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GLOBAL ATTRACTOR FOR COUPLED TWO-COMPARTMENT GRAY-SCOTT EQUATIONS

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Abstract. This paper is concerned with the long time behavior for the solution semiflow of the coupled two-compartment Gray-Scott equations with the homogeneous Neumann boundary condition on a bounded domain of space dimension $n \leq 3$. Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, we prove that the equations possesses a global attractor in $H^k(\Omega)^4$ $(k \geq 0)$ space.

1. Introduction

In this paper, we study a coupled two-compartment Gray-Scott equation, which is a four-component reaction-diffusion system [4, 9, 15].

$$u_t = d_1 \Delta u - (F+k)u + u^2 v + D_1(w-u), \tag{1.1}$$

$$v_t = d_2 \Delta v + F(1-v) - u^2 v + D_2(z-v), \tag{1.2}$$

$$w_t = d_1 \Delta w - (F+k)w + w^2 z + D_1(u-w), \tag{1.3}$$

$$z_t = d_2 \Delta z + F(1-z) - w^2 z + D_2(v-z), \tag{1.4}$$

for $t>0$, on a bounded domain $\Omega \subset \mathbb{R}^n, n \leq 3$, the equations (1.1)-(1.4) have the following boundary conditions

$$\frac{\partial u}{\partial \nu}(t,x) = \frac{\partial v}{\partial \nu}(t,x) = \frac{\partial w}{\partial \nu}(t,x) = \frac{\partial z}{\partial \nu}(t,x) = 0,\ t>0,\ x \in \partial\Omega, \tag{1.5}$$

and initial conditions

$$\begin{aligned} &u(0,x) = u_0(x),\ v(0,x) = v_0(x), \\ &w(0,x) = w_0(x),\ z(0,x) = z_0(x),\ x \in \Omega, \end{aligned} \tag{1.6}$$

where d_1, d_2, F, k, D_1, and D_2 are positive constants.

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As is well-known, the Gray-Scott model was originally a system of two ordinary differential equations describing the kinetics of cubic autocatalytic chemical or biochemical reactions (see [5, 6, 7]). Assume A is an autocatalytic reactant which decays to form a product P in the irreversible reactions shown above, while B is another reactant for which higher concentration beyond a certain level increases the rate of its own removal, then the kinetics describes the following scheme of chemical reactions:

$$2A + B \xrightarrow{k_1} 3A, \;\; A \xrightarrow{k_1} P,$$

where k_1 is the reaction rate constant. Then by the law of mass action and the Fick's law, one obtain a system of two nonlinear reaction-diffusion equations which called the Gray-Scott equations:

$$\begin{aligned} u_t &= d_1 \Delta u - (F + k)u + u^2 v, \\ v_t &= d_2 v + F(1 - v) - u^2 v, \end{aligned}$$

where k is called the effective production rate constant and $1/F$ is called the mean residence time in dimensionless units. The known examples of isothermal autocatalytic reactions which can be modeled by Gray-Scott equations, including the ferrocyanide-iodate-sulphite reaction, the chlorite-iodide-malonic acid (CIMA) reaction, and quite a few enzyme catalytic reactions (see [1, 2, 10]). During the past years, there are many papers were denoted to the Gray-Scott equations, for example [8, 13, 14] and so on.

The dynamic properties of diffusion equation and diffusion system such as the global asymptotical behaviors of solutions and global attractors are important for the study of diffusion model, which ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. There are many studies on the existence of global attractors for diffusion equations. For the classical results we refer the reader to [3, 11, 19, 21, 22, 24]. Recently, based on the iteration technique for regularity estimates, combining with the classical existence theorem of global attractors, Song et al. [17, 18] considered the global attractor for some parabolic equations, such as Cahn-Hilliard equation, Swift-Hohenberg equation and so on, in H^k $(0 \le k \le \infty)$ space. Zhao and Liu [23] studied the global attractor for a fourth order parabolic equation modeling epitaxial thin-film growth in H^k $(0 \le k < 5)$ space.

In this paper, we are interested in the existence of global attractors for the diffusion system (1.1)-(1.4). Based on You's recent paper [20] and Ma and Wang's work [12], we shall prove that the problem (1.1)-(1.6) possesses a global attractor in $H^k(\Omega)^4$ $(0 \le k < \infty)$ space.

The outline of this paper is as follows: In the next section, we give some preparations for our consideration, we also give the main result on the existence of global attractor for the problem (1.1)-(1.6); In Section 3, the main result is proved.

In the following, the letters $C, C_i, \ (i = 0, 1, 2, \ldots)$ will always denote positive constants different in various occurrences.

2. Preliminary

Assume X and X_1 are two Banach spaces, $X_1 \subset X$ a compact and dense inclusion. Consider the following equation defined on X,

$$\begin{cases} U_t = LU + QU, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where U is an unknown function, $L : X_1 \to X$ a linear operator and $Q : X_1 \to X$ a nonlinear operator. Then the solution of (2.1) can be expressed as

$$U(t, U_0) = S(t)U_0,$$

where $S(t) : X \to X \ (t \geq 0)$ is a semiflows generated by (2.1).

We used to assume that the linear operator $L : X_1 \to X$ in (2.1) is a sectorial operator, which generates an analytic semiflows e^{tL}, and L induces the fractional power operators $\mathscr{L}^\alpha$ and fractional order spaces X_α as follows,

$$\mathscr{L}^\alpha = (-L)^\alpha : X_\alpha \to X, \ \alpha \in R, \tag{2.2}$$

where $X_\alpha = D(\mathscr{L}^\alpha)$ is the domain of $\mathscr{L}^\alpha$. By the semiflows theory of linear operators, $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$. If you want to know more about the space H_α, I recommend you read [12].

Now, we introduce a lemma on the existence of global attractor which can be founded in [12, 17, 18, 23].

Lemma 2.1. *Assume that $U(t, U_0) = S(t)U_0$ $(U_0 \in X, t \geq 0)$ is a solution of* (2.1) *and $S(t)$ the semiflows generated by* (2.1). *Assume further that X_α is the fractional order space generated by L and*

(B1) *For some $\alpha \geq 0$ there is a bounded set $B \subset X_\alpha$, which means for any $U_0 \in X_\alpha$, there exists $t_{U_0} > 0$ such that*

$$U(t, U_0) \in B, \ \forall t > t_{U_0};$$

(B2) *There is a $\beta > \alpha$, for any bounded set $E \subset X_\beta$, $\exists T > 0$ and $C > 0$ such that*

$$\|U(t, U_0)\|_{X_\beta} \leq C, \ \forall t > T, \ U_0 \in E.$$

Then (2.1) *has a global attractor $\mathscr{A} \subset X_\alpha$ which attracts any bounded set of X_α in the X_α-norm.*

We also have the following lemma which can be founded in [12, 17, 18, 23].

Lemma 2.2. *Assume that $L : X_1 \to X_\alpha$ is a sectorial operator which generates an analytic semiflows $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy* $\text{Re}\lambda < -\lambda_0$ *for some real number $\lambda_0 > 0$, then for $\mathscr{L}^\alpha (\mathscr{L} = -L)$ we have*

(C1) *$T(t) : X \to X_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$;*

(C2) *$T(t)\mathscr{L}^\alpha x = \mathscr{L}^\alpha T(t)x, \forall x \in X_\alpha$;*

(C3) *For each* $t > 0$, $\mathscr{L}^\alpha T(t) : X \to X$ *is bounded, and*

$$\|\mathscr{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t};$$

where some $\delta > 0$ *and* $C_\alpha > 0$ *is a constant depending only on* α;

(C4) *The* X_α*-norm can be defined by* $\|x\|_{X_\alpha} = \|\mathscr{L}^\alpha x\|_X$.

Now, we introduce the space as follows

$$(2.3)\quad \begin{cases} \mathcal{H} = H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \\ \mathcal{H}_{\frac{1}{2}} = \{(u,v,w,z) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega); \\ \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0\} \\ \mathcal{H}_1 = (H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)) \bigcap \mathcal{H}_{\frac{1}{2}}. \end{cases}$$

We also define the operators L_i $(i = 1, 2)$ and Q_i $(i = 1, 2, 3, 4)$ by

$$(2.4)\quad \begin{cases} L_1 u = d_1 \Delta u, \ L_2 v = d_2 \Delta v, \ L_1 w = d_1 \Delta w, \ L_2 z = d_2 \Delta z, \\ Q_1 g = q_1(u,v,w,z) = -(F+k)u + u^2 v + D_1(w-u), \\ Q_2 g = q_2(u,v,w,z) = F(1-v) - u^2 v + D_2(z-v), \\ Q_3 g = q_3(u,v,w,z) = -(F+k)w + w^2 z + D_1(u-w), \\ Q_4 g = q_4(u,v,w,z) = F(1-z) - w^2 z + D_2(v-z), \end{cases}$$

where $g = \mathrm{col}(u,v,w,,z)$. It is easy to check that $q_i (i = 1,2,3,4)$ are nonlinear functions, $q_1(u,v,w,z) = q_3(w,z,u,v)$ and $q_2(u,v,w,z) = q_4(w,z,u,v)$. Obviously, the linear operator $L_i : H^2(\Omega) \to L^2(\Omega)$, $(i = 1, 2)$ given by (2.4) are sectorial operators.

By using the Lumer-Phillips theorem and the analytic semiflows generation theorem [16], we obtain the linear operator

$$(2.5)\quad L = \begin{pmatrix} d_1\Delta & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 \\ 0 & 0 & d_1\Delta & 0 \\ 0 & 0 & 0 & d_2\Delta \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_1 \\ L_2 \end{pmatrix} : H^2(\Omega)^4 \to L^2(\Omega)^4,$$

which is the generator of an analytic C_0-semiflows on the Hilbert space $L^2(\Omega)^4$. Define

$$Qg = \begin{pmatrix} Q_1 g \\ Q_2 g \\ Q_3 g \\ Q_4 g \end{pmatrix} = \begin{pmatrix} q_1(u,v,w,z) \\ q_2(u,v,w,z) \\ q_3(u,v,w,z) \\ q_4(u,v,w,z) \end{pmatrix} = \begin{pmatrix} -(F+k)u + u^2 v + D_1(w-u) \\ F(1-v) - u^2 v + D_2(z-v) \\ -(F+k)w + w^2 z + D_1(u-w) \\ F(1-z) - w^2 z + D_2(v-z) \end{pmatrix},$$

then the initial boundary value problem (1.1)-(1.6) is formulated into the following problem:

$$(2.6)\quad \frac{dg}{dt} = Lg + Qg, \ t > 0,$$

where $g = \mathrm{col}(u,v,w,z)$, or written as (u,v,w,z), for any initial data $g(0) = g_0 = \mathrm{col}(u_0,v_0,w_0,z_0)$, or written as (u_0,v_0,w_0,z_0). It is easy to see that in (2.6) L is a linear operator and Q a nonlinear operator.

Compared with (2.1), it is easy to see that $X = \mathcal{H}$, $X_1 = \mathcal{H}_1$, $L : \mathcal{H}_1 \to \mathcal{H}$ is a linear sectorial operator and Q a nonlinear operator in (2.6). We can define the fractional order spaces $\mathscr{L}^\alpha$ as (2.2), where $\mathcal{H}_\alpha = D(\mathscr{L}^\alpha) = \mathrm{H}_\alpha \times \mathrm{H}_\alpha \times \mathrm{H}_\alpha \times \mathrm{H}_\alpha = D((-L_1)^\alpha) \times D((-L_2)^\alpha) \times D((-L_1)^\alpha) \times D((-L_2)^\alpha)$ is the domain of $\mathscr{L}^\alpha$.

The following propositions on the existence and uniqueness of strong solution and global weak solution for problem (2.6) can be found in [20].

Proposition 2.3. *For any given initial data $g_0 \in \mathcal{H}$, there exists a unique, local weak solution $g(t) = (u(t), v(t), w(t), z(t))$, $t \in [0, \tau]$ for some $\tau > 0$, of the problem* (2.6)*, which becomes a strong solution on* $(0, \tau]$ *and satisfies*

$$g \in C([0, T_{\max}); \mathcal{H}) \bigcap C^1((0, T_{\max}); \mathcal{H}) \bigcap L^2(0, T_{\max}; \mathcal{H}_{\frac{1}{2}}).$$

Proposition 2.4. *For any given initial data $g_0 \in \mathcal{H}$, there exists a unique, global, weak solution $g(t) = (u(t), v(t), w(t), z(t))$, $t \in [0, \infty)$, of the problem* (2.6)*.*

Based on Proposition 2.4, we can define a semiflow $\{S(t)\}_{t\geq 0}$ on $L^2(\Omega)^4$, where

$$S(t) : g_0 \longmapsto g(t, g_0), \ g_0 \in L^2(\Omega)^4, \ t \geq 0,$$

which will be called coupled Gray-Scott semiflow generated by the two-compartment Gray-Scott evolutionary equations (1.1)-(1.6).

We summarize the results in [20].

Proposition 2.5. *For any given positive parameters d_1, d_2, F, k, D_1 and D_2, there exists a constant $K_1 > 0$ such that the set*

$$B_0 = \{g \in L^2(\Omega)^4 : \|g\|^2 \leq K_1\}$$

is a bounded absorbing set in $L^2(\Omega)^4$ for the coupled Gray-Scott semiflow $\{S(t)\}_{t\geq 0}$.

Proposition 2.6. *For any given positive parameters d_1, d_2, F, k, D_1, D_2 and initial data $(u_0, v_0, w_0, z_0) \in B_0$, the $(u(t), w(t))$ components of the solution trajectory $g(t) = S(t)g_0$ of the initial value problem* (2.6) *satisfy*

$$\|u(t)\|^2_{H^1} + \|w(t)\|^2_{H^1} \leq M_1, \ \forall t > T_1,$$

where $M_1 > 0$ is a constant depending on $|\Omega|$ but independent of the initial data, and $T_1 > 0$ is finite and only depends on K_1 and $|\Omega|$.

Proposition 2.7. *For any given positive parameters d_1, d_2, F, k, D_1, D_2 and initial data $(u_0, v_0, w_0, z_0) \in B_0$, the $(v(t), z(t))$ components of the solution trajectory $g(t) = S(t)g_0$ of the initial value problem* (2.6) *satisfy*

$$\|v(t)\|^2_{H^1} + \|z(t)\|^2_{H^1} \leq M_2, \ \forall t > T_2,$$

where $M_2 > 0$ is a constant depending on $|\Omega|$ but independent of the initial data, and $T_2 > 0$ is finite and only depends on K_1 and $|\Omega|$.

Proposition 2.8. *For any given positive parameters d_1, d_2, F, k, D_1, D_2 and the constant $R > 0$, there exists a constant $M(R) > 0$ such that if the initial datum $(u_0, v_0, w_0, z_0) \in H^1(\Omega)^4$ and*

$$\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|w_0\|^2_{H^1(\Omega)} + \|z_0\|^2_{H^1(\Omega)} \leq R,$$

then for all $t \geq 0$, we have

$$\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)} + \|z\|^2_{H^1(\Omega)} \leq M(R).$$

Proposition 2.9. *For any given positive parameters d_1, d_2, F, k, D_1 and D_2, there exists a global attractor in the phase space $L^2(\Omega)^4$ for the solution semiflow $\{S(t)\}_{t\geq 0}$ of the problem* (2.5).

In the following, we give the main result, which provides the existence of global attractors of the equations (1.1)-(1.4) in any kth space $H^k(\Omega)^4$, where $0 \leq k < \infty$.

Theorem 2.10. *For any given positive parameters d_1, d_2, F, k, D_1, D_2 and $g_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}_\alpha$ $(\forall \alpha \geq 0)$, the semiflows $S(t)$ associated with problem* (1.1)-(1.6) *possesses a global attractor $\mathscr{A}$ in $\mathcal{H}_\alpha$ space and $\mathscr{A}$ attractors any bounded set of $\mathcal{H}_\alpha$ in the $\mathcal{H}_\alpha$-norm.*

3. Proof of Theorem 2.10

We are now in a position to state and prove the main theorem in this paper, which provides the existence of a global attractor of the equations (1.1)-(1.6) in spaces $\mathcal{H}_\alpha$ of any αth differentiable function.

For any $g_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$, the solution (u, v, w, z) of the problem (1.1)-(1.6) can be written as

$$\begin{aligned} u(t, u_0) &= e^{tL_1} u_0 + \int_0^t e^{(t-\tau)L_1} Q_1 g d\tau \\ &= e^{tL_1} u_0 + \int_0^t e^{(t-\tau)L_1} q_1(u, v, w, z) d\tau, \end{aligned} \tag{3.1}$$

$$\begin{aligned} v(t, v_0) &= e^{tL_2} v_0 + \int_0^t e^{(t-\tau)L_2} Q_2 g d\tau \\ &= e^{tL_2} v_0 + \int_0^t e^{(t-\tau)L_2} q_2(u, v, w, z) d\tau, \end{aligned} \tag{3.2}$$

$$\begin{aligned} w(t, w_0) &= e^{tL_1} w_0 + \int_0^t e^{(t-\tau)L_1} Q_3 g d\tau \\ &= e^{tL_1} w_0 + \int_0^t e^{(t-\tau)L_1} q_3(u, v, w, z) d\tau, \end{aligned} \tag{3.3}$$

$$z(t, z_0) = e^{tL_2} z_0 + \int_0^t e^{(t-\tau)L_2} Q_4 g d\tau$$

$$= e^{tL_2}z_0 + \int_0^t e^{(t-\tau)L_2} q_4(u,v,w,z)d\tau, \tag{3.4}$$

By Lemma 2.1, in order to prove Theorem 2.10, we first prove the following lemma.

Lemma 3.1. *Suppose d_1, d_2, F, k, D_1 and D_2 are given positive parameters, $g = (u,v,w,z)$ is a solution to the problem* (1.1)-(1.6)*, $g_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}_\alpha$ ($\forall \alpha \geq 0$), then, the semiflows $S(t)$ associated with problem* (1.1)-(1.6) *is uniformly compact in $\mathcal{H}_\alpha$.*

Proof. It suffices to prove that for any bounded set $B \subset \mathcal{H}_\alpha$ with initial value $g_0 = (u_0, v_0, w_0, z_0) \in B \subset \mathcal{H}_\alpha$, there exists $C > 0$ such that

$$\|g(u, g_0)\|_{\mathcal{H}_\alpha} \leq C, \ \forall t \geq 0, \ \alpha \geq 0. \tag{3.5}$$

Obviously, if we get

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \leq C. \ \forall t \geq 0, \ \alpha \geq 0,$$

then, we obtain (3.5) immediately.

For $\alpha = \frac{1}{2}$, this follows from Proposition 2.8, i.e., for any bounded set $B \subset \mathcal{H}_{\frac{1}{2}}$ with initial value $(u_0, v_0, w_0, z_0) \in B \subset \mathcal{H}_{\frac{1}{2}}$, there exists a constant $C > 0$ such that $\forall t \geq 0$,

$$\|u(t,u_0)\|_{\mathrm{H}_{\frac{1}{2}}} + \|v(t,v_0)\|_{\mathrm{H}_{\frac{1}{2}}} + \|w(t,w_0)\|_{\mathrm{H}_{\frac{1}{2}}} + \|z(t,z_0)\|_{\mathrm{H}_{\frac{1}{2}}} \leq C. \tag{3.6}$$

So, we only need to show (3.5) for any $\alpha \geq \frac{1}{2}$. There are three steps for us to prove it.

Step 1. We prove that for any bounded set $B \subset \mathcal{H}_\alpha$ ($\frac{1}{2} \leq \alpha < 1$), there exists a positive constant C such that $\forall t \geq 0, \ \frac{1}{2} \leq \alpha < 1$,

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \leq C. \tag{3.7}$$

For the dimension $n \leq 3$, we have $H^1(\Omega) \hookrightarrow L^6(\Omega)$. It then follows from Proposition 2.5 and Proposition 2.8 that

$$\begin{aligned}
\|q_1(u,v,w,z)\|^2_{L^2} &= \int_\Omega |q_1(u,v,w,z)|^2 dx \\
&= \int_\Omega [-(F+k)u + u^2 v + D_1(w-u)]^2 dx \\
&\leq C \left(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \int_\Omega u^4 v^2 dx \right) \\
&\leq C \left(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \big(\int_\Omega u^6 dx \big)^{\frac{4}{3}} + \big(\int_\Omega v^6 dx \big)^{\frac{2}{3}} \right) \\
&\leq C (\|u\|^2 + \|w\|^2 + \|u\|^{\frac{8}{3}}_{H^1} + \|v\|^{\frac{4}{3}}_{H^1}) \leq C
\end{aligned} \tag{3.8}$$

and

$$\|q_2(u,v,w,z)\|^2_{L^2} = \int_\Omega |q_2(u,v,w,z)|^2 dx$$

$$
\begin{aligned}
&= \int_\Omega [F(1-v) - u^2 v + D_2(z-v)]^2 dx \\
&\le C\left(C + \int_\Omega v^2 dx + \int_\Omega u^4 v^2 dx + \int_\Omega z^2 dx\right) \\
&\le C\left(C + \int_\Omega v^2 dx + \big(\int_\Omega u^6 dx\big)^{\frac{4}{3}} + \big(\int_\Omega v^6 dx\big)^{\frac{2}{3}} + \int_\Omega z^2 dx\right) \\
&\le C(\|v\|^2 + \|z\|^2 + \|u\|_{H^1}^{\frac{8}{3}} + \|v\|_{H^1}^{\frac{4}{3}}) \le C. \qquad (3.9)
\end{aligned}
$$

Note that $q_1(u,v,w,z) = q_3(w,z,u,v)$ and $q_2(u,v,w,z) = q_4(w,z,u,v)$, simple calculations show that

$$\|q_3(u,v,w,z)\|_{L^2}^2 \le C, \tag{3.10}$$

$$\|q_4(u,v,w,z)\|_{L^2}^2 \le C. \tag{3.11}$$

By (3.1), (3.6) and (3.8), we obtain

$$
\begin{aligned}
\|u(t,u_0)\|_{\mathrm{H}_\alpha} &= \|e^{tL_1}u_0 + \int_0^t e^{(t-\tau)L_1} q_1(u,v,w,z) d\tau\|_{\mathrm{H}_\alpha} \\
&\le \|e^{tL_1}u_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)L_1} q_1(u,v,w,z) d\tau\|_{\mathrm{H}_\alpha} \\
&\le C\|u_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-L_1)^\alpha e^{(t-\tau)L_1}\| \cdot \|q_1(u,v,w,z)\|_{L^2(\Omega)} d\tau \\
&\le C\|u_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau} d\tau \\
&\le C, \ \forall t \ge 0, \ g_0 \in B, \qquad (3.12)
\end{aligned}
$$

where $0 < \alpha < 1$. By (3.2), (3.6) and (3.9), we obtain

$$
\begin{aligned}
\|v(t,v_0)\|_{\mathrm{H}_\alpha} &= \|e^{tL_2}v_0 + \int_0^t e^{(t-\tau)L_2} q_2(u,v,w,z) d\tau\|_{\mathrm{H}_\alpha} \\
&\le \|e^{tL_2}v_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)L_2} q_2(u,v,w,z) d\tau\|_{\mathrm{H}_\alpha} \\
&\le C\|v_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-L_2)^\alpha e^{(t-\tau)L_2}\| \cdot \|q_2(u,v,w,z)\|_{L^2(\Omega)} d\tau \\
&\le C\|v_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau} d\tau \\
&\le C, \ \forall t \ge 0, \ g_0 \in B, \qquad (3.13)
\end{aligned}
$$

where $0 < \alpha < 1$. By (3.3), (3.4), (3.6), (3.10) and (3.11), simple calculations shows that

$$\|w(t,w_0)\|_{\mathrm{H}_\alpha} \le C, \ \forall t \ge 0, \ g_0 \in B, \tag{3.14}$$

$$\|z(t,z_0)\|_{\mathrm{H}_\alpha} \le C, \ \forall t \ge 0, \ g_0 \in B, \tag{3.15}$$

where $0 < \alpha < 1$. By (3.12), (3.13), (3.14) and (3.15), we obtain (3.7) immediately.

Step 2. We prove that for any bounded set $B \subset \mathcal{H}_\alpha$ $(1 \leq \alpha < \frac{3}{2})$, there exists a positive constant C such that $\forall t \geq 0,\ 1 \leq \alpha < \frac{3}{2}$,

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \leq C. \tag{3.16}$$

By Proposition 2.5, Proposition 2.8 and the following embedding theorems of fractional order spaces

$$\mathrm{H}_\alpha \hookrightarrow C^0(\Omega) \bigcap H^1(\Omega),\ \forall \alpha > \frac{3}{4}, \tag{3.17}$$

we obtain

$$\begin{aligned} &\|q_1(u,v,w,z)\|^2_{\mathrm{H}_{\frac{1}{2}}} \\ &\leq C\int_\Omega |\nabla q_1(u,v,w,z)|^2 dx + C_0 \\ &\leq C\int_\Omega |\nabla(-(F+k)u + u^2 v + D_1(w-u))|^2 dx + C_0 \\ &\leq C\int_\Omega \big(-(F+k)\nabla u + 2uv\nabla u + u^2\nabla v + D_1\nabla w - D_1\nabla u\big)^2 dx + C_0 \\ &\leq C\int_\Omega \big(|\nabla u|^2 + u^2v^2|\nabla u|^2 + u^4|\nabla v|^2 + |\nabla w|^2 + |\nabla u|^2\big) dx + C_0 \\ &\leq C\int_\Omega \left(|\nabla u|^2 + \sup_{x\in\Omega} u^2v^2 \cdot |\nabla u|^2 + \sup_{x\in\Omega} u^4 \cdot |\nabla v|^2 + |\nabla w|^2 + |\nabla u|^2\right) dx \\ &\quad + C_0 \\ &\leq C\int_\Omega \big(|\nabla u|^2 + |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 + |\nabla u|^2\big) dx + C_0 \\ &\leq C(\|u\|^2_{\mathrm{H}_\alpha} + \|v\|^2_{\mathrm{H}_\alpha} + \|w\|^2_{\mathrm{H}_\alpha}) + C_0 \leq C \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\|q_2(u,v,w,z)\|^2_{\mathrm{H}_{\frac{1}{2}}} \\ &\leq C\int_\Omega |\nabla q_2(u,v,w,z)|^2 dx + C_0 \\ &\leq C\int_\Omega [\nabla(F(1-v) - u^2 v + D_2(z-v))]^2 dx + C_0 \\ &\leq C\int_\Omega \big(-F\nabla v - 2uv\nabla u - u^2\nabla v + D_2\nabla z - D_2\nabla v\big)^2 dx + C_0 \\ &\leq C\int_\Omega \big(|\nabla v|^2 + u^2v^2|\nabla u|^2 + u^4|\nabla v|^2 + |\nabla z|^2 + |\nabla v|^2\big) dx + C_0 \\ &\leq C\int_\Omega \left(|\nabla v|^2 + \sup_{x\in\Omega} u^2v^2 \cdot |\nabla u|^2 + \sup_{x\in\Omega} u^4 \cdot |\nabla v|^2 + |\nabla z|^2 + |\nabla v|^2\right) dx \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&+ C_0\\
&\le C\int_\Omega \left(|\nabla v|^2+|\nabla u|^2+|\nabla v|^2+|\nabla z|^2+|\nabla v|^2\right)dx + C_0\\
&\le C(\|u\|^2_{\mathrm{H}_\alpha}+\|v\|^2_{\mathrm{H}_\alpha}+\|z\|^2_{\mathrm{H}_\alpha})+C_0\le C.
\end{aligned}
$$

Note that $q_1(u,v,w,z)=q_3(w,z,u,v)$ and $q_2(u,v,w,z)=q_4(w,z,u,v)$, simple calculations show that

$$\|q_3(u,v,w,z)\|^2_{\mathrm{H}_{\frac{1}{2}}}\le C, \tag{3.20}$$

$$\|q_4(u,v,w,z)\|^2_{\mathrm{H}_{\frac{1}{2}}}\le C. \tag{3.21}$$

By (3.1), (3.7) and (3.18), we obtain

$$
\begin{aligned}
\|u(t,u_0)\|_{\mathrm{H}_\alpha} &= \|e^{tL_1}u_0+\int_0^t e^{(t-\tau)L_1}q_1(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}\\
&\le \|e^{tL_1}u_0\|_{\mathrm{H}_\alpha}+\|\int_0^t e^{(t-\tau)L_1}q_1(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}\\
&\le C\|u_0\|_{\mathrm{H}_\alpha}+\int_0^t \|(-L_1)^{-\frac{1}{2}+\alpha}e^{(t-\tau)L_1}\|\cdot\|q_1(u,v,w,z)\|_{\mathrm{H}_{\frac{1}{2}}}d\tau\\
&\le C\|u_0\|_{\mathrm{H}_\alpha}+C\int_0^t \tau^{\frac{1}{2}-\alpha}e^{-\delta\tau}d\tau\\
&\le C,\ \forall t\ge 0,\ g_0\in B,
\end{aligned}
\tag{3.22}
$$

where $\frac{1}{2}<\alpha<\frac{3}{2}$. By (3.2), (3.7) and (3.19), we obtain

$$
\begin{aligned}
\|v(t,v_0)\|_{\mathrm{H}_\alpha} &= \|e^{tL_2}v_0+\int_0^t e^{(t-\tau)L_2}q_2(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}\\
&\le \|e^{tL_2}v_0\|_{\mathrm{H}_\alpha}+\|\int_0^t e^{(t-\tau)L_2}q_2(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}\\
&\le C\|v_0\|_{\mathrm{H}_\alpha}+\int_0^t \|(-L_2)^{-\frac{1}{2}+\alpha}e^{(t-\tau)L_2}\|\cdot\|q_2(u,v,w,z)\|_{\mathrm{H}_{\frac{1}{2}}}d\tau\\
&\le C\|v_0\|_{\mathrm{H}_\alpha}+C\int_0^t \tau^{\frac{1}{2}-\alpha}e^{-\delta\tau}d\tau\\
&\le C,\ \forall t\ge 0,\ g_0\in B,
\end{aligned}
\tag{3.23}
$$

where $\frac{1}{2}<\alpha<\frac{3}{2}$. By (3.3), (3.4), (3.7), (3.20) and (3.21), simple calculations shows that

$$\|w(t,w_0)\|_{\mathrm{H}_\alpha}\le C,\ \forall t\ge 0,\ g_0\in B, \tag{3.24}$$

$$\|z(t,z_0)\|_{\mathrm{H}_\alpha}\le C,\ \forall t\ge 0,\ g_0\in B, \tag{3.25}$$

where $\frac{1}{2}<\alpha<\frac{3}{2}$. By (3.22), (3.23), (3.24) and (3.25), we obtain (3.16) immediately.

Step 3. We prove that for any bounded set $B \subset \mathcal{H}_\alpha$ ($\frac{3}{2} \le \alpha < 2$), there exists a positive constant C such that $\forall t \ge 0,\ \frac{3}{2} \le \alpha < 2$,

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \le C. \tag{3.26}$$

By Proposition 2.5 and Proposition 2.8 and the following embedding theorems of fractional order spaces

$$\mathrm{H}_\alpha \hookrightarrow H^2(\Omega),\ H^2(\Omega) \hookrightarrow C^0(\Omega) \bigcap W^{1,4}(\Omega),\ \forall \alpha > 1,$$

we obtain

$$\begin{aligned}
&\|q_1(u,v,w,z)\|^2_{\mathrm{H}_1} \\
&\le C \int_\Omega |\Delta q_1(u,v,w,z)|^2 dx + C_0 \\
&\le C \int_\Omega |\Delta(-(F+k)u + u^2 v + D_1(w-u))|^2 dx + C_0 \\
&\le C \int_\Omega \big(-(F+k)\Delta u + 2uv\Delta u + 2v|\nabla u|^2 + 4u\nabla u \nabla v \\
&\quad + u^2 \Delta v + D_1 \Delta w - D_1 \Delta u\big)^2 dx + C_0 \\
&\le C \int_\Omega \big(|\Delta u|^2 + u^2 v^2 |\Delta u|^2 + v^2 |\nabla u|^4 + u^2 |\nabla u \nabla v|^2 \\
&\quad + u^4 |\Delta v|^2 + |\Delta w|^2 + |\Delta u|^2\big) dx + C_0 \\
&\le C \int_\Omega \Bigg(|\Delta u|^2 + \sup_{x\in\Omega} u^2 v^2 \cdot |\Delta u|^2 + \sup_{x\in\Omega} v^2 \cdot |\nabla u|^4 \\
&\quad + \sup_{x\in\Omega} u^2 \cdot |\nabla u \nabla v|^2 + \sup_{x\in\Omega} u^4 \cdot |\Delta v|^2 + |\Delta w|^2 + |\Delta u|^2\Bigg) dx + C_0 \\
&\le C \int_\Omega (|\Delta u|^2 + |\nabla v|^4 + |\nabla u|^4 + |\Delta v|^2 + |\Delta w|^2) dx + C_0 \\
&\le C(\|u\|^2_{\mathrm{H}_\alpha} + \|v\|^4_{\mathrm{H}_\alpha} + \|u\|^4_{\mathrm{H}_\alpha} + \|v\|^2_{\mathrm{H}_\alpha} + \|w\|^2_{\mathrm{H}_\alpha}) + C_0 \le C
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\|q_2(u,v,w,z)\|^2_{\mathrm{H}_1} \\
&\le C \int_\Omega |\Delta q_2(u,v,w,z)|^2 dx + C_0 \\
&\le C \int_\Omega [\Delta(F(1-v) - u^2 v + D_2(z-v))]^2 dx + C_0 \\
&\le C \int_\Omega (-F\Delta v - 2uv\Delta u - 2v|\nabla u|^2 - 4u\nabla u \nabla v \\
&\quad - u^2 \Delta v + D_2 \Delta z - D_2 \Delta v)^2 dx + C_0 \\
&\le C \int_\Omega (|\Delta u|^2 + u^2 v^2 |\Delta u|^2 + v^2 |\nabla u|^4 + u^2 |\nabla u \nabla v|^2
\end{aligned}$$

$$+ u^4|\Delta v|^2 + |\Delta z|^2 + |\Delta v|^2)dx + C_0$$

$$\leq C\int_\Omega (|\Delta u|^2 + \sup_{x\in\Omega} u^2v^2 \cdot |\Delta u|^2 + \sup_{x\in\Omega} v^2 \cdot |\nabla u|^4$$

$$+ \sup_{x\in\Omega} u^2 \cdot |\nabla u\nabla v|^2 + \sup_{x\in\Omega} u^4 \cdot |\Delta v|^2 + |\Delta z|^2 + |\Delta v|^2)dx + C_0$$

$$\leq C\int_\Omega (|\Delta u|^2 + |\nabla u|^4 + |\nabla v|^4 + |\Delta v|^2 + |\Delta z|^2)dx + C_0$$

$$\leq C(\|\Delta u\|^2 + \|\Delta v\|^2 + \|\nabla u\|^4 + \|\nabla v\|^4 + \|\Delta z\|^2) + C_0$$

$$\leq C(\|u\|^2_{\mathrm{H}_\alpha} + \|v\|^2_{\mathrm{H}_\alpha} + \|u\|^4_{\mathrm{H}_\alpha} + \|v\|^4_{\mathrm{H}_\alpha} + \|z\|^2_{\mathrm{H}_\alpha}) + C_0 \leq C. \tag{3.28}$$

Note that $q_1(u,v,w,z) = q_3(w,z,u,v)$ and $q_2(u,v,w,z) = q_4(w,z,u,v)$, simple calculations show that

$$\|q_3(u,v,w,z)\|^2_{\mathrm{H}_1} \leq C, \tag{3.29}$$

$$\|q_4(u,v,w,z)\|^2_{\mathrm{H}_1} \leq C. \tag{3.30}$$

By (3.1), (3.16) and (3.27), we obtain

$$\|u(t,u_0)\|_{\mathrm{H}_\alpha} = \|e^{tL_1}u_0 + \int_0^t e^{(t-\tau)L_1}q_1(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}$$

$$\leq \|e^{tL_1}u_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)L_1}q_1(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}$$

$$\leq C\|u_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-L_1)^{\alpha-1}e^{(t-\tau)L_1}\| \cdot \|q_1(u,v,w,z)\|_{\mathrm{H}_1}d\tau$$

$$\leq C\|u_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{1-\alpha}e^{-\delta\tau}d\tau$$

$$\leq C, \ \forall t \geq 0, \ g_0 \in B, \tag{3.31}$$

where $1 < \alpha < 2$. By (3.2), (3.16) and (3.28), we obtain

$$\|v(t,v_0)\|_{\mathrm{H}_\alpha} = \|e^{tL_2}v_0 + \int_0^t e^{(t-\tau)L_2}q_2(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}$$

$$\leq \|e^{tL_2}v_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)L_2}q_2(u,v,w,z)d\tau\|_{\mathrm{H}_\alpha}$$

$$\leq C\|v_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-L_2)^{\alpha-1}e^{(t-\tau)L_2}\| \cdot \|q_2(u,v,w,z)\|_{\mathrm{H}_1}d\tau$$

$$\leq C\|v_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{1-\alpha}e^{-\delta\tau}d\tau$$

$$\leq C, \ \forall t \geq 0, \ g_0 \in B, \tag{3.32}$$

where $1 < \alpha < 2$. By (3.3), (3.4), (3.16), (3.29) and (3.30), simple calculations shows that

$$\|w(t,w_0)\|_{\mathrm{H}_\alpha} \leq C, \ \forall t \geq 0, \ g_0 \in B, \tag{3.33}$$

$$\|z(t,z_0)\|_{\mathrm{H}_\alpha} \le C,\ \forall t \ge 0,\ g_0 \in B, \tag{3.34}$$

where $1 < \alpha < 2$. By (3.31), (3.32), (3.33) and (3.34) together, we obtain (3.26) immediately.

In the same fashion as in the proof of (3.26), by iteration we can prove that for any bounded set $B \subset \mathcal{H}_\alpha$, there exists a positive constant C such that

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \le C,\ \forall t \ge 0,\ \alpha \ge 0.$$

It then follows from the above inequality that

$$\|g(t,g_0)\|_{\mathcal{H}_\alpha} \le C,\ \forall t \ge 0,\ \alpha \ge 0,$$

that is, for all $\alpha \ge 0$, the solution $g = (u, v, w, z)$ of (1.1)-(1.6) is uniformly bounded in $\mathcal{H}_\alpha$.

Hence, Lemma 3.1 is proved. □

Lemma 3.2. *Suppose d_1, d_2, F, k, D_1 and D_2 are given positive parameters, $g = (u, v, w, z)$ is a solution to the problem (1.1)-(1.6), $g_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}_\alpha$ ($\forall\alpha \ge 0$), then, the problem (1.1)-(1.6) has a bounded absorbing set in $\mathcal{H}_\alpha$.*

Proof. It suffices to prove that for any bounded set $B \subset \mathcal{H}_\alpha$ ($\alpha \ge 0$) with initial value $g_0 = (u_0, v_0, w_0, z_0) \in B$, there exist $T > 0$ and a constant $C > 0$ independent of (u_0, v_0, w_0, z_0), such that

$$\|g(t,g_0)\|_{\mathcal{H}_\alpha} \le C, \quad \forall t \ge T. \tag{3.35}$$

Obviously, if we get

$$\|u(t,u_0)\|^2_{\mathrm{H}_\alpha} + \|v(t,v_0)\|^2_{\mathrm{H}_\alpha} + \|w(t,w_0)\|^2_{\mathrm{H}_\alpha} + \|z(t,z_0)\|^2_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge T,$$

then we obtain (3.35) immediately.

For $\alpha = \frac{1}{2}$, this follows from Propositions 2.5, 2.6 and 2.7. So we shall prove (3.35) for any $\alpha \ge \frac{1}{2}$. We prove the lemma in the following steps:

Step 1. we prove that for any $\frac{1}{2} \le \alpha < 1$, the problem (1.1)-(1.6) has a bounded absorbing set in $\mathcal{H}_\alpha$.

It then follows from (3.1)-(3.4) that

$$u(t,u_0) = e^{(t-T)L_1}u(T,u_0) + \int_T^t e^{(t-\tau)L_1} q_1(u,v,w,z)d\tau, \tag{3.36}$$

$$v(t,v_0) = e^{(t-T)L_2}v(T,v_0) + \int_T^t e^{(t-\tau)L_2} q_2(u,v,w,z)d\tau, \tag{3.37}$$

$$w(t,w_0) = e^{(t-T)L_1}w(T,w_0) + \int_T^t e^{(t-\tau)L_1} q_3(u,v,w,z)d\tau, \tag{3.38}$$

$$z(t,z_0) = e^{(t-T)L_2}z(T,z_0) + \int_T^t e^{(t-\tau)L_2} q_4(u,v,w,z)d\tau. \tag{3.39}$$

Assume $\mathbb{B}$ is the bounded absorbing set of the problem (1.1)-(1.6) and satisfy $\mathbb{B} \subset \mathcal{H}$, we also assume $T_0 > 0$ the time such that $\forall t > T_0$, $(u_0, v_0, w_0, z_0) \in B \subset \mathcal{H}_\alpha$,

$$(u(t,u_0), v(t,v_0), w(t,w_0), z(t,z_0)) \in \mathbb{B}, \quad \alpha \geq \frac{1}{2}. \tag{3.40}$$

It is easy to check that

$$\|e^{tL_i}\| \leq Ce^{-d\lambda_1 t}, \tag{3.41}$$

here, $i = 1, 2$, $\lambda_1 > 0$ is the first eigenvalue of the equation

$$\begin{cases} -\Delta\Sigma = \lambda\Sigma, \\ \frac{\partial\Sigma}{\partial\nu}|_{\partial\Omega} = 0, \end{cases} \tag{3.42}$$

where $\Sigma = u, v, w, z$.

Then, for any given $T > 0$ and $(u_0, v_0, w_0, z_0) \in B \subset \mathcal{H}_\alpha$ ($\alpha \geq \frac{1}{2}$), we deduce that

$$\lim_{t\to\infty} \|e^{(t-T)L_1} u(T,u_0)\|_{\mathrm{H}_\alpha} = 0, \quad \lim_{t\to\infty} \|e^{(t-T)L_2} v(T,v_0)\|_{\mathrm{H}_\alpha} = 0,$$
$$\lim_{t\to\infty} \|e^{(t-T)L_1} w(T,w_0)\|_{\mathrm{H}_\alpha} = 0, \quad \lim_{t\to\infty} \|e^{(t-T)L_2} z(T,z_0)\|_{\mathrm{H}_\alpha} = 0.$$

Then, by (3.36) and (3.40), we obtain

$$\begin{aligned} &\|u(t,u_0)\|_{\mathrm{H}_\alpha} \\ \leq\ & \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^t \|(-L_1)^\alpha e^{(t-\tau)L_1}\| \|q_1(u,v,w,z)\|_{\mathrm{H}} d\tau \\ \leq\ & \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^t \|(-L_1)^\alpha e^{(t-\tau)L_1}\| d\tau \\ \leq\ & \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + C\int_0^{T-T_0} \tau^{-\alpha} e^{-\delta\tau} d\tau \\ \leq\ & \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + C_1, \end{aligned} \tag{3.43}$$

where C_1 is a positive constant. By (3.37) and (3.40), we have

$$\begin{aligned} &\|v(t,v_0)\|_{\mathrm{H}_\alpha} \\ \leq\ & \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^t \|(-L_2)^\alpha e^{(t-\tau)L_2}\| \|q_2(u,v,w,z)\|_{\mathrm{H}} d\tau \\ \leq\ & \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^t \|(-L_2)^\alpha e^{(t-\tau)L_2}\| d\tau \\ \leq\ & \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + C\int_0^{T-T_0} \tau^{-\alpha} e^{-\delta\tau} d\tau \\ \leq\ & \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + C_2, \end{aligned} \tag{3.44}$$

where C_2 is a positive constant. Using the same method, we can also obtain

$$\|w(t,w_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)L_1} w(T_0,w_0)\|_{\mathrm{H}_\alpha} + C_3, \tag{3.45}$$

$$\|r(t,r_0)\|_{\mathrm{H}_\alpha} \le \|e^{(t-T_0)L_2} r(T_0,r_0)\|_{\mathrm{H}_\alpha} + C_4, \tag{3.46}$$

where $C_3,\ C_4$ are positive constants.

Then, by (3.43)-(3.46), we obtain (3.35) holds for all $\frac{1}{2} \le \alpha < 1$.

Step 2. We prove that for any $1 \le \alpha < \frac{3}{2}$, the problem (1.1)-(1.6) has a bounded absorbing set in $\mathcal{H}_\alpha$.

By (3.36) and (3.18), we obtain

(3.47)

$$\begin{aligned}
&\|u(t,u_0)\|_{\mathrm{H}_\alpha}\\
&\le \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^{t} \|(-L_1)^{\alpha-\frac{1}{2}} e^{(t-\tau)L_1}\| \|q_1(u,v,w,z)\|_{\mathrm{H}_{\frac{1}{2}}} d\tau\\
&\le \|e^{(t-T_0)L_1} u(t,u_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \|(-L_1)^{\alpha-\frac{1}{2}} e^{(t-\tau)L_1}\| d\tau\\
&\le \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \tau^{-(\alpha-\frac{1}{2})} e^{-\delta\tau} d\tau\\
&\le \|e^{(t-T_0)L_1} u(T_0,u_0)\|_{\mathrm{H}_\alpha} + C_5,
\end{aligned}$$

where C_5 is a positive constant. By (3.37) and (3.19), we have

$$\begin{aligned}
&\|v(t,v_0)\|_{\mathrm{H}_\alpha} \qquad (3.48)\\
&\le \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^{t} \|(-L_2)^{\alpha-\frac{1}{2}} e^{(t-\tau)L_2}\| \|q_2(u,v,w,z)\|_{\mathrm{H}_{\frac{1}{2}}} d\tau\\
&\le \|e^{(t-T_0)L_2} v(t,v_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \|(-L_2)^{\alpha-\frac{1}{2}} e^{(t-\tau)L_2}\| d\tau\\
&\le \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \tau^{-(\alpha-\frac{1}{2})} e^{-\delta\tau} d\tau\\
&\le \|e^{(t-T_0)L_2} v(T_0,v_0)\|_{\mathrm{H}_\alpha} + C_6,
\end{aligned}$$

where C_6 is a positive constant. Using the same method, we can also obtain

$$\|w(t,w_0)\|_{\mathrm{H}_\alpha} \le \|e^{(t-T_0)L_1} w(T_0,w_0)\|_{\mathrm{H}_\alpha} + C_7, \tag{3.49}$$

$$\|z(t,z_0)\|_{\mathrm{H}_\alpha} \le \|e^{(t-T_0)L_2} z(T_0,z_0)\|_{\mathrm{H}_\alpha} + C_8, \tag{3.50}$$

where $C_7,\ C_8$ are positive constants.

Then, by (3.47)-(3.50), we obtain (3.35) holds for all $1 \le \alpha < \frac{3}{2}$.

By iteration, we can prove that for any $\alpha \ge \frac{3}{2}$, (3.35) holds. Therefore, the problem (1.1) has a bounded absorbing set in $\mathcal{H}_\alpha$.

Then, Lemma 3.2 is proved. □

Now, we give the proof the main result.

Proof of Theorem 2.10. By Lemma 2.1, Lemma 3.1, Lemma 3.2, we immediately conclude that the proof of Theorem 2.10 is completed. □

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