frac{[\sqrt{\lambda_1}s_j - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\}d\vec{w}d\vec{s}.$$

Carrying out the integration with respect to $s_1, \dots, s_m$ in the above expression, we obtain

(3.9)
$$\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_2}} F(x+\lambda_1^{-1/2}y)d\mu(x)d\mu(y)$$
$$= \bigg(\prod_{j=1}^m \frac{-iq_2}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\bigg(\prod_{j=1}^m \frac{\lambda_1}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\int_{\mathbb{R}^m} f(\vec{w})\exp\bigg\{-\sum_{j=1}^m \frac{[\sqrt{-iq_2}w_j - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\}$$
$$\cdot \bigg[\int_{\mathbb{R}^m}\exp\bigg\{-\sum_{j=1}^m \bigg(\frac{(-iq_2)+\lambda_1}{2B_{\alpha_j}}\bigg)s_j^2$$
$$+\sum_{j=1}^m \bigg(\frac{\sqrt{\lambda_1}A_{\alpha_j} + \sqrt{-iq_2}(\sqrt{-iq_2}w_j - A_{\alpha_j})}{B_{\alpha_j}}\bigg)s_j + \sum_{j=1}^m \bigg(-\frac{A_{\alpha_j}^2}{2B_{\alpha_j}}\bigg)\bigg\}d\vec{s}\bigg]d\vec{w}$$
$$= \bigg(\prod_{j=1}^m \frac{1}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}}\bigg(\frac{(-iq_2)\lambda_1}{(-iq_2)+\lambda_1}\bigg)^{\frac{m}{2}}$$
$$\cdot \int_{\mathbb{R}^m} f(\vec{w})\exp\bigg\{-\sum_{j=1}^m \frac{1}{2B_{\alpha_j}}\bigg(\bigg(\frac{(-iq_2)\lambda_1}{(-iq_2)+\lambda_1}\bigg)^{\frac{1}{2}}w_j - A_{\alpha_j}\bigg)^2\bigg\}$$
$$\cdot \exp\bigg\{\sum_{j=1}^m \bigg(-\frac{A_{\alpha_j}}{B_{\alpha_j}} + \frac{((-iq_2)^{-1/2}+\lambda_1^{-1/2})A_{\alpha_j}}{B_{\alpha_j}\sqrt{(-iq_2)^{-1}+\lambda_1^{-1}}}\bigg)\bigg(\frac{(-iq_2)\lambda_1}{(-iq_2)+\lambda_1}\bigg)^{\frac{1}{2}}w_j$$

$$-\sum_{j=1}^{m}\frac{((-iq_2)^{-1/2}\lambda_1^{-1/2})A_{\alpha_j}^2}{B_{\alpha_j}((-iq_2)^{-1}+\lambda_1^{-1})}\bigg\}d\vec{w}.$$

Let

$$\Gamma_{q_1}=\Big\{\lambda_1\in\tilde{\mathbb{C}}_+:\mathrm{Re}(\lambda_1^{1/2})\le\big((1+|q_1|)/2\big)^{1/2}\Big\}$$

and let $J^*(\lambda_1)$ be a function given by the last expression of (3.9) for all $\lambda_1\in\Gamma_{q_1}$. Then J^* is well-defined on the region Γ_{q_1} and is an analytic function of λ_1 throughout the domain $\mathrm{Int}(\Gamma_{q_1})$ so that $\int_\Delta J^*(\lambda_1)d\lambda_1=0$ for every rectifiable simple closed curve Δ lying in $\mathrm{Int}(\Gamma_{q_1})$. Thus by using the Fubini theorem and Morera's theorem, the last expression of (3.9) above is an analytic function of λ_1 throughout the domain $\mathrm{Int}(\Gamma_{q_1})$ and so letting $\lambda_1\to -iq_1$, we can see that the first expression of (3.6) is equal to the expression (3.7).

Next, using equation (3.5) with q replaced with Q_2, we obtain that

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}}F(z)\prod_{j=1}^{m}E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,z\rangle)d\mu(z)\\
&=\bigg(\prod_{j=1}^{m}\frac{-iQ_2}{2\pi B_{\alpha_j}}\bigg)^{\frac12}\int_{\mathbb{R}^m}f(\vec{w})\prod_{j=1}^{m}E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(w_j)\\
&\quad\cdot\exp\bigg\{-\sum_{j=1}^{m}\frac{[\sqrt{-iQ_2}w_j-A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\}d\vec{w}.\end{aligned}$$

Finally, using (3.5) with q replaced with q_1, (3.8) and (3.9) with q_2 replaced with q_1, respectively, and similar arguments as above, we can see that

$$\begin{aligned}&\int_{C_{a,b}[0,T]}\bigg[\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}}F(\lambda_2^{-1/2}x+y)d\mu(y)\bigg]d\mu(x)\\
&=\bigg(\prod_{j=1}^{m}\frac{1}{2\pi B_{\alpha_j}}\bigg)^{\frac12}\bigg(\frac{(-iq_1)\lambda_2}{(-iq_1)+\lambda_2}\bigg)^{\frac{m}{2}}\\
&\quad\cdot\int_{\mathbb{R}^m}f(\vec{w})\exp\bigg\{-\sum_{j=1}^{m}\frac{1}{2B_{\alpha_j}}\bigg(\bigg(\frac{(-iq_1)\lambda_2}{(-iq_1)+\lambda_2}\bigg)^{\frac12}w_j-A_{\alpha_j}\bigg)^2\bigg\}\\
&\quad\cdot\exp\bigg\{\sum_{j=1}^{m}\bigg(-\frac{A_{\alpha_j}}{B_{\alpha_j}}+\frac{((-iq_1)^{-1/2}+\lambda_2^{-1/2})A_{\alpha_j}}{B_{\alpha_j}\sqrt{(-iq_1)^{-1}+\lambda_2^{-1}}}\bigg)\bigg(\frac{(-iq_1)\lambda_2}{(-iq_1)+\lambda_2}\bigg)^{\frac12}w_j\\
&\qquad-\sum_{j=1}^{m}\frac{((-iq_1)^{-1/2}\lambda_2^{-1/2})A_{\alpha_j}^2}{B_{\alpha_j}((-iq_1)^{-1}+\lambda_2^{-1})}\bigg\}d\vec{w}\end{aligned}$$

is an analytic function of λ_2 through the domain $\mathrm{Int}(\Gamma_{q_2})$ where $\Gamma_{q_2}=\{\lambda_2\in\tilde{\mathbb{C}}_+:\mathrm{Re}(\lambda_2^{1/2})\le((1+|q_2|)/2)^{1/2}\}$, and so letting $\lambda_2\to -iq_2$, we can see that the third expression of (3.6) is equal to the expression (3.7).

The condition (3.4) with $|q|$ replaced with $\max\{|q_1|, |q_2|, |Q_2|\}$ will ensure the existence of all expressions in (3.6) and (3.7). □

Next corollary is easily obtained from Theorem 3.2.

Corollary 3.3. *Let F be as in Theorem 3.1. Then*

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x+y)d\mu(x)d\mu(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q/2}} F(z) \prod_{j=1}^{m} E_{q,q}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z).$$

In our next theorem, we establish a Fubini theorem for the multiple generalized analytic Feynman integral.

Theorem 3.4. *Let $q_1, \dots, q_n \in \mathbb{R} - \{0\}$. Let F be given by equation* (3.1) *whose associated function f satisfies the condition* (3.4) *with $|q|$ replaced with* $\max\{|q_1|, \dots, |q_n|, |Q_n|\}$ *where Q_n is given by equation* (3.2)*. Then*

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(x_1 + \cdots + x_n) d\mu(x_1) \cdots d\mu(x_n) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_n}} F(z) \prod_{j=1}^{m} E_{q_1,\dots,q_n}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z). \tag{3.10}$$

Proof. By using equations (3.2), (3.3) and (3.6), we obtain inductively that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(x_1 + \cdots + x_n) d\mu(x_1) \cdots d\mu(x_n) \\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_3}} \\
&\quad \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(y_1 + x_3 + \cdots + x_n) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, y_1\rangle) d\mu(y_1) d\mu(x_3) \cdots d\mu(x_n) \\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_4}} \\
&\quad \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_3}} F(y_2 + x_4 + \cdots + x_n) \prod_{j=1}^{m} E_{q_1,q_2,q_3}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, y_2\rangle) d\mu(y_2) d\mu(x_4) \cdots d\mu(x_n) \\
&= \cdots \\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_n}} F(z) \prod_{j=1}^{m} E_{q_1,\dots,q_n}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z).
\end{aligned}$$

Thus we have the desired result. □

Choosing $q_j = q$ for $j = 1, \dots, n$, we can easily obtain the following corollary of Theorem 3.4.

Corollary 3.5. *Let F be as in Theorem 3.1. Then*

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x_1 + \cdots + x_n) d\mu(x_1) \cdots d\mu(x_n)\\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q/n}} F(z) \prod_{j=1}^{m} E_{q,\dots,q}^{A_{\alpha_j}, B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z).\end{aligned}$$

Lemma 3.6. *Let $q_0 \in \mathbb{R} - \{0\}$. Let F be given by equation* (3.1) *with*

$$\int_{\mathbb{R}^m} |f(\vec{u})| \exp\Big\{ \Big(\frac{1+|q_0|}{2}\Big)^{\frac{1}{2}} \sum_{j=1}^{m} \frac{|A_{\alpha_j} u_j|}{B_{\alpha_j}} \Big\} d\vec{u} < +\infty.$$

Then for all real q and $\gamma > 0$ with $|\gamma q| \le |q_0|$,

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q}} F(x) d\mu(x) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(\gamma^{-1/2} x) d\mu(x). \tag{3.11}$$

Proof. By using equation (3.5) with q replaced with γq, we have that

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q}} F(x) d\mu(x)\\ &= \Big(\prod_{j=1}^{n} \frac{-i(\gamma q)}{2\pi B_{\alpha_j}}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{ -\sum_{j=1}^{n} \frac{[\sqrt{-i(\gamma q)} u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}} \Big\} d\vec{u}\\ &= \Big(\prod_{j=1}^{n} \frac{-iq}{2\pi B_{\alpha_j}}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\gamma^{-1/2}\vec{u}) \exp\Big\{ -\sum_{j=1}^{n} \frac{[\sqrt{-iq} u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}} \Big\} d\vec{u}\\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(\gamma^{-1/2} x) d\mu(x).\end{aligned}$$

□

Theorem 3.7. *Let F be as in Lemma 3.6. Then for all $q_1, q_2 \in \mathbb{R} - \{0\}$ and $\gamma, \tau > 0$ with $|q_1/\tau^2| \le |q_0|$, $|q_2/\gamma^2| \le |q_0|$ and $|q_1 q_2/(\gamma^2 q_1 + \tau^2 q_2)| \le |q_0|$,*

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} F(\gamma x + \tau y) d\mu(x) d\mu(y)\\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1 q_2/(\gamma^2 q_1 + \tau^2 q_2)}} F(z) \prod_{j=1}^{m} E_{q_2/\gamma^2, q_1/\tau^2}^{A_{\alpha_j}, B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z).\end{aligned}$$

Proof. By using equations (3.6) and (3.11), we have that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\text{anf}_{q_1}}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_2}} F(\gamma x+\tau y)d\mu(x)d\mu(y)\\
&= \int_{C_{a,b}[0,T]}^{\text{anf}_{q_1}}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_2/\gamma^2}} F(x+\tau y)d\mu(x)d\mu(y)\\
&= \int_{C_{a,b}[0,T]}^{\text{anf}_{q_2/\gamma^2}}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_1}} F(x+\tau y)d\mu(y)d\mu(x)\\
&= \int_{C_{a,b}[0,T]}^{\text{anf}_{q_2/\gamma^2}}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_1/\tau^2}} F(x+y)d\mu(y)d\mu(x)\\
&= \int_{C_{a,b}[0,T]}^{\text{anf}_{q_1q_2/(\gamma^2q_1+\tau^2q_2)}} F(z)\prod_{j=1}^{m} E_{q_2/\gamma^2,q_1/\tau^2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,z\rangle)d\mu(z).
\end{aligned}$$

Thus we have the desired result. □

In the following examples, we exhibit functionals F to apply our results above.

In Example 3.9 below, for simplicity, we will let $T=2$ and choose $a(t)=2t$ and $b(t)=t^2+t$ on $[0,2]$. Next, let

$$\alpha_1(t)=\begin{cases}\frac{1}{2}, & 0\le t\le 1\\ 0, & 1<t\le 2.\end{cases}$$

Then, it follows easily that $\|\alpha_1\|_{a,b}=1$, $A_{\alpha_1}=1$, $B_{\alpha_1}=1/2$.

Example 3.8. Let $F: C_{a,b}[0,2]\to\mathbb{R}$ be defined by the formula

$$F(x)=\exp\{-4\langle\alpha_1,x\rangle^2\}.$$

By using equation (3.6) and a direct calculation, we obtain

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\text{anf}_{Q_2}} F(z)E_{q_1,q_2}^{1,\frac{1}{2}}(\langle\alpha_1,z\rangle)d\mu(z)\\
&=\left(\frac{-iQ_2}{\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}} E_{q_1,q_2}^{1,\frac{1}{2}}(w)\exp\Big\{-4w^2-[\sqrt{-iQ_2}w-1]^2\Big\}dw
\end{aligned}$$

and so

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\text{anf}_{q_2}}\int_{C_{a,b}[0,T]}^{\text{anf}_{q_1}} F(x+y)d\mu(x)d\mu(y)\\
&=\left(\frac{-iQ_2}{\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}} E_{q_1,q_2}^{1,\frac{1}{2}}(w)\exp\Big\{-4w^2-[\sqrt{-iQ_2}w-1]^2\Big\}dw.
\end{aligned}$$

Let $\mathcal{M}(\mathbb{R}^m)$ denote the space of complex-valued Borel measures on $\mathcal{B}(\mathbb{R}^m)$. It is well known that a complex-valued Borel measure ν necessarily has a finite total variation $\|\nu\|$, and $\mathcal{M}(\mathbb{R}^m)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

For $\nu \in \mathcal{M}(\mathbb{R}^m)$, the Fourier transform $\hat{\nu}$ of ν is a complex-valued function defined on $\mathbb{R}^m$ by the formula

$$\hat{\nu}(\vec{u}) = \int_{\mathbb{R}^m} \exp\left\{ i \sum_{j=1}^{m} u_j v_j \right\} d\nu(\vec{v}),$$

where $\vec{u} = (u_1, \ldots, u_m)$ and $\vec{v} = (v_1, \ldots, v_m)$ are in $\mathbb{R}^m$.

Example 3.9. Let $\{\alpha_1, \ldots, \alpha_m\}$ be an orthonormal set of functions from $L^2_{a,b}[0,T]$. Define the functional $F : C_{a,b}[0,T] \to \mathbb{C}$ by

$$F(x) = \hat{\nu}(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_m, x\rangle),$$

where $\hat{\nu}$ is the Fourier transform of ν in $\mathcal{M}(\mathbb{R}^m)$.

Now by using equation (3.6) and a direct calculation, we obtain

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j}, B_{\alpha_j}}(\langle \alpha_j, z\rangle) d\mu(z)$$
$$= \int_{\mathbb{R}^m} \exp\left\{ -\frac{i}{2Q_2} \sum_{j=1}^{m} B_{\alpha_j} v_j^2 + i\left(\frac{i}{Q_2}\right)^{\frac{1}{2}} \frac{\sqrt{-iq_1} + \sqrt{-iq_2}}{\sqrt{(-iq_1) + (-iq_2)}} \sum_{j=1}^{m} A_{\alpha_j} v_j \right\} d\nu(\vec{v})$$

and hence

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(x+y) d\mu(x) d\mu(y)$$
$$= \int_{\mathbb{R}^m} \exp\left\{ -\frac{i}{2Q_2} \sum_{j=1}^{m} B_{\alpha_j} v_j^2 + i\left(\frac{i}{Q_2}\right)^{\frac{1}{2}} \frac{\sqrt{-iq_1} + \sqrt{-iq_2}}{\sqrt{(-iq_1) + (-iq_2)}} \sum_{j=1}^{m} A_{\alpha_j} v_j \right\} d\nu(\vec{v}).$$

4. An L_1 generalized analytic Fourier-Feynman transforms

In Section 3 above, we studied a Fubini theorem for generalized analytic Feynman integral. In this section, we establish a Fubini theorem for L_1 generalized analytic Fourier-Feynman transform(GFFT). We then establish the generalized analytic Feynman integration formulas involving L_1 analytic GFFT.

We state the definition of the L_1 analytic GFFT.

Definition 4.1. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(y+x) d\mu(x).$$

Then for $q \in \mathbb{R} - \{0\}$, the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F is defined by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$ whenever the limit exists. That is to say,

$$T_q^{(1)}(F)(y) = \int_{C_{a,b}[0,T]}^{\text{anf}_q} F(y+x)d\mu(x) \tag{4.1}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

We note that if $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(G) \approx T_q^{(1)}(F)$.

Now we present our results for the L_1 analytic GFFT and multiple L_1 analytic GFFT for the functional F without proof.

Theorem 4.2. *Let q and F be as in Theorem 3.1. Then the L_1 analytic GFFT of F, $T_q^{(1)}(F)$ exists and is given by the formula*

$$\begin{aligned} T_q^{(1)}(F)(y) = \bigg(\prod_{j=1}^{m} \frac{-iq}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}} &\int_{\mathbb{R}^m} f(\vec{u} + \langle \vec{\alpha}, y\rangle) \\ &\cdot \exp\bigg\{ -\sum_{j=1}^{m} \frac{[\sqrt{-iq}u_j - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\} d\vec{u} \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Theorem 4.3. *Let $q_1, \ldots, q_n$ and F be as in Theorem 3.4. Then the multiple L_1 analytic GFFT of F, $T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))$ exists and is given by the formula*

$$\begin{aligned} &T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))(y) \\ = &\prod_{l=1}^{n}\bigg(\prod_{j=1}^{m} \frac{-iq_l}{2\pi B_{\alpha_j}}\bigg)^{\frac{1}{2}} \int_{\mathbb{R}^m}\cdots\int_{\mathbb{R}^m} f(\vec{u}_1 + \cdots + \vec{u}_n + \langle \vec{\alpha}, y\rangle) \\ &\cdot \exp\bigg\{-\sum_{j=1}^{m} \frac{[\sqrt{-iq_1}u_{1j} - A_{\alpha_j}]^2}{2B_{\alpha_j}} - \cdots - \sum_{j=1}^{m} \frac{[\sqrt{-iq_n}u_{nj} - A_{\alpha_j}]^2}{2B_{\alpha_j}}\bigg\} d\vec{u}_1\cdots d\vec{u}_n \end{aligned} \tag{4.2}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where $\vec{u}_1 = (u_{11}, \ldots, u_{1m}), \ldots, \vec{u}_n = (u_{n1}, \ldots, u_{nm}) \in \mathbb{R}^m$.

In our next theorem, we establish a formula for the generalized analytic Feynman integral involving the L_1 analytic GFFT.

Theorem 4.4. *Let q_0 and F be as in Lemma 3.6. Then for all $q_1, q_2 \in \mathbb{R} - \{0\}$ and all $\gamma > 0$ with $|\gamma q_j| \le |q_0|$ for $j = 1, 2$,*

$$\begin{aligned} \int_{C_{a,b}[0,T]}^{\text{anf}_{\gamma q_2}} (T_{q_1}^{(1)}(F))(\sqrt{\gamma}y)d\mu(y) &= \int_{C_{a,b}[0,T]}^{\text{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle \alpha_j, z\rangle)d\mu(z) \\ &= \int_{C_{a,b}[0,T]}^{\text{anf}_{\gamma q_1}} (T_{q_2}^{(1)}(F))(\sqrt{\gamma}x)d\mu(x). \end{aligned}$$

Proof. By using equations (3.6), (3.11) and (4.1), we have that

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q_2}} (T_{q_1}^{(1)}(F))(\sqrt{\gamma}y)d\mu(y)\\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q_2}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(\sqrt{\gamma}y + x)d\mu(x)d\mu(y)\\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q_2}} F(\sqrt{\gamma}y + x)d\mu(y)d\mu(x) \qquad (4.3)\\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} F(y + x)d\mu(y)d\mu(x)\\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j, z\rangle)d\mu(z).
\end{aligned}
$$

Also, by using the similar method used in (4.3), we obtain that

$$
\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{\gamma q_1}} (T_{q_2}^{(1)}(F))(\sqrt{\gamma}x)d\mu(x) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j, z\rangle)d\mu(z).
$$

Thus we have the desired result. □

Our next corollary below says that the generalized analytic Feynman integral with parameter q_2 of the L_1 analytic GFFT with parameter q_1 equals the generalized analytic Feynman integral with parameter q_1 of the L_1 analytic GFFT with parameter q_2.

Corollary 4.5. *Let q_0 and F be as in Lemma 3.6. Then for all $q_1, q_2 \in \mathbb{R}-\{0\}$ with $|q_j| \le |q_0|$ for $j = 1, 2$,*

$$
\begin{aligned}
\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_2}} (T_{q_1}^{(1)}(F))(y)d\mu(y) &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(z) \prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j, z\rangle)d\mu(z)\\
&= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} (T_{q_2}^{(1)}(F))(x)d\mu(x).
\end{aligned}
$$

Also, choosing $q_j = q$ for $j = 1, 2$,

$$
\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q}} (T_{q}^{(1)}(F))(y)d\mu(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q/2}} F(z) \prod_{j=1}^{m} E_{q,q}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j, z\rangle)d\mu(z).
$$

In our next theorem, we establish a formula for the multiple L_1 analytic GFFT.

Theorem 4.6. *Let $q_1, \ldots, q_n$ and F be as in Theorem 3.4. Then*

$$\begin{aligned} &T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_n}} F(y+x)\prod_{j=1}^{m} E_{q_1,\ldots,q_n}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,x\rangle)d\mu(x) \end{aligned} \tag{4.4}$$

for s-a.e. $y \in C_{a,b}[0,T]$. In addition, both expressions in (4.4) *are given by the right-hand side of equation* (4.2) *above.*

Proof. By using equations (3.10) and (4.1), we have

$$\begin{aligned} &T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}} F(y+z_1+\cdots+z_n)d\mu(z_1)\cdots d\mu(z_n) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_n}} F(y+x)\prod_{j=1}^{m} E_{q_1,\ldots,q_n}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,x\rangle)d\mu(x) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. □

Choosing $q_j = q$ for $j = 1, \ldots, n$, we can easily obtain the following corollary.

Corollary 4.7. *Let q and F be as in Theorem 4.2. Then*

(i) *for s-a.e. $y \in C_{a,b}[0,T]$,*

$$(T_q^{(1)}(T_q^{(1)}(F)))(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q/2}} F(x+y)\prod_{j=1}^{m} E_{q,q}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,x\rangle)d\mu(x),$$

and

(ii) *for s-a.e. $y \in C_{a,b}[0,T]$,*

$$\begin{aligned} &T_q^{(1)}(T_q^{(1)}(\cdots(T_q^{(1)}(T_q^{(1)}(F)))\cdots))(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q/n}} F(y+x)\prod_{j=1}^{m} E_{q,\ldots,q}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,x\rangle)d\mu(x). \end{aligned}$$

Corollary 4.8. *Let q_1, q_2 and F be as in Theorem 3.2. Then*

$$\begin{aligned} T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))(y) &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{Q_2}} F(x+y)\prod_{j=1}^{m} E_{q_1,q_2}^{A_{\alpha_j},B_{\alpha_j}}(\langle\alpha_j,x\rangle)d\mu(x) \\ &= T_{q_1}^{(1)}(T_{q_2}^{(1)}(F))(y) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

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