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WEIGHTED COMPOSITION OPERATORS BETWEEN H^∞ AND $BMOA$

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Dedicated to John Conway for his retirement

Abstract. We study the bounded and the compact weighted composition operators from the Hardy space H^∞ into $BMOA$ and into $VMOA$, from $BMOA$ into H^∞, as well as from $BMOA$ into the Bloch space. We also provide new boundedness and compactness criteria for the weighted composition operators on $BMOA$ and on $VMOA$.

1. Introduction

Let X and Y be Banach spaces of analytic functions on a domain Ω in $\mathbb{C}$, ψ analytic on Ω and φ an analytic self-map of Ω such that $\psi(f\circ\varphi)\in Y$ for each $f\in X$. The *weighted composition operator with symbols* ψ *and* φ from X to Y is the operator $W_{\psi,\varphi}$ defined by

$$W_{\psi,\varphi}f = M_\psi C_\varphi f = \psi(f\circ\varphi) \quad \text{for } f\in X,$$

where M_ψ and C_φ denote the multiplication and the composition operators.

In recent years, a considerable interest in the study of the weighted composition operators has emerged. The motivation has been primarily because such operators arise naturally in the study of the isometries of many functional Banach spaces.

The space $BMOA$ is the space of functions f on the open unit disk $\mathbb{D}$ in the Hardy space H^2 such that

$$\|f\|_* = \sup_{a\in\mathbb{D}} \|f\circ L_a - f(a)\|_{H_2} < \infty,$$

where $L_a(z) = \frac{a-z}{1-\overline{a}z}$ for $z\in\mathbb{D}$ and, for $g\in H^p$,

$$\|g\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |g(re^{i\theta})|^p \, dm(\theta),$$

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having denoted by m the normalized Lebesgue measure on $\partial\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The correspondence $f \mapsto \|f\|_*$ is a seminorm and $\|f\|_{BMOA} = |f(0)| + \|f\|_*$ yields a Banach space structure on $BMOA$.

The *Bloch space* is the space $\mathcal{B}$ consisting of the analytic functions f on $\mathbb{D}$ such that $\|f\|_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$. The Bloch space is a Banach space under the Bloch norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$. The Hardy space H^∞ is properly contained in $BMOA$, which is in turn a proper subset of $\mathcal{B}$. In fact, $\|f\|_{\mathcal{B}} \le \|f\|_{BMOA}$. Thus, the inclusion of $BMOA$ into $\mathcal{B}$ is continuous. Furthermore, $\|f\|_\beta \le \|f\|_\infty$ and $\|f\|_* \le 2\|f\|_\infty$ for $f \in H^\infty$, where $\|f\|_\infty$ denotes the supremum norm of f. Moreover, if $f \in BMOA$, then

$$(1)\quad |f(z)| \le |f(0)| + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_\beta \le |f(0)| + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_*, \quad z \in \mathbb{D}.$$

For more information on the spaces BMO, $BMOA$ and $\mathcal{B}$, we suggest [7], [8] and [2].

The compact composition operators on $BMOA$ and on its subspace $VMOA$, defined as the space consisting of the function $f \in BMOA$ such that

$$\lim_{|a|\to 1} \|f \circ L_a - f(a)\|_{H^2} = 0,$$

have been characterized in [4], [11], [15], [17] and [19]. In particular, in [19] Wulan, Zheng and Zhu obtained a compactness criterion in terms of a 'little oh' condition on the seminorms of the symbol's powers. They also obtained an analogous characterization of the compact composition operators on the Bloch space in terms of the sequence $\{\|\varphi^n\|_\beta\}$. In [6], this result was suitably extended to the weighted composition operators from the spaces H^∞, $\mathcal{B}$ and the Dirichlet space $\mathcal{D}$ into $\mathcal{B}$, although for the latter two spaces, an additional condition was needed. Moreover, it was shown that in the case of $W_{\psi,\varphi} : H^\infty \to \mathcal{B}$, $\|W_{\psi,\varphi}\| \sim \sup \|\psi\varphi^n\|_{\mathcal{B}}$, where, as customary, by $A \sim B$ we mean $c_1 A \le B \le c_2 A$ for some positive constants c_1, c_2.

The interest in a limit formula involving the norms of $\psi\varphi^n$ is that it may lead to an expression of the essential norm. Indeed, in [20], Zhao obtained a formula for the essential norm of the composition operators C_φ between certain Bloch type spaces in terms of the norm of the nth powers of the symbol. Therefore, a natural question is whether the norm of the sequence $\{\psi\varphi^n\}$ may likewise lead in some cases to a formula for the essential norm of the weighted composition operator $W_{\psi,\varphi}$.

In this paper, our primary aim is to characterize the bounded and the compact weighted composition operators between H^∞ and $BMOA$, as well as on $BMOA$ and on $VMOA$, in terms of $\{\|\psi\varphi^n\|_{BMOA}\}_{n\in\mathbb{N}}$. Specifically, in Section 2, we characterize the bounded weighted composition operators from H^∞ to $BMOA$ and to $VMOA$. In particular, we show that $\|W_{\psi,\varphi}\| \sim \sup \|\psi\varphi^n\|_{BMOA}$. We also give sufficient conditions for compactness of $W_{\psi,\varphi} : H^\infty \to BMOA$ and characterize the compact operators from $H^\infty \to VMOA$.

In Section 3, we characterize the bounded and the compact weighted composition operators from $BMOA$ to H^∞.

The bounded and the compact weighted composition operators on $BMOA$ and on $VMOA$ have been characterized by Laitila in [12]. In Section 4, we give new characterizations of such operators. Theorem 4.3 yields Theorem 1 in [19] in the case when the multiplication symbol is the constant 1. Finally, in Section 5, we show that a weighted composition operator from $BMOA$ to the Bloch space is bounded (respectively, compact) if and only if it is bounded (respectively, compact) as an operator from the Bloch space to itself.

2. Boundedness and compactness of $W_{\psi,\varphi} : H^\infty \to BMOA$

We begin by recalling some useful results by Laitila we shall need later.

Lemma 2.1 ([12], (1.1)). *For $2 \leq p < \infty$, there exists a positive constant K_p such that*

$$\|f \circ L_a - f(a)\|_{H^p} \leq K_p \|f\|_* \quad \text{for each } a \in \mathbb{D}, f \in BMOA.$$

Lemma 2.2 ([12], (2.2)). *For $f \in BMOA$ and φ an analytic self-map of $\mathbb{D}$, $\|f \circ \varphi\|_* \leq \|f\|_*$.*

For ψ, φ analytic functions on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $a \in \mathbb{D}$, define

$$\alpha(\psi, \varphi, a) = |\psi(a)| \|L_{\varphi(a)} \circ \varphi \circ L_a\|_{H^2}.$$

Lemma 2.3 ([12], Lemma 3.4 and Proposition 2.3). *For $\psi \in BMOA$ and an analytic self-map φ of $\mathbb{D}$, there exist constants $C_1 > 0$ and $C_2 \geq 1$ such that for all $f \in BMOA$ and all $a \in \mathbb{D}$,*

$$\|(\psi \circ L_a - \psi(a))(f \circ \varphi \circ L_a - f(\varphi(a)))\|_{H^2}^2 \leq C_1 \|\psi\|_* \|\psi \circ L_a - \psi(a)\|_{H^2} \quad \text{and}$$
$$|\psi(a)| \, \|f \circ \varphi \circ L_a - f(\varphi(a))\|_{H^2} \leq C_2 \, \alpha(\psi, \varphi, a) \|f\|_*.$$

We are now ready to prove one of our main results in this section.

Theorem 2.1. *Let ψ be an analytic function on $\mathbb{D}$ and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*

(a) *The operator $W_{\psi,\varphi} : H^\infty \to BMOA$ is bounded.*

(b) $M := \sup_{n \in \mathbb{N} \cup \{0\}} \|\psi \varphi^n\|_{BMOA} < \infty$.

(c) $\psi \in BMOA$ *and* $\sup_{a \in \mathbb{D}} \alpha(\psi, \varphi, a) < \infty$.

Proof. (a) $\Rightarrow$ (b) Suppose $W_{\psi,\varphi}$ is bounded. For each nonnegative integer n, let $p_n(z) = z^n$. Then $p_n \in H^\infty$ with $\|p_n\|_\infty = 1$, so $\psi\varphi^n = \psi(p_n \circ \varphi) \in BMOA$ and $\|\psi\varphi^n\|_{BMOA} = \|W_{\psi,\varphi} p_n\|_{BMOA} \leq \|W_{\psi,\varphi}\|$. Thus, $M < \infty$.

(b) $\Rightarrow$ (c) Assume (b) holds. In particular then, $\psi \in BMOA$. Fix $a \in \mathbb{D}$ and for $z \in \mathbb{D}$, define $h_a(z) = L_{\varphi(a)}(z) - \varphi(a)$. Then $h_a(z) = (|\varphi(a)|^2 - 1) \sum_{n=1}^\infty \overline{\varphi(a)}^{\,n-1} z^n$ and so

$$\|W_{\psi,\varphi} h_a\|_* \leq (1 - |\varphi(a)|^2) \sum_{n=1}^\infty |\varphi(a)|^{n-1} \|\psi\varphi^n\|_*$$

$$\leq M(1-|\varphi(a)|^2)\,\frac{1}{1-|\varphi(a)|} \leq 2M.$$

Since $\|h_a\|_\infty \leq 2$ and $\|\psi\|_* \leq M$, we have

$$\begin{aligned}\alpha(\psi,\varphi,a) \leq{}& \|(\psi\circ L_a - \psi(a))(h_a\circ\varphi\circ L_a)\|_{H^2}\\ &+ \|W_{\psi,\varphi}h_a\circ L_a - W_{\psi,\varphi}h_a(a)\|_{H^2} \leq 4M.\end{aligned}$$

(c) $\Rightarrow$ (a) Assume (c) holds. Set $L = \sup_{a\in\mathbb{D}}\alpha(\psi,\varphi,a)$ and let $f \in H^\infty$ with $\|f\|_\infty = 1$. We need to show that $\|W_{\psi,\varphi}f\|_{BMOA}$ is bounded by a constant independent of f. Since $|\psi(0)f(\varphi(0))| \leq |\psi(0)|$, we only need to prove the boundedness of $\|W_{\psi,\varphi}f\|_*$.

Fix $a\in\mathbb{D}$. Then, using Lemma 2.3, we have

$$\begin{aligned}&\|W_{\psi,\varphi}f\circ L_a - \psi(a)f(\varphi(a))\|_{H^2}\\ &\leq \|(\psi\circ L_a-\psi(a))(f\circ\varphi\circ L_a)\|_{H^2} + |\psi(a)|\|f\circ\varphi\circ L_a - f(\varphi(a))\|_{H^2}\\ &\leq \|\psi\|_*\|f\|_\infty + C_2\,\alpha(\psi,\varphi,a)\|f\|_*\\ &\leq \|\psi\|_* + 2C_2\,\alpha(\psi,\varphi,a).\end{aligned}$$

Thus, $\|W_{\psi,\varphi}f\|_* \leq \|\psi\|_* + 2C_2\,L$, proving the boundedness of the operator. Furthermore, from the above estimates, we obtain $M \leq \|W_{\psi,\varphi}\| \leq (1+8C_2)M$, so $\|W_{\psi,\varphi}\| \sim M$. □

From Theorem 2.1, we deduce the following result.

Corollary 2.1. *Let φ be an analytic self-map of $\mathbb{D}$. If $\psi \in BMOA$, then the sequence $\{\|\psi\varphi^n\|_{BMOA}\}$ is bounded if and only if $\sup_{a\in\mathbb{D}}\alpha(\psi,\varphi,a) < \infty$.*

For an analytic self-map φ of $\mathbb{D}$, for $t, R \in (0,1)$, and for $a\in\mathbb{D}$, define

$$\tilde{E}(\varphi,a,t) = \{\zeta\in\partial\mathbb{D} : |\varphi(L_a(\zeta))| > t\} \quad\text{and}\quad \Omega_R = \{a\in\mathbb{D} : |\varphi(a)| \leq R\}.$$

Proposition 2.1. *Let $\psi \in BMOA$, φ an analytic self-map of $\mathbb{D}$ such that $\psi\varphi^n \in BMOA$ for each $n\in\mathbb{N}$ and $\lim_{n\to\infty}\|\psi\varphi^n\|_{BMOA} = 0$. Then for all $R\in(0,1)$,*

$$\lim_{t\to 1}\sup_{a\in\Omega_R}\int_{\tilde{E}(\varphi,a,t)}|\psi\circ L_a(\zeta)|^2\,dm(\zeta) = 0. \tag{2}$$

Proof. Fix $R\in(0,1)$, $\varepsilon > 0$ and set $L = \max\{1,\|\psi\|_*\}$. Choose $N\in\mathbb{N}$ such that $\|\psi\varphi^n\|_* < \varepsilon/2$ and $R^n < \frac{\varepsilon}{2L}$ for all $n \geq N$. Then, by the triangle inequality, for all $n\geq N$ and all $a\in\Omega_R$, we have

$$\begin{aligned}\|(\psi\circ L_a)(\varphi^n\circ L_a - \varphi(a)^n)\|_{H^2} &\leq \|(\psi\circ L_a)(\varphi^n\circ L_a) - \psi(a)\varphi(a)^n\|_{H^2}\\ &\quad+ \|(\psi\circ L_a - \psi(a))\varphi(a)^n\|_{H^2}\\ &\leq \|\psi\varphi^n\|_* + \|\psi\|_*R^n < \varepsilon.\end{aligned}$$

For $n\geq N$ and $\varepsilon < t < 1$, we have

$$\Big(t^n - \frac{\varepsilon}{2}\Big)^2\sup_{a\in\Omega_R}\int_{\tilde{E}(\varphi,a,t)}|\psi\circ L_a(\zeta)|^2\,dm(\zeta)$$

$$\leq \sup_{a\in\Omega_R} \int_{\tilde{E}(\varphi,a,t)} |(\psi \circ L_a(\zeta))(\varphi^n \circ L_a(\zeta) - \varphi(a)^n)|^2 \, dm(\zeta)$$
$$\leq \sup_{a\in\Omega_R} \|(\psi \circ L_a)(\varphi^n \circ L_a - \varphi(a)^n)\|_{H^2}^2 < \varepsilon^2.$$

Taking the limit as $t \to 1$, we obtain

$$\Big(1 - \frac{\varepsilon}{2}\Big)^2 \lim_{t\to 1} \sup_{a\in\Omega_R} \int_{\tilde{E}(\varphi,a,t)} |\psi \circ L_a(\zeta)|^2 \, dm(\zeta) \leq \varepsilon^2.$$

The conclusion follows at once by taking the limit as $\varepsilon \to 0$. □

Remark 2.1. For $a \in \mathbb{D}$ and $t \in (0,1)$ define

$$E(\varphi, a, t) = \{\zeta \in \partial\mathbb{D} : |(L_{\varphi(a)} \circ \varphi \circ L_a)(\zeta)| > t\}.$$

In [12], Remark 3.3, it was shown that for $R \in (0,1)$, if $|\varphi(a)| \leq R$, $\zeta \in \partial\mathbb{D}$, and $|(\varphi \circ L_a)(\zeta)| < 1$, then

$$\frac{1-R}{1+R} \leq \frac{1 - |(L_{\varphi(a)} \circ \varphi \circ L_a)(\zeta)|^2}{1 - |(\varphi \circ L_a)(\zeta)|^2} \leq \frac{1+R}{1-R}.$$

From these estimates it is easy to show that

$$\lim_{t\to 1} \sup_{a\in\Omega_R} \int_{E(\varphi,a,t)} |\psi \circ L_a(\zeta)|^2 \, dm(\zeta) = 0$$
$$\iff \lim_{t\to 1} \sup_{a\in\Omega_R} \int_{\tilde{E}(\varphi,a,t)} |\psi \circ L_a(\zeta)|^2 \, dm(\zeta) = 0.$$

The following compactness criterion will be used to show some of our results. Its proof is based on a direct application of Lemma 3.7 of [18].

Lemma 2.4. *A bounded weighted composition operator $W_{\psi,\varphi}$ from H^∞ (respectively, $BMOA$ or $VMOA$) to $BMOA$ or $VMOA$ is compact if and only if, for every bounded sequence $\{f_n\}$ in H^∞ (respectively, $BMOA$ or $VMOA$) converging to 0 uniformly on compact subsets of $\mathbb{D}$, the sequence $\{\|W_{\psi,\varphi} f_n\|_{BMOA}\}$ approaches 0 as $n \to \infty$.*

We now present a characterization of compactness under a restriction on the symbols.

Theorem 2.2. *Suppose $W_{\psi,\varphi} : H^\infty \to BMOA$ is bounded and*

$$\lim_{|\varphi(a)|\to 1} \|\psi \circ L_a - \psi(a)\|_{H^2} = 0.$$

Then, the following statements are equivalent:

(a) *$W_{\psi,\varphi}$ is compact.*

(b) $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$.

(c) $\lim_{|\varphi(a)|\to 1} \alpha(\psi, \varphi, a) = 0$ *and* (2) *holds for each* $R \in (0,1)$.

Proof. (a) $\Rightarrow$ (b) Assume $W_{\psi,\varphi}$ is compact. For $n \in \mathbb{N}$, let $p_n(z) = z^n$, $z \in \mathbb{D}$. Then by the boundedness in H^∞, the uniform convergence on compact subsets of $\{p_n\}$ to 0, and using Lemma 2.4, it follows that $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$.

(b) $\Rightarrow$ (c) Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$. For $n \in \mathbb{N}$, let

$$h_n(z) = L_{\varphi(a_n)} - \varphi(a_n) = (|\varphi(a_n)|^2 - 1) \sum_{k=1}^{\infty} \overline{\varphi(a_n)}^{\,k-1} z^k, \quad z \in \mathbb{D}.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|\psi\varphi^k\|_* < \varepsilon/2$ for all $k > N$. Then, arguing as in the proof of Theorem 2.1, we have

$$\begin{aligned}\|W_{\psi,\varphi} h_n\|_* &\le (1 - |\varphi(a_n)|^2) \sum_{k=1}^{\infty} |\varphi(a_n)|^{k-1} \|\psi\varphi^k\|_* \\ &\le (1 - |\varphi(a_n)|^2) \sum_{k=1}^{N} |\varphi(a_n)|^{k-1} \|\psi\varphi^k\|_* + \varepsilon.\end{aligned}$$

Since $\|\psi\varphi^k\|_*$ is bounded, we obtain $\lim_{n\to\infty} \|W_{\psi,\varphi} h_n\|_* \le \varepsilon$. Since ε is arbitrary, it follows that $\lim_{n\to\infty} \|W_{\psi,\varphi} h_n\|_* = 0$. Observing that $\|h_n\|_\infty \le 2$, we deduce

$$\begin{aligned}\alpha(\psi, \varphi, a_n) &\le \|(\psi \circ L_{a_n} - \psi(a_n))(h_n \circ \varphi \circ L_{a_n})\|_{H^2} \\ &\quad + \|W_{\psi,\varphi} h_n \circ L_{a_n} - W_{\psi,\varphi} h_n(a_n)\|_{H^2} \\ &\le 2\|\psi \circ L_{a_n} - \psi(a_n)\|_{H^2} + \|W_{\psi,\varphi} h_n\|_* \to 0\end{aligned}$$

as $n \to \infty$. Hence, $\alpha(\psi, \varphi, a) \to 0$ as $|\varphi(a)| \to 1$.

Finally, observe that condition (2) holds for each $R \in (0, 1)$ by Proposition 2.1.

(c) $\Rightarrow$ (a) By Lemma 2.4, to prove that $W_{\psi,\varphi}$ is compact, it suffices to show that $\|W_{\psi,\varphi} f_n\|_{BMOA}$ approaches 0 as $n \to \infty$ for any sequence $\{f_n\}$ in H^∞ converging to 0 uniformly on compact subsets of $\mathbb{D}$ such that $\|f_n\|_\infty \le 1$ for all $n \in \mathbb{N}$. Let $\{f_n\}$ be such a sequence and fix $\varepsilon > 0$. Then there exists $R \in (0, 1)$ such that $\|\psi \circ L_a - \psi(a)\|_{H^2} < \varepsilon/2$ and $\alpha(\psi, \varphi, a) < \varepsilon/(4C_2)$ for all $a \in \mathbb{D}$ such that $|\varphi(a)| > R$, where C_2 is the constant in Lemma 2.3. Thus, by Lemma 2.3, for any $n \in \mathbb{N}$, if $|\varphi(a)| > R$, then

$$\begin{aligned}\|W_{\psi,\varphi} f_n \circ L_a - \psi(a) f_n(\varphi(a))\|_{H^2} &\le \|(\psi \circ L_a - \psi(a))(f_n \circ \varphi \circ L_a)\|_{H^2} \\ &\quad + |\psi(a)| \|f_n \circ \varphi \circ L_a - f_n(\varphi(a))\|_{H^2} \\ &\le \|\psi \circ L_a - \psi(a)\|_{H^2} + C_2\, \alpha(\psi, \varphi, a) \|f_n\|_* \\ &< \frac{\varepsilon}{2} + 2C_2\, \alpha(\psi, \varphi, a) \|f_n\|_\infty < \varepsilon. \end{aligned} \tag{3}$$

On the other hand, by the convergence of f_n to 0 on compact subsets of $\mathbb{D}$, if $|\varphi(a)| \le R$, then $|f_n(\varphi(a))| \le \max_{|w|\le R} |f_n(w)| < \varepsilon/(3\|\psi\|_*)$ for n sufficiently large. By assumption, there exists $t_0 \in (0,1)$ such that

$$\sup_{a\in\Omega_R} \int_{\tilde{E}(\varphi,a,t)} |\psi \circ L_a(\zeta)|^2\, dm(\zeta) < \frac{\varepsilon^2}{18}$$

for all t, $t_0 \le t < 1$. We are going to show that

$$\sup_{a\in\Omega_R} \int_{\partial\mathbb{D}\setminus\tilde{E}(\varphi,a,t)} |\psi \circ L_a(\zeta)|^2 |(f_n \circ \varphi \circ L_a)(\zeta) - f_n(\varphi(a))|^2\, dm(\zeta) < c\varepsilon^2 \tag{4}$$

for some positive constant c and for all n sufficiently large. Assuming (4) holds, we deduce that for $|\varphi(a)| \le R$,

$$\begin{aligned} &\|W_{\psi,\varphi} f_n \circ L_a - \psi(a) f_n(\varphi(a))\|^2_{H^2} \\ &\le 2\|(\psi\circ L_a)(f_n\circ\varphi\circ L_a - f_n(\varphi(a)))\|^2_{H^2} \\ &\quad + 2\|(\psi\circ L_a - \psi(a)) f_n(\varphi(a))\|^2_{H^2} \\ &\le 2\int_{\partial\mathbb{D}\setminus\tilde{E}(\varphi,a,t)} |\psi\circ L_a(\zeta)|^2 |f_n\circ\varphi\circ L_a(\zeta) - f_n(\varphi(a))|^2\, dm(\zeta) \\ &\quad + 8\int_{\tilde{E}(\varphi,a,t)} |\psi\circ L_a(\zeta)|^2\, dm(\zeta) + \frac{2\varepsilon^2}{9} < (2c+1)\varepsilon^2. \end{aligned} \tag{5}$$

From (3) and (5), taking the supremum over all $a \in \mathbb{D}$, we obtain $\|W_{\psi,\varphi} f_n\|_* < C\varepsilon$ for some constant $C > 0$ and all n sufficiently large. Noting that

$$|\psi(0) f_n(\varphi(0))| \to 0 \text{ as } n \to \infty,$$

we obtain $\lim\limits_{n\to\infty} \|W_{\psi,\varphi} f_n\|_{BMOA} = 0$.

To prove (4), we shall argue as in the proof of Theorem 3.1 of [12]. For $n \in \mathbb{N}$ and $a \in \Omega_R$, let $G_{n,a} = f_n \circ L_{\varphi(a)} - f_n(\varphi(a))$ and $\lambda_a = L_{\varphi(a)} \circ \varphi \circ L_a$. Then, as shown on p. 37 of [12], $f_n \circ \varphi \circ L_a - f_n(\varphi(a)) = G_{n,a}\lambda_a$, and

$$\|(\psi\circ L_a)\lambda_a\|_{H^2} \le \|\psi\|_* \|\lambda_a\|_\infty + \sup_{b\in\mathbb{D}} \alpha(\psi,\varphi,b) = c < \infty.$$

Furthermore, for $s = \sqrt{\frac{1-R}{1+R}t^2 + \frac{2R}{1+R}}$, with $t_0 \le t < 1$, and for $\zeta \in \partial\mathbb{D}\setminus E(\varphi,a,s)$,

$$\begin{aligned} |f_n\circ\varphi\circ L_a(\zeta) - f_n(\varphi(a))| &= |G_{n,a}(\lambda_a(\zeta))| \le 2|\lambda_a(\zeta)| \max_{|w|\le s} |G_{n,a}(w)| \\ &\le 2\max_{|w|\le s} |f_n(L_{\varphi(a)}(w))| + |f_n(\varphi(a))| < \varepsilon \end{aligned}$$

for all n sufficiently large. Thus,

$$\int_{\partial\mathbb{D}\setminus E(\varphi,a,s)} |\psi\circ L_a(\zeta)|^2 |(f_n\circ\varphi\circ L_a(\zeta) - f_n(\varphi(a)))|^2\, dm(\zeta) < c\varepsilon^2.$$

From the inclusion $E(\varphi,a,s) \subseteq \tilde{E}(\varphi,a,t)$, which can be verified using Remark 2.1, we deduce that

$$\int_{\partial\mathbb{D}\setminus\tilde{E}(\varphi,a,t)} |\psi\circ L_a(\zeta)|^2 |(f_n\circ\varphi\circ L_a(\zeta) - f_n(\varphi(a)))|^2\, dm(\zeta) < c\varepsilon^2.$$

The proof is now complete. □

Remark 2.2. Theorem 2.2 yields characterizations of compactness in the case $\psi \in VMOA$. We have not been able to prove or disprove whether the compactness of $W_{\psi,\varphi} : H^\infty \to BMOA$ implies that

$$\lim_{|\varphi(a)|\to 1} \|\psi \circ L_a - \psi(a)\|_{H^2} = 0.$$

From Theorem 2.2, taking $\psi = 1$ and using Theorem 1 of [19], we deduce the following result.

Corollary 2.2. *For an analytic self-map φ of $\mathbb{D}$ the following propositions are equivalent*:
(a) *$C_\varphi : H^\infty \to BMOA$ is compact.*
(b) *$C_\varphi : BMOA \to BMOA$ is compact.*
(c) $\lim_{n\to\infty} \|\varphi^n\|_{BMOA} = 0$.

We now derive a characterization of the bounded weighted composition operators from H^∞ to $VMOA$ similar to Theorem 2.1.

Theorem 2.3. *Let ψ be an analytic function on $\mathbb{D}$ and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent*:
(a) *The operator $W_{\psi,\varphi} : H^\infty \to VMOA$ is bounded.*
(b) *For each integer $n \geq 0$, $\psi\varphi^n \in VMOA$ and $\sup_{n\in\mathbb{N}} \|\psi\varphi^n\|_{BMOA} < \infty$.*
(c) *For each integer $n \geq 0$, $\psi\varphi^n \in VMOA$ and $\sup_{a\in\mathbb{D}} \alpha(\psi, \varphi, a) < \infty$.*

Proof. (a) $\Rightarrow$ (b) follows immediately from the boundedness in H^∞ of the sequence $p_n(z) = z^n$.

(b) $\Rightarrow$ (a) The boundedness of $W_{\psi,\varphi}$ as an operator from H^∞ to $BMOA$ follows immediately from Theorem 2.1. Thus, we only need to show that for each $f \in VMOA$, $W_{\psi,\varphi} f \in VMOA$. From (b) it follows that this property holds for the polynomials. Since $VMOA$ is the closure in $BMOA$ of the polynomials in $BMOA$, the result follows.

The equivalence of (b) and (c) follows immediately from Corollary 2.1. □

We end the section by characterizing the compact operators $W_{\psi,\varphi} : H^\infty \to VMOA$.

Theorem 2.4. *Suppose $W_{\psi,\varphi} : H^\infty \to VMOA$ is bounded. Then $W_{\psi,\varphi} : H^\infty \to VMOA$ is compact if and only if $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$.*

Proof. Assume $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$. Since $W_{\psi,\varphi} : H^\infty \to VMOA$ is bounded, $\psi \in VMOA$. Thus, condition (b) of Theorem 2.2 is satisfied. Therefore, $W_{\psi,\varphi}$ is compact as an operator from H^∞ to $BMOA$. Since it is bounded as an operator mapping H^∞ into $VMOA$, it is also compact as an operator mapping H^∞ into $VMOA$.

The proof of the converse is immediate. □

3. Boundedness and compactness of $W_{\psi,\varphi} : BMOA \to H^\infty$

Let φ be an analytic self-map of $\mathbb{D}$. For $a \in \mathbb{D}$ such that $\varphi(a) \neq 0$, define

$$k_a(z) = \varphi(a)\frac{\left(\frac{1}{2}\log\frac{1+\overline{\varphi(a)}z}{1-\overline{\varphi(a)}z}\right)^2}{\rho(0,\varphi(a))}, \quad z \in \mathbb{D},$$

where ρ denotes the hyperbolic distance on $\mathbb{D}$: $\rho(0,z) = \frac{1}{2}\log\frac{1+|z|}{1-|z|}$. Then k_a is bounded, and hence in $BMOA$, $k_a(0) = 0$, $k_a(\varphi(a)) \leq |\varphi(a)|\rho(0,\varphi(a))$, and a straightforward calculation shows that

$$\frac{1}{6}\,\rho(0,\varphi(a)) \leq \frac{1}{4}\,|\varphi(a)|\rho(0,\varphi(a)) \leq k_a(\varphi(a)) \tag{6}$$

for $\frac{2}{3} \leq |\varphi(a)| < 1$. We now show that

$$\sup\{\|k_a\|_{BMOA} : a \in \mathbb{D}, \varphi(a) \neq 0\} < \infty. \tag{7}$$

For $\varphi(a) \neq 0$, define the functions

$$\ell_a(z) = \frac{1}{2}\log\frac{1+\overline{\varphi(a)}z}{1-\overline{\varphi(a)}z}, \;\; h_a(z) = \frac{\varphi(a)\ell_a(z)}{\rho(0,\varphi(a))}, \; z \in \mathbb{D}.$$

Since logarithms of non-vanishing univalent functions are in $BMOA$ (see [3] or [5]), $\ell_a \in BMOA$ and by Theorem 5 of [9], $\|\ell_a\|_{BMOA}$ is bounded above by a constant independent of a. Moreover, for $z \in \mathbb{D}$,

$$|h_a(z)| \leq \frac{|\varphi(a)|(\rho(0,\varphi(a)) + \frac{\pi}{4})}{\rho(0,\varphi(a))},$$

which is bounded above by a constant independent of a as one can see by looking at the cases when $\varphi(a)$ is near 0 and $|\varphi(a)|$ is near 1. Thus $\sup\{\|h_a\|_\infty : a \in \mathbb{D}, \varphi(a) \neq 0\} < \infty$. Therefore, noting that $k_a = h_a\ell_a = \frac{\varphi(a)\ell_a^2}{\rho(0,\varphi(a))}$, for $b \in \mathbb{D}$, we obtain

$$\begin{aligned}
&\|k_a \circ L_b - k_a(b)\|_{H^2}^2 \\
&= \frac{|\varphi(a)|^2}{\rho(0,\varphi(a))^2}\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|\ell_a(L_b(re^{i\theta}))^2 - \ell_a(b)^2|^2\,d\theta \\
&= \frac{|\varphi(a)|^2}{\rho(0,\varphi(a))^2}\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|\ell_a(L_b(re^{i\theta})) - \ell_a(b)|^2|\ell_a(L_b(re^{i\theta})) + \ell_a(b)|^2\,d\theta\right) \\
&\leq \frac{|\varphi(a)|^2}{\rho(0,\varphi(a))^2}\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|\ell_a(L_b(re^{i\theta})) - \ell_a(b)|^2 4\|\ell_a\|_\infty^2\,d\theta\right) \\
&= 4\|h_a\|_\infty^2\|\ell_a \circ L_b - \ell_a(b)\|_{H^2}^2 \leq 4\|h_a\|_\infty^2\|\ell_a\|_{BMOA}^2.
\end{aligned}$$

Therefore, by the above remarks, taking the supremum over all $b \in \mathbb{D}$, we deduce (7).

For an analytic function ψ on $\mathbb{D}$ and an analytic self-map of $\mathbb{D}$, define

$$\theta_{\psi,\varphi} = \sup_{z\in\mathbb{D}}|\psi(z)|\rho(0,\varphi(z)).$$

Theorem 3.1. *For an analytic function ψ on $\mathbb{D}$ and an analytic self-map of $\mathbb{D}$, the following propositions are equivalent*:

(a) $W_{\psi,\varphi} : BMOA \to H^\infty$ *is bounded.*

(b) ψ, $W_{\psi,\varphi}k_a \in H^\infty$ *for* $\varphi(a) \neq 0$, *and* $\sup\{\|W_{\psi,\varphi}k_a\|_\infty : a \in \mathbb{D}, \varphi(a) \neq 0\} < \infty$.

(c) $\psi \in H^\infty$ *and* $\theta_{\psi,\varphi} < \infty$.

(d) $W_{\psi,\varphi} : \mathcal{B} \to H^\infty$ *is bounded.*

Proof. (a) $\Rightarrow$ (b) follows immediately from (7).

(b) $\Rightarrow$ (c) By (6), if $\frac{2}{3} \leq |\varphi(a)| < 1$, then $|\psi(a)|\rho(0,\varphi(a)) \leq 6|\psi(a)|k_a(\varphi(a)) \leq 6\|W_{\psi,\varphi}k_a\|_\infty$ which is bounded by assumption. On the other hand, if $|\varphi(a)| \leq \frac{2}{3}$, then

$$|\psi(a)\rho(0,\varphi(a))| \leq \frac{1}{2}\log 5 \max\left\{|\psi(z)| : z \in \mathbb{D}, |\varphi(z)| \leq \frac{2}{3}\right\}.$$

Therefore, taking the supremum over all $a \in \mathbb{D}$, we deduce that $\theta_{\psi,\varphi} < \infty$.

The equivalence of (c) and (d) was established in [1], Theorem 7.1, for the case of a general bounded homogeneous domain D, which reduces to (b) when $D = \mathbb{D}$. For an equivalent characterization, see [10]. For the case of the ball, see [16] and for the case of the polydisk, see [13].

Finally, the implication (d) $\Rightarrow$ (a) is an immediate consequence of the continuous inclusion of $BMOA$ into $\mathcal{B}$. □

The following standard compactness criterion can be proved using Lemma 3.7 of [18].

Lemma 3.1. *A bounded weighted composition operator $W_{\psi,\varphi}$ from $BMOA$ to H^∞ (respectively, from $BMOA$ to $\mathcal{B}$) is compact if and only if, for every bounded sequence $\{f_n\}$ in $BMOA$ converging to* 0 *uniformly on compact subsets of $\mathbb{D}$, $\|W_{\psi,\varphi}f_n\|_\infty \to 0$ (respectively, $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}} \to 0$) as $n \to \infty$.*

Theorem 3.2. *For $W_{\psi,\varphi} : BMOA \to H^\infty$ bounded, the following propositions are equivalent*:

(a) $W_{\psi,\varphi} : BMOA \to H^\infty$ *is compact.*

(b) $\lim_{|\varphi(a)|\to 1} \|W_{\psi,\varphi}k_a\|_\infty = 0$.

(c) $\lim_{|\varphi(a)|\to 1} |\psi(a)|\rho(0,\varphi(a)) = 0$.

(d) $W_{\psi,\varphi} : \mathcal{B} \to H^\infty$ *is compact.*

Proof. (a) $\Rightarrow$ (b) If $\{a_n\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$, then the sequence $\{k_{a_n}\}$ converges to 0 uniformly on compact subsets and is bounded in $BMOA$. Thus, by Lemma 3.1, the compactness of $W_{\psi,\varphi} : BMOA \to H^\infty$ implies that $\|W_{\psi,\varphi}k_{a_n}\|_\infty \to 0$.

(b) $\Rightarrow$ (c) By (6), for $a \in \mathbb{D}$ such that $\frac{2}{3} \leq |\varphi(a)| < 1$, we have

$$|\psi(a)|\rho(0,\varphi(a)) \leq 6|\psi(a)k_a(\varphi(a))| \leq 6\|W_{\psi,\varphi}k_a\|_\infty \to 0 \text{ as } |\varphi(a)| \to 1.$$

(c) $\Rightarrow$ (d) Let $\{f_n\}$ be a sequence in $\mathcal{B}$ converging to 0 uniformly on compact subsets whose norm in $\mathcal{B}$ is bounded by $M > 0$. By Lemma 3.1, it suffices to show that $\|W_{\psi,\varphi} f_n\|_\infty \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$. By part (c), there exists $r \in (0,1)$ such that $|\psi(a)| < \varepsilon/M$ and $|\psi(a)|\rho(0,\varphi(a)) < \varepsilon/M$ for all $a \in \mathbb{D}$ such that $|\varphi(a)| > r$. Thus, by (1), if $|\varphi(a)| > r$, then

$$|\psi(a) f_n(\varphi(a))| \le |\psi(a)| \left(|f_n(0)| + \rho(0,\varphi(a))\|f_n\|_\beta\right) < \frac{\varepsilon}{M}\|f_n\|_{\mathcal{B}} \le \varepsilon.$$

Moreover, if $|\varphi(a)| \le r$, then $|\psi(a) f_n(\varphi(a))| \le \max_{z\in\Omega_r} |\psi(z) f_n(\varphi(z))| < \varepsilon$ for n sufficiently large. Therefore, $\|W_{\psi,\varphi} f_n\|_\infty \le \varepsilon$ for all n sufficiently large. Hence, $\|W_{\psi,\varphi} f_n\|_\infty \to 0$ as $n \to \infty$.

(d) $\Rightarrow$ (a) If $\{f_n\}$ is a bounded sequence in $BMOA$, then it is also bounded in $\mathcal{B}$, so by the compactness of $W_{\psi,\varphi} : \mathcal{B} \to H^\infty$, there exists a subsequence $\{f_{n_k}\}$ such that $W_{\psi,\varphi} f_{n_k}$ converges in H^∞. Thus, $W_{\psi,\varphi}$ is also compact as an operator from $BMOA$ to H^∞. □

For a characterization of the compactness of $W_{\psi,\varphi} : \mathcal{B} \to H^\infty$ equivalent to (c), see [10].

4. Boundedness and compactness of $W_{\psi,\varphi}$ on $BMOA$

In this section, we provide new characterizations of the bounded and the compact weighted composition operators on $BMOA$. The following result was proved by Laitila in [12].

Theorem 4.1 ([12], Theorems 2.1 and 3.1)**.** *Let ψ and φ be analytic on $\mathbb{D}$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then*

(a) *$W_{\psi,\varphi} : BMOA \to BMOA$ is bounded if and only if $\sup_{a\in\mathbb{D}} \alpha(\psi,\varphi,a) < \infty$ and $\sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty$, where $\beta(\psi,\varphi,a) = \left(\log \frac{2}{1-|\varphi(a)|^2}\right) \|\psi \circ L_a - \psi(a)\|_{H^2}$.*

(b) *If $W_{\psi,\varphi}$ is bounded on $BMOA$, then $W_{\psi,\varphi}$ is compact if and only if*

$$\lim_{|\varphi(a)|\to 1} \alpha(\psi,\varphi,a) = 0, \tag{8}$$

$$\lim_{|\varphi(a)|\to 1} \beta(\psi,\varphi,a) = 0, \text{ and} \tag{9}$$

$$\lim_{t\to 1} \sup_{a\in\Omega_R} \int_{E(\varphi,a,t)} |(\psi\circ L_a)(\zeta)|^2 \, dm(\zeta) = 0, \tag{10}$$

for all $R \in (0,1)$, where $E(\varphi,a,t) = \{\zeta \in \partial\mathbb{D} : |(L_{\varphi(a)} \circ \varphi \circ L_a)(\zeta)| > t\}$.

For $a \in \mathbb{D}$, define $g_a(z) = \log \frac{2}{1-\overline{\varphi(a)}z}$, $z \in \mathbb{D}$.

Theorem 4.2. *Let ψ and φ be analytic on $\mathbb{D}$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then the following propositions are equivalent:*

(a) *$W_{\psi,\varphi} : BMOA \to BMOA$ is bounded.*

(b) *$\sup_{n\in\mathbb{N}\cup\{0\}} \|\psi\varphi^n\|_{BMOA} < \infty$ and $\sup_{a\in\mathbb{D}} \|W_{\psi,\varphi} g_a\|_{BMOA} < \infty$.*

(c) *$\sup_{n\in\mathbb{N}\cup\{0\}} \|\psi\varphi^n\|_{BMOA} < \infty$ and $\sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty$.*

Proof. The equivalence of (a) and (c) follows immediately from Theorem 4.1 and Corollary 2.1, while the implication (b) $\Rightarrow$ (c) follows at once from Theorem 2.1 in [12]. Thus, it suffices to show that (a) implies (b).

Assume $W_{\psi,\varphi}$ is bounded on $BMOA$. Then $\psi = W_{\psi,\varphi}1 \in BMOA$ and for each $n \in \mathbb{N}$, $\|p_n\|_{BMOA} \leq 2\|p_n\|_\infty = 2$, where $p_n(z) = z^n$ for $z \in \mathbb{D}$. Thus,

$$\sup_{n\in\mathbb{N}} \|\psi\varphi^n\|_{BMOA} = \sup_{n\in\mathbb{N}} \|W_{\psi,\varphi}p_n\|_{BMOA} \leq 2\|W_{\psi,\varphi}\|.$$

On the other hand, by Theorem 11.4 of [8], $\sup_{a\in\mathbb{D}} \|g_a\|_{BMOA} < \infty$, so by the boundedness of $W_{\psi,\varphi}$, $\sup_{a\in\mathbb{D}} \|W_{\psi,\varphi}g_a\|_{BMOA}$ is finite as well. □

To obtain a new characterization of the compact weighted composition operators on $BMOA$, we consider the family $\{f_a : a \in \mathbb{D}\}$ defined by

$$f_a(z) = \left(\log \frac{2}{1-|\varphi(a)|^2}\right)^{-1}\left(\log \frac{2}{1-\overline{\varphi(a)}z}\right)^2, \; z \in \mathbb{D}. \tag{11}$$

Then $f_a(\varphi(a)) = \log \frac{2}{1-|\varphi(a)|^2}$, $f_a'(\varphi(a)) = \frac{2\overline{\varphi(a)}}{1-|\varphi(a)|^2}$, $f_a \in BMOA$ and $M = \sup_{a\in\mathbb{D}} \|f_a\|_{BMOA} < \infty$ (see [12], p. 35).

Theorem 4.3. *Let ψ and φ be analytic on $\mathbb{D}$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and suppose $W_{\psi,\varphi}$ is bounded on $BMOA$. Then the following propositions are equivalent*:

(a) *$W_{\psi,\varphi}$ is compact.*

(b) $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$ *and* $\lim_{|\varphi(a)|\to1} \|W_{\psi,\varphi}f_a\|_{BMOA} = 0$.

(c) $\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0$ *and* $\lim_{|\varphi(a)|\to1} \beta(\psi,\varphi,a) = 0$.

Proof. (a) $\Rightarrow$ (b) Suppose $W_{\psi,\varphi}$ is compact. Since $\{p_n\}$ is a bounded sequence in $BMOA$ converging to 0 on compact subsets of $\mathbb{D}$, by Lemma 2.4, $\|\psi\varphi^n\|_{BMOA} \to 0$ as $n \to \infty$.

Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$. Then f_{a_n} converges to 0 uniformly on compact subsets and $\|f_{a_n}\|_{BMOA} \leq M$. By Lemma 2.4, it follows that $\|W_{\psi,\varphi}f_{a_n}\|_{BMOA} \to 0$ as $n \to \infty$.

(b) $\Rightarrow$ (c) Suppose (b) holds. By Theorem 2.2, the assumption $\|\psi\varphi^n\|_{BMOA} \to 0$ as $n \to \infty$ implies that $\alpha(\psi,\varphi,a) \to 0$ as $|\varphi(a)| \to 1$. The convergence to 0 of $\beta(\psi,\varphi,a)$ as $|\varphi(a)| \to 1$ now follows from the proof of Theorem 3.1 in [12] (see pp. 35–36).

(c) $\Rightarrow$ (a) Suppose (c) holds. Since by assumption, (9) is valid, by part (b) of Theorem 4.1, to prove that $W_{\psi,\varphi}$ is compact, it suffices to show that conditions (8) and (10) hold.

First observe that $W_{\psi,\varphi}$ is bounded as an operator from H^∞ to $BMOA$. Since $\log \frac{2}{1-|\varphi(a_n)|^2} \to \infty$, it follows that $\|\psi \circ L_{a_n} - \psi(a_n)\|_{H^2} \to 0$, as $n \to \infty$. Conditions (8) and (10) now follow from the equivalence of (b) and (c) in Theorem 2.2 and Remark 2.1. □

In [12], Laitila characterized the bounded and the compact weighted composition operators on $VMOA$ and gave an approximation of the essential norm.

We now provide new characterizations of boundedness and compactness in the spirit of Theorems 4.2 and 4.3. Their proofs follow from these results and from Proposition 4.1, Theorem 2.1 and Theorem 4.3 in [12]. We omit the details.

Theorem 4.4. *Let ψ and φ be analytic on $\mathbb{D}$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$.*

(a) *$W_{\psi,\varphi}$ is bounded on $VMOA$ if and only if for each integer $n \geq 0$, $\psi\varphi^n \in VMOA$,*

$$\sup_{n\in\mathbb{N}} \|\psi\varphi^n\|_{BMOA} < \infty \quad \text{and} \quad \sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty.$$

(b) *If $W_{\psi,\varphi}$ is bounded on $VMOA$, then $W_{\psi,\varphi}$ is compact on $VMOA$ if and only if*

$$\lim_{n\to\infty} \|\psi\varphi^n\|_{BMOA} = 0 \quad \text{and} \quad \lim_{|a|\to 1} \beta(\psi,\varphi,a) = 0.$$

5. Boundedness and compactness of $W_{\psi,\varphi} : BMOA \to \mathcal{B}$

Recall from Section 1 that the inclusion map of $BMOA$ into $\mathcal{B}$ is continuous. Thus, if $W_{\psi,\varphi}$ is a bounded weighted composition operator on $\mathcal{B}$, it is also bounded as an operator from $BMOA$ into $\mathcal{B}$. We are going to show that, in fact, the converse holds as well. We shall make use of the following result whose proof appears in Corollary 2.1 of [6].

Lemma 5.1. *For $\psi \in \mathcal{B}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty$ if and only if $\sup_{n\in\mathbb{N}} \|\psi\varphi^n\|_{\mathcal{B}} < \infty$.*

Next, we consider the one-parameter family of functions on $\mathbb{D}$ defined in (11), which will be used to characterize the bounded and the compact weighted composition operator from $BMOA$ into $\mathcal{B}$.

Theorem 5.1. *For ψ and φ analytic on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, the following conditions are equivalent:*

(a) *$W_{\psi,\varphi} : \mathcal{B} \to \mathcal{B}$ is bounded.*

(b) *$W_{\psi,\varphi} : BMOA \to \mathcal{B}$ is bounded.*

(c) *$\sup_{n\geq 0} \|\psi\varphi^n\|_{\mathcal{B}} < \infty$, and $\sup_{a\in\mathbb{D}} \|W_{\psi,\varphi} f_a\|_{\mathcal{B}} < \infty$.*

(d) *$\tau_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty$ and*
$s_{\psi,\varphi} = \sup_{z\in\mathbb{D}} (1-|z|^2)|\psi'(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty$.

Proof. (a) $\Rightarrow$ (b) follows at once from the continuous inclusion of $BMOA$ into $\mathcal{B}$.

(b) $\Rightarrow$ (c) follows at once from the fact that the sequence $p_n(z) = z^n$ is bounded in $BMOA$, so that $\sup_{n\in\mathbb{N}} \|\psi\varphi^n\|_{\mathcal{B}} \leq \|W_{\psi,\varphi}\| \sup_{n\geq 0} \|p_n\|_{BMOA} < \infty$, where $W_{\psi,\varphi}$ is regarded as an operator from $BMOA$ to $\mathcal{B}$. Similarly, $\sup_{\alpha\in\mathbb{D}} \|W_{\psi,\varphi} f_a\|_{\mathcal{B}} \leq \|W_{\psi,\varphi}\| \sup_{a\in\mathbb{D}} \|f_a\|_{BMOA} < \infty$.

(c) $\Rightarrow$ (d) By Lemma 5.1, $\tau_{\psi,\varphi} < \infty$. To show that $s_{\psi,\varphi} < \infty$, fix $a \in \mathbb{D}$. Then

$$\|W_{\psi,\varphi} f_a\|_{\mathcal{B}} \geq (1-|a|^2) \left| \psi'(a) \log \frac{2}{1-|\varphi(a)|^2} + \frac{2\psi(a)\overline{\varphi(a)}\varphi'(a)}{1-|\varphi(a)|^2} \right|. \tag{12}$$

Hence $(1-|a|^2)|\psi'(a)| \log \frac{2}{1-|\varphi(a)|^2} \leq \|W_{\psi,\varphi} f_a\|_{\mathcal{B}} + 2\tau_{\psi,\varphi} < +\infty$.

(d) $\Rightarrow$ (a) follows from Theorem 1 of [14]. □

We now show that the weighted multiplication operators from $BMOA$ to $\mathcal{B}$ are compact if and only if they are compact as operators acting on $\mathcal{B}$. We shall need the following result.

Lemma 5.2 ([6], Corollary 2.2). *Let $\psi \in \mathcal{B}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic such that $(1-|z|^2)|\psi'(z)| \to 0$ as $|\varphi(z)| \to 1$. Then* $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0$ *if and only if* $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0$.

Theorem 5.2. *For $a \in \mathbb{D}$, let f_a be defined as in* (11). *For a bounded operator $W_{\psi,\varphi} : BMOA \to \mathcal{B}$ the following propositions are equivalent*:

(a) *$W_{\psi,\varphi} : BMOA \to \mathcal{B}$ is compact.*

(b) $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0$, *and* $\lim_{|\varphi(a)|\to 1} \|W_{\psi,\varphi} f_a\|_{\mathcal{B}} = 0$.

(c) $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0$ *and*
$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|\psi'(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0$.

(d) *$W_{\psi,\varphi} : \mathcal{B} \to \mathcal{B}$ is compact.*

Proof. (a) $\Rightarrow$ (b) The sequence $\{p_n\}$ defined by $p_n(z) = z^n$ for $n \in \mathbb{N}$ is bounded in $BMOA$ and converges to zero uniformly on compact subsets, so by Lemma 3.1, $\|\psi\varphi^n\|_{\mathcal{B}} \to 0$. On the other hand, if $\{a_n\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(a_n)| \to 1$, then the sequence $\{f_{a_n}\}$ defined as in (11) converges to 0 uniformly on compact subsets and is bounded in $BMOA$. Thus, again by Lemma 3.1, the compactness of $W_{\psi,\varphi}$ implies that $\|W_{\psi,\varphi} f_{a_n}\|_{\mathcal{B}} \to 0$.

(b) $\Rightarrow$ (c) By (12), for $a \in \mathbb{D}$, we have

$$(1-|a|^2)|\psi'(a)| \log \frac{2}{1-|\varphi(a)|^2} \leq \|W_{\psi,\varphi} f_a\|_{\mathcal{B}} + \frac{2(1-|a|^2)|\psi(a)\varphi'(a)|}{1-|\varphi(a)|^2}. \tag{13}$$

Then (c) follows immediately from (b), Lemma 5.2, and (13).

(c) $\Rightarrow$ (d) follows from Theorem 2 of [14] while (d) $\Rightarrow$ (a) is immediate due to the continuous inclusion of $BMOA$ into $\mathcal{B}$. □

In the special case when φ is the identity, we see that the only analytic function ψ satisfying (c) is the constant 0. Thus, there are no nontrivial compact multiplication operators from $BMOA$ to $\mathcal{B}$. When $\psi = 1$, from Theorem 2 of [19], Corollary 6.1 of [6], and Theorem 5.2, we obtain the following result.

Corollary 5.1. *For an analytic self map φ of $\mathbb{D}$, the following propositions are equivalent*:

(a) *$C_\varphi : H^\infty \to \mathcal{B}$ is compact.*

(b) $C_\varphi : \mathcal{B} \to \mathcal{B}$ *is compact.*
(c) $C_\varphi : \mathcal{D} \to \mathcal{B}$ *is compact.*
(d) $C_\varphi : BMOA \to \mathcal{B}$ *is compact.*
(e) $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0$.

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