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ON THE ISOPERIMETRIC DEFICIT UPPER LIMIT

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Abstract. In this paper, the reverse Bonnesen style inequalities for convex domain in the Euclidean plane $\mathbb{R}^2$ are investigated. The Minkowski mixed convex set of two convex sets K and L is studied and some new geometric inequalities are obtained. From these inequalities obtained, some isoperimetric deficit upper limits, that is, the reverse Bonnesen style inequalities for convex domain K are obtained. These isoperimetric deficit upper limits obtained are more fundamental than the known results of Bottema ([5]) and Pleijel ([22]).

1. Introductions and preliminaries

Perhaps the best known geometric inequality is the classical isoperimetric inequality. And its analytic proofs root back to centuries ago. One can find some simplified and beautiful proofs that lead to generalizations of higher dimensions and applications to other branches of mathematics (cf. [1], [6], [10], [11]-[17], [16], [20], [21], [24]-[25], [27], [29], [30], [31], [32], [34]-[35], [37], [39]).

The classical isoperimetric inequality says that: the circle is the only curve of constant perimeter enclosing the maximum area. This property is most precisely expressed in the following inequality:

$$P^2 - 4\pi A \geq 0, \tag{1}$$

where P and A are, respectively, the perimeter of curve Γ and the area Γ encloses. The equality sign holds if and only if Γ is a circle.

Let K be a domain with the boundary composing of the simple curve ∂K of perimeter P_K and area A_K in the Euclidean plane $\mathbb{R}^2$. The isoperimetric deficit of K is defined as

$$\Delta_2(K) = P_K^2 - 4\pi A_K. \tag{2}$$

The isoperimetric deficit $\Delta_2(K)$ measures the deficit between a domain K and a disc of radius $\frac{P_K}{2\pi}$. During the 1920's, Bonnesen proved a series of inequalities

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of the form

$$\Delta_2(K) \geq B_K, \tag{3}$$

where the quantity B_K is an invariant of geometric significance having the following basic properties:

1. B_K is non-negative;
2. B_K is vanish only when K is a disc.

Many Bs are found in the last century and mathematicians are still working on those unknown invariants of geometric significance. An inequality of type (3) is called a Bonnesen style inequality. See [2], [3], [4], [6], [10], [16], [19], [21], [24], [25], [34] and [33] for more detailed references.

A set of points K in the Euclidean space $\mathbb{R}^n$ is convex if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)y \in K$. The convex hull K^* of a set of points K in $\mathbb{R}^n$ is the intersection of all convex sets that contain K. A domain is a compact set with nonempty interiors. Since for any domain K in $\mathbb{R}^2$, its convex hull K^* increases the area A^* and decreases the perimeter P^*. Then we have $P^2 - 4\pi A \geq P^{*2} - 4\pi A^*$, that is, $\Delta_2(K) \geq \Delta_2(K^*)$. Therefore the isoperimetric inequality and the Bonnesen style inequality are valid for all domains in $\mathbb{R}^2$ if these inequalities are valid for convex domains.

The following known inequality is so-called the Bonnesen isoperimetric inequality:

Proposition 1. *Let K be a convex domain of area A_K and perimeter P_K. Let $r_I(K)$ and $r_E(K)$ be the radius of the maximum inscribed disc and the radius of the minimum circumscribed disc, respectively, of K. Then*

$$\Delta_2(K) = P_K^2 - 4\pi A_K \geq \pi^2(r_E(K) - r_I(K))^2,$$

where the equality holds if and only if K is a disc.

When mathematicians are mainly interested in and focus on the lower bounds of the isoperimetric deficit, that is, the Bonnesen-type inequalities, there is another question: Is there invariant U_K of geometric significance such that

$$\Delta_2(K) \leq U_K? \tag{4}$$

That is, to find the reverse Bonnesen style inequality. We also expect that the upper bound be attained when K is a disc. We are not aware of any general upper bound up today except for few results for special convex domains, that is, for oval domains ([5], [22], [25]).

Let K be an oval domain in $\mathbb{R}^2$ with the continues radius of curvature ρ of the boundary ∂K. Bottema (see [5], [25]) finds the following reverse Bonnesen isoperimetric inequality:

Proposition 2. *Let K be a convex domain of area A_K and perimeter P_K with the continuous radius of curvature ρ of ∂K. Let ρ_m and ρ_M be the smallest and the greatest values, respectively, of ρ. Then*

$$\Delta_2(K) = P_K^2 - 4\pi A_K \leq \pi^2(\rho_M - \rho_m)^2. \tag{5}$$

The equality sign holds if and only if $\rho_M = \rho_m$, that is, K is a disk.

Recently, the Bottema's inequality (5) has been generalized to the plane of constant curvature in [19].

Pleijel (see [22], [25]) has an improvement of Bottema's result as follows:

Proposition 3. *Let K be a convex domain of area A_K and perimeter P_K with the continuous curvature radius ρ of ∂K. Let ρ_m and ρ_M be the smallest and the greatest values, respectively, of ρ. Then*

$$\Delta_2(K) = P_K^2 - 4\pi A_K \leq \pi(4-\pi)(\rho_M - \rho_m)^2. \tag{6}$$

The equality sign holds if and only if K is a disc.

In this paper, we first investigate the Minkowski mixed area of two convex sets K, L in the Euclidean plane $\mathbb{R}^2$ and obtain some geometric inequalities involving the mixed area $A_{K,L}$ of K and L. By these inequalities obtained we derive some reverse Bonnesen style inequalities for a convex domain K. These reverse Bonnesen style inequalities, that is, the isoperimetric deficit upper bounds obtained, are invariants involving area A_K, circum length P_K, radius r_I of the inscribed disc and the radius r_E of the circumscribed disk of K. As we expected, those reverse Bonnesen style inequalities are held as equalities if and only if K is a disc. These upper limits obtained are analogues of Bottema's (5) and Pleijel's (6). One of the main results is Theorem 3 that improves the isoperimetric deficit upper limit of Bokowski and Heil in [2].

2. The mixed convex set of convex sets in $\mathbb{R}^2$

A line G in the Euclidean plane $\mathbb{R}^2$ can be determined by its distance p from the origin O and the angle ϕ of the normal with the x-axis. The line equation can be expressed as

$$G(p,\phi): \quad x\cos\phi + y\sin\phi - p = 0, \quad 0 \leq p < +\infty, \quad 0 \leq \phi \leq 2\pi. \tag{7}$$

If function $p = p(\phi)$ is of class C^2 and of periodic, then the envelope of the family of lines is:

$$x = p\cos\phi - p'\sin\phi. \quad y = p\sin\phi + p'\cos\phi. \tag{8}$$

If the envelope is the boundary ∂K of a convex set K and O is an interior point of K, then $p = p(\phi)$ is called the *support function* of K (or the support function of the convex curve ∂K with reference to the origin O). The lines (7) are called the support lines of K. Therefore we can prove that $p + p'' > 0$, and the arc-parameter is

$$ds = (p + p'')d\phi. \tag{9}$$

It is well-known that a necessary and sufficient condition that a function of period 2π should be the support function of a convex set K is that

$$p(\phi) + p''(\phi) > 0, \quad 0 \leq \phi < 2\pi. \tag{10}$$

From the Cauchy's formula for convex sets, we have the perimeter of K is

$$P_K = \int_0^{2\pi} p(\phi)\, d\phi, \tag{11}$$

and the area of a convex set K can also be evaluated by its support function, that is,

$$A_K = \frac{1}{2}\int_{\partial K} p\, ds = \frac{1}{2}\int_0^{2\pi} p(p + p'')\, d\phi \tag{12}$$

or

$$A_K = \frac{1}{2}\int_0^{2\pi} (p^2 - {p'}^2)\, d\phi. \tag{13}$$

Let K, L be compact convex sets with, respectively, support functions p_K, p_L assumed of class C^2. Then $p_K + p_L$ determines a convex set $M(K, L)$ called the *mixed convex set of* K *and* L.

The perimeter of $M(K, L)$ is

$$P_{M(K,L)} = P_K + P_L, \tag{14}$$

and the area of $M(K, L)$ is given by

$$A_{M(K,L)} = A_K + A_L + 2A_{K,L}.$$

The Minkowski *mixed area* of K and L is defined as

$$A_{K,L} = \frac{1}{2}\int_0^{2\pi} (p_K p_L - p'_K p'_L)\, d\phi. \tag{15}$$

Integration by parts gives,

$$A_{K,L} = \frac{1}{2}\int_0^{2\pi} p_K(p_L + p''_L) d\phi = \frac{1}{2}\int_0^{2\pi} p_L(p_K + p''_K) d\phi = A_{L,K}. \tag{16}$$

Therefore we have the following results:

Lemma 1. *For convex sets* K *and* L, $A_{K,L}$ *is rigid invariant.*

Lemma 2. *For any convex set* K,

$$A_{K,K} = A_K. \tag{17}$$

Lemma 3. *The mixed area* $A_{K,L}$ *is monotonic, that is, for convex sets* K, L_1, L_2 *such that* $L_1 \subset L_2$, *then*

$$A_{K,L_1} \le A_{K,L_2}. \tag{18}$$

Lemma 4. *Let* K *be a convex set. Then for the disc* B_r *of radius* r, *we have*

$$A_{K,B_r} = \frac{r}{2} P_K. \tag{19}$$

Lemma 5 (Minkowski)**.** *Let* K *and* L *be convex sets. Then*

$$A_{K,L}^2 - A_K A_L \ge 0, \tag{20}$$

where equality holds if and only if K *and* L *are homothetic.*

Let K be a convex set and B_r be disc of radius r. Then by Lemma 4 we have

$$A^2_{K,B_r} - A_K A_{B_r} = \left(\frac{r}{2}P_K\right)^2 - \pi r^2 A_K = \frac{r^2}{4}(P_K^2 - 4\pi A_K). \tag{21}$$

Then by Lemma 5 we immediately obtain the known classical isoperimetric inequality (1).

3. The isoperimetric deficit upper limit for convex domains in $\mathbb{R}^2$

Let K be a convex domain of length P_K and area A_K in $\mathbb{R}^2$. Assume that K encloses a maximum inscribed circle of radius r_I and is circumscribed in a smallest circle of radius r_E. Then we have the following inequalities that will lead to our isoperimetric upper limits.

Theorem 1. *Let K be a convex domain in the Euclidean plane $\mathbb{R}^2$ and d_K be the diameter of K. Then the invariants P_K, A_K, r_I, r_E of K satisfy the following inequalities*:

$$r_I \le \frac{2A_K}{P_K} \le \sqrt{\frac{A_K}{\pi}} \le \frac{P_K}{2\pi} \le \frac{d_K}{2} \le r_E. \tag{22}$$

Each equality sign holds if and only if K is a disc.

Proof. Since the inequalities $\frac{2A_K}{P_K} \le \sqrt{\frac{A_K}{\pi}} \le \frac{P_K}{2\pi}$ are just the variant forms of the isoperimetric inequality. The inequality $\frac{P_K}{2\pi} \le \frac{d_K}{2} \le r_E$ is known ([25], [24]). Therefore we just need to prove the first inequality $r_I \le \frac{2A_K}{P_K}$.

The inequality

$$r_I \le \frac{2A_K}{P_K} \tag{23}$$

comes immediately from Lemma 2, Lemma 3 and Lemma 4. We complete the proof of Theorem 1. □

Remark 1. The inequality (23) holds for all convex domains, with C^2-smooth or non C^2-smooth boundary ([36], [37]). Therefore all inequalities in (22) are valid for all convex domains, even for convex domains with non C^2-smooth boundaries.

Via inequalities

$$r_I \le \frac{2A_K}{P_K} \le \frac{P_K}{2\pi} \le r_E$$

in (22), we have

$$\frac{P_K}{2\pi} - \frac{2A_K}{P_K} \le r_E - r_I,$$

that is,

$$P_K^2 - 4\pi A_K \le 2\pi P_K(r_E - r_I).$$

We have proved the following:

Theorem 2. *Let K be a convex domain of perimeter P_K, and area A_K in the Euclidean plane $\mathbb{R}^2$. Let r_I and r_E be, respectively, the in-radius and the circum-radius of K. Then we have the following reverse Bonnesen style inequality*:

$$\Delta_2(K) = P_K^2 - 4\pi A_K \leq 2\pi P_K(r_E - r_I), \tag{24}$$

where the equality holds if and only if K is a disc.

The inequality (24) is also obtained by Bokowski and Heil for convex domain K with C^2-smooth boundary ∂K (see [2]) by a different approach. And they state that the inequality is better than Favard's inequality (cf. page 83 in [4]) for the minimal circular annulus (minimalkreisring).

Via inequalities

$$r_I \leq \frac{2A_K}{P_K} \leq \frac{P_K}{2\pi} \leq \frac{d_K}{2}$$

from (22), we have

$$\frac{P_K}{2\pi} - \frac{2A_K}{P_K} \leq \frac{d_K}{2} - r_I(K),$$

that is,

$$P_K^2 - 4\pi A_K \leq 2\pi P_K\left(\frac{d_K}{2} - r_I\right).$$

We obtain the following isoperimetric deficit upper limit that is an improvement of Bokowski and Heil's inequality (24).

Theorem 3. *Let K be a convex domain of perimeter P_K and area A_K in the Euclidean plane $\mathbb{R}^2$. Let r_I and d_K be, respectively, the in-radius and the diameter of K. Then we have the following reverse Bonnesen style inequality*:

$$\Delta_2(K) = P_K^2 - 4\pi A_K \leq 2\pi P_K\left(\frac{d_K}{2} - r_I\right), \tag{25}$$

where the equality holds if and only if K is a disc.

From inequalities

$$r_I \leq \frac{2A_K}{P_K} \leq \sqrt{\frac{A_K}{\pi}} \leq r_E$$

in (22), we have

$$r_I^2 \leq \frac{4A_K^2}{P_K^2} \leq \frac{A_K}{\pi} \leq r_E^2,$$

and then

$$\frac{A_K}{\pi} - \frac{4A_K^2}{P_K^2} \leq r_E^2 - r_I^2.$$

Therefore we have

$$P_K^2 - 4\pi A_K \leq \frac{\pi P_K^2}{A_K}(r_E^2 - r_I^2). \tag{26}$$

By inequalities

$$r_I \leq \sqrt{\frac{A_K}{\pi}} \leq \frac{P_K}{2\pi} \leq r_E$$

in (22), we have

$$r_I^2 \leq \frac{A_K}{\pi} \leq \frac{P_K^2}{4\pi^2} \leq r_E^2,$$

and then

$$\frac{P_K^2}{4\pi^2} - \frac{A_K}{\pi} \leq r_E^2 - r_I^2,$$

that is,

$$P_K^2 - 4\pi A_K \leq 4\pi^2(r_E^2 - r_I^2). \tag{27}$$

We have proved the following theorem.

Theorem 4. *Let K be a convex domain of perimeter P_K and area A_K in the Euclidean plane $\mathbb{R}^2$. Let r_I and r_E be, respectively, the in-radius and the circum-radius of K. Then we have the following reverse Bonnesen style inequalities*:

$$\begin{aligned} \Delta_2(K) = P_K^2 - 4\pi A_K &\leq \tfrac{\pi P_K^2}{A_K}(r_E^2 - r_I^2); \\ \Delta_2(K) = P_K^2 - 4\pi A_K &\leq 4\pi^2(r_E^2 - r_I^2). \end{aligned} \tag{28}$$

Each equality sign holds if and only if K is a disc.

Also by inequalities

$$r_I \leq \frac{2A_K}{P_K} \leq \frac{P_K}{2\pi} \leq \frac{d_K}{2},$$

we have:

Theorem 5. *Let K be a convex domain of the perimeter P_K and area A_K in the Euclidean plane $\mathbb{R}^2$. Let r_I and d_K be, respectively, the in-radius and the diameter of K. Then we have the following reverse Bonnesen style inequalities*:

$$\begin{aligned} \Delta_2(K) = P_K^2 - 4\pi A_K &\leq \tfrac{\pi P_K^2}{A_K}\left(\tfrac{d_K^2}{4} - r_I^2\right); \\ \Delta_2(K) = P_K^2 - 4\pi A_K &\leq 4\pi^2\left(\tfrac{d_K^2}{4} - r_I^2\right). \end{aligned} \tag{29}$$

Each equality holds as an equality if and only if K is a disc.

Remark 2. By the isoperimetric inequality (1) and inequalities (22) we have

$$\begin{aligned} 2\pi P_K\left(\frac{d_K}{2} - r_I\right) &\leq 2\pi P_K(r_E - r_I) \leq 4\pi^2 r_E(r_E - r_I) \\ &\leq 4\pi^2(r_E^2 - r_I^2) \leq \frac{\pi P_K^2}{A_K}(r_E^2 - r_I^2). \end{aligned}$$

Therefore the inequality (25), that is,

$$\Delta_2(K) = P_K^2 - 4\pi A_K \leq 2\pi P_K\left(\frac{d_K}{2} - r_I\right) \tag{30}$$

is the best isoperimetric deficit upper limit.

Remark 3. Our isoperimetric deficit upper limits do not assume K is an oval domain, that is, we do not assume K with the continuous radius of curvature ρ. Therefore those isoperimetric deficit upper limits obtained are more fundamental than Bottema and Pleijel's results.

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