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THE RIGIDITY OF MINIMAL SUBMANIFOLDS IN A LOCALLY SYMMETRIC SPACE

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Abstract. In the present paper, we discuss the rigidity phenomenon of closed minimal submanifolds in a locally symmetric Riemannian manifold with pinched sectional curvature. We show that if the sectional curvature of the submanifold is no less than an explicitly given constant, then either the submanifold is totally geodesic, or the ambient space is a sphere and the submanifold is isometric to a product of two spheres or the Veronese surface in S^4.

1. Introduction

Let M^n be an n-dimensional closed minimal submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We denote by S the squared norm of the second fundamental form of M. If N^{n+p} is the $(n+p)$-dimensional unit sphere S^{n+p}, a famous rigidity theorem due to Simons [11], Lawson [8] and Chern-do Carmo-Kobayashi [1] says that if $S \leq \frac{n}{2-1/p}$, then either M is totally geodesic, or M is one of the Clifford minimal hypersurfaces $S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$, $k = 1, \ldots, n-1$, or $n = 2$, $p = 2$, and M is the Veronese surface in S^4. Further discussions have been carried out by many other authors [2, 9, 12, 13, 14, 15], etc.

Denote by K_M the sectional curvature of M. In 1975, Yau [16] proved the following rigidity theorem.

Theorem 1.1 ([16]). *Let M^n be an n-dimensional oriented closed minimal submanifold in S^{n+p}. If $K_M \geq \frac{p-1}{2p-1}$, then either M is the totally geodesic sphere, the standard immersion of the product of two spheres, or M is the Veronese surface in S^4.*

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Later, Itoh [7] proved that if $K_M \geq \frac{n}{2(n+1)}$, then M is the totally geodesic sphere, or the Veronese surface in S^4. Recently, Gu-Xu [5] made an improvement of Yau's rigidity theorem. They obtained the following theorem.

Theorem 1.2 ([5]). *Let M^n be an n-dimensional oriented closed minimal submanifold in S^{n+p}. If $K_M \geq \frac{\mathrm{sgn}(p-1)p}{2(p+1)}$, then either M is the totally geodesic sphere, the standard immersion of the product of two spheres, or M is the Veronese surface in S^4. Here $\mathrm{sgn}(\cdot)$ is the standard sign function.*

In this paper, we discuss the rigidity of minimal submanifolds in a Riemannian manifold. We assume that the ambient space is locally symmetric and δ-pinched. We obtain the following theorem.

Theorem 1.3. *Let M^n be an n-dimensional oriented closed minimal submanifold in an n-dimensional simply connected and locally symmetric Riemannian manifold N^{n+p}. Suppose the sectional curvature K_N of N satisfies $\delta \leq K_N \leq 1$. If*

$$K_M \geq \frac{4}{3n(p+1)}(n-1)^{\frac{1}{2}}(p-1)(p+2)(1-\delta) + \left(\frac{p+2}{2(p+1)} - \frac{\delta}{p+1}\right)\mathrm{sgn}(p-1),$$

then either M is totally geodesic, or $N^{n+p} = S^{n+p}$ and M is isometric to the standard immersion of the product of two spheres or the Veronese surface in S^4.

2. Preliminaries

Let M^n be an n-dimensional closed minimal submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p;\ 1 \leq i, j, k, \ldots \leq n;\ n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in N such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of N, respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j,\ h^\alpha_{ij} = h^\alpha_{ji},$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,\ \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{1}$$

$$R_{\alpha\beta kl} = \bar{R}_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}), \tag{2}$$

where $h, \xi, R_{ijkl}, R_{\alpha\beta kl}$, and $\bar{R}_{ABCD}$ are the second fundamental form, the mean curvature vector, the curvature tensor, the normal curvature tensor of M, and the curvature tensor of N, respectively. We set

$$S = |h|^2, \ H = |\xi|, \ H_\alpha = (h^\alpha_{ij})_{n\times n}.$$

Then M is minimal if and only if $H = 0$.

Denote the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl}, respectively. We have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha},$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Then

$$h^\alpha_{ijk} - h^\alpha_{ikj} = -\bar{R}_{\alpha ijk}, \tag{3}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mi}R_{mjkl} + h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}. \tag{4}$$

Consider $\bar{R}_{\alpha ijk}$ as a section of the bundle $NM \otimes T^*M \otimes T^*M \otimes T^*M$. Its covariant derivative $\bar{R}_{\alpha ijkl}$ is defined by

$$\sum_l \bar{R}_{\alpha ijkl}\omega_l$$

$$= d\bar{R}_{\alpha ijk} - \sum_m \bar{R}_{\alpha mjk}\omega_{mi} - \sum_m \bar{R}_{\alpha imk}\omega_{mj} - \sum_m \bar{R}_{\alpha ijm}\omega_{mk} - \sum_\beta \bar{R}_{\beta ijk}\omega_{\beta\alpha}.$$

Let $\bar{R}_{ABCD,E}$ be the covariant derivative of $\bar{R}_{ABCD}$ as a curvature tensor of N. Restricted to M, $\bar{R}_{\alpha ijk,l}$ is given by

$$\bar{R}_{\alpha ijk,l} = \bar{R}_{\alpha ijkl} - \sum_\beta \bar{R}_{\alpha\beta jk}h^\beta_{il} - \sum_\beta \bar{R}_{\alpha i\beta k}h^\beta_{jl} - \sum_\beta \bar{R}_{\alpha ij\beta}h^\beta_{kl} + \sum_m \bar{R}_{mijk}h^\alpha_{ml}.$$

Now we assume that the ambient space N is locally symmetric, i.e., $\bar{R}_{ABCD,E} = 0$. Then we have

$$\bar{R}_{\alpha ijkl} = \sum_\beta \bar{R}_{\alpha\beta jk}h^\beta_{il} + \sum_\beta \bar{R}_{\alpha i\beta k}h^\beta_{jl} + \sum_\beta \bar{R}_{\alpha ij\beta}h^\beta_{kl} - \sum_m \bar{R}_{mijk}h^\alpha_{ml}. \tag{5}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h is defined by $\triangle h^\alpha_{ij} = \sum_k \triangle h^\alpha_{ijkk}$. Then from (1), (4), (5) and the minimality of M, we have for any real number a,

$$\begin{aligned}\sum_{i,j,\alpha} h^\alpha_{ij}\triangle h^\alpha_{ij} \quad = \quad & 4\sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{jk}h^\beta_{il}\bar{R}_{\alpha\beta ij} - \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ij}\bar{R}_{\alpha k\beta k} \\ & + (1-a)\sum_\alpha\sum_{i,j,k,m} h^\alpha_{ij}(h^\alpha_{mk}\bar{R}_{mijk} + h^\alpha_{mi}\bar{R}_{mkjk})\end{aligned}$$

$$
\begin{aligned}
&+ a\sum_{\alpha,\beta}[\mathrm{tr}(H_\alpha H_\beta)]^2\\
&+ (1+a)\sum_{\alpha}\sum_{i,j,k,m} h_{ij}^\alpha(h_{mk}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk})\\
&- \sum_{\alpha,\beta}[\mathrm{tr}(H_\alpha^2 H_\beta^2) - \mathrm{tr}(H_\alpha H_\beta)^2]. \qquad (6)
\end{aligned}
$$

The DDVV inequality proved by Lu [10], Ge and Tang [3] is stated as follows.

Lemma 2.1 (DDVV Inequality)**.** *Let $B_1, \ldots, B_m$ be symmetric $(n\times n)$ real matrices. Then*

$$\sum_{r,s=1}^{m} \|[B_r, B_s]\|^2 \le \Big(\sum_{r=1}^{m}\|B_r\|^2\Big)^2,$$

where the equality holds if and only if under some rotation all B_r's are zero except two matrices which can be written as

$$\tilde{B}_r = P\begin{pmatrix} 0 & \mu & 0 & \cdots & 0\\ \mu & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}P^t, \quad \tilde{B}_s = P\begin{pmatrix} \mu & 0 & 0 & \cdots & 0\\ 0 & -\mu & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}P^t,$$

where P is an orthogonal $(n\times n)$-matrix. Here $\|\cdot\|^2$ denotes the sum of squares of entries of the matrix and $[A,B] = AB - BA$ is the commutator of the matrices A, B.

In the proof of the theorem, we also use the following Berger's inequalities.

Lemma 2.2 ([4])**.** *Let N be an $(n+p)$-dimensional Riemannian manifold satisfying $a \le K_N \le b$. Let $\{e_A\}$ be a local orthonormal basis. Then*
(1) $\bar{R}_{ABCD} \le \frac{2}{3}(b-a)$ *for all distinct A, B, C, D.*
(2) $\bar{R}_{ACBC} \le \frac{1}{2}(b-a)$ *for $A \ne B$.*

3. Proof of the main theorem

To prove Theorem 1.3, we first give some estimates.
For a fixed α, we choose $\{e_i\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha\delta_{ij}$. Then

$$
\begin{aligned}
&4\sum_{\beta}\sum_{i,j,k} h_{jk}^\alpha h_{il}^\beta \bar{R}_{\alpha\beta ij}\\
=\ &4\sum_{\beta\ne\alpha}\sum_{i\ne k}\lambda_k^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta ik}\\
\ge\ &-\frac{8}{3}(1-\delta)\sum_{\beta\ne\alpha}\sum_{i\ne k}|\lambda_k^\alpha h_{ik}^\beta|
\end{aligned}
$$

$$
\begin{aligned}
&\geq -\frac{4}{3}(1-\delta)\sum_{\beta\neq\alpha}\sum_{i\neq k}[(n-1)^{-\frac{1}{2}}(\lambda_k^\alpha)^2+(n-1)^{\frac{1}{2}}(h_{ik}^\beta)^2]\\
&= -\frac{4}{3}(1-\delta)(n-1)^{\frac{1}{2}}(p-1)\mathrm{tr}H_\alpha^2-\frac{4}{3}(1-\delta)(n-1)^{\frac{1}{2}}\sum_{\beta\neq\alpha}\mathrm{tr}H_\beta^2.
\end{aligned}
$$

Then we have

$$
4\sum_{\alpha,\beta}\sum_{i,j,k}h_{jk}^\alpha h_{il}^\beta \bar{R}_{\alpha\beta ij}\geq -\frac{8}{3}(1-\delta)(n-1)^{\frac{1}{2}}(p-1)S. \tag{7}
$$

Since $(\mathrm{tr}(H_\alpha H_\beta))$ is a symmetric $(p\times p)$-matrix, we can choose the normal frame fields $\{e_\alpha\}$ such that

$$
\mathrm{tr}(H_\alpha H_\beta)=\mathrm{tr}H_\alpha^2\cdot\delta_{\alpha\beta}.
$$

Then

$$
\begin{aligned}
-\sum_{\alpha,\beta}\sum_{i,j,k}h_{ij}^\alpha h_{ij}^\beta\bar{R}_{\alpha k\beta k} &= -\sum_{\alpha,\beta}\mathrm{tr}(H_\alpha H_\beta)\sum_k\bar{R}_{\alpha k\beta k}\\
&= -\sum_\alpha \mathrm{tr}(H_\alpha^2)\sum_k\bar{R}_{\alpha k\alpha k}\\
&\geq -nS. \qquad (8)
\end{aligned}
$$

We also have

$$
\begin{aligned}
\sum_{i,j,k,m}h_{ij}^\alpha(h_{mk}^\alpha\bar{R}_{mijk}+h_{mi}^\alpha\bar{R}_{mkjk}) &= \frac{1}{2}\sum_{i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2\bar{R}_{ikik}\\
&\geq \frac{1}{2}\delta\sum_{i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2\\
&= n\delta\mathrm{tr}(H_\alpha^2).
\end{aligned}
$$

Hence

$$
\sum_\alpha\sum_{i,j,k,m}h_{ij}^\alpha(h_{mk}^\alpha\bar{R}_{mijk}+h_{mi}^\alpha\bar{R}_{mkjk})\geq n\delta S. \tag{9}
$$

Similarly, we have

$$
\sum_\alpha\sum_{i,j,k,m}h_{ij}^\alpha(h_{mk}^\alpha R_{mijk}+h_{mi}^\alpha R_{mkjk})\geq nK_{\min}S, \tag{10}
$$

where $K_{\min}$ is the minimum of the sectional curvature at a point.

On the other hand, by a direct computation and the DDVV inequality, we obtain

$$
\begin{aligned}
\sum_{\alpha,\beta}\mathrm{tr}(H_\alpha^2H_\beta^2)-\mathrm{tr}(H_\alpha H_\beta)^2 &= \frac{1}{2}\sum_{\alpha,\beta}\mathrm{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2\\
&\leq \frac{1}{2}\mathrm{sgn}(p-1)\Big(\sum_\alpha\mathrm{tr}H_\alpha^2\Big)^2
\end{aligned}
$$

$$= \frac{1}{2}\mathrm{sgn}(p-1)S^2. \tag{11}$$

We also have

$$\sum_{\alpha,\beta}[\mathrm{tr}(H_\alpha H_\beta)]^2 \geq \frac{S^2}{p}. \tag{12}$$

Proof of Theorem 1.3. For any $0 \leq a \leq 1$, from (6)-(12) we have

$$\begin{aligned}\sum_{i,j,\alpha} h_{ij}^\alpha \triangle h_{ij}^\alpha \ &\geq\ -\frac{8}{3}(1-\delta)(n-1)^{\frac{1}{2}}(p-1)S - nS + (1-a)n\delta S \\ &\quad +(1+a)nK_{\min}S + \Big(-\frac{(1-a)}{2}\mathrm{sgn}(p-1) + \frac{a}{p}\Big)S^2. \end{aligned} \tag{13}$$

Take $a = \frac{p}{p+2}\cdot \mathrm{sgn}(p-1)$. From (13) we have

$$\begin{aligned} \frac{1}{2}\triangle S &= \sum_\alpha \sum_{i,j,k}(h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \triangle h_{ij}^\alpha \qquad (14)\\ &\geq \sum_\alpha \sum_{i,j,k}(h_{ijk}^\alpha)^2 - \frac{8}{3}(1-\delta)(n-1)^{\frac{1}{2}}(p-1)S - nS \\ &\quad + (1 - \frac{p}{p+2}\cdot \mathrm{sgn}(p-1))n\delta S \\ &\quad + (1 + \frac{p}{p+2}\cdot \mathrm{sgn}(p-1))nK_{\min}S \\ &\geq \sum_\alpha \sum_{i,j,k}(h_{ijk}^\alpha)^2 + nS\Big[\Big(1 + \frac{p}{p+2}\cdot \mathrm{sgn}(p-1)\Big)K_{\min} \\ &\quad - \frac{8}{3n}(1-\delta)(n-1)^{\frac{1}{2}}(p-1) - 1 + \Big(1 - \frac{p}{p+2}\cdot \mathrm{sgn}(p-1)\Big)\delta\Big]. \qquad (15) \end{aligned}$$

It follows from our assumption and the maximum principle that S is a constant. Hence $\sum_\alpha \sum_{i,j,k}(h_{ijk}^\alpha)^2 = 0$ and all the inequalities in (7)-(12) are equalities. Let

$$A = -\sum_\alpha \sum_{i,j,k} h_{ij}^\alpha(\bar{R}_{\alpha kikj} + \bar{R}_{\alpha ijkk}).$$

Since (7)-(12) are equalities, from (5) we have

$$A = -2(1-\delta)(n-1)^{\frac{1}{2}}(p-1)S - nS + n\delta S. \tag{16}$$

We let

$$\omega = \sum_k \sum_\alpha \sum_{i,j}(h_{ik}^\alpha \bar{R}_{\alpha jij} + h_{ij}^\alpha \bar{R}_{\alpha ijk})\omega_k.$$

Then

$$\mathrm{div}\omega \ = \ \sum_k \sum_\alpha \sum_{i,j}(h_{ik}^\alpha \bar{R}_{\alpha jij} + h_{ij}^\alpha \bar{R}_{\alpha ijk})_k$$

$$= \sum_{\alpha}\sum_{i,j,k} h_{ij}^{\alpha}(\bar{R}_{\alpha kikj} + \bar{R}_{\alpha ijkk})$$

$$= -A. \tag{17}$$

By (16) and (17) we have

$$\begin{aligned} 0 &= \int_M [-2(1-\delta)(n-1)^{\frac{1}{2}}(p-1)S - nS + n\delta S]d\mu \\ &= \int_M [-2(n-1)^{\frac{1}{2}}(p-1) - n](1-\delta)Sd\mu. \end{aligned} \tag{18}$$

Since the integrand of (18) is non-positive, we have $(1-\delta)S = 0$. Hence $S = 0$ or $\delta = 1$. If $S = 0$, then M is totally geodesic. If $\delta = 1$, then $N = S^{n+p}$, and our assumption reduces to $K_M \geq \frac{p}{2p+1}\text{sgn}(p-1)$. From Theorem 1.2 we see that either M is the totally geodesic sphere, the standard immersion of the product of two spheres, or M is the Veronese surface in S^4. This completes the proof of Theorem 1.3. □

References

[1] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, In: Functional analysis and relathed fields, pp. 59–75. Berlin, Heidelberg, New York, Springer, 1970.

[2] Q. Ding and Y. L. Xin, *On Chern's problem for rigidity of minimal hypersurfaces in the spheres*, Adv. Math. **227** (2011), no. 1, 131–145.

[3] J. Q. Ge and Z. Z. Tang, *A proof of the DDVV conjecture and its equality case*, Pacific J. Math. **237** (2008), no. 1, 87–95.

[4] S. I. Goldberg, *Curvature and Homology*, Academic Press, London, 1998.

[5] J. R. Gu and H. W. Xu, *On Yau rigidity theorem for minimal submanifolds in spheres*, preprint, arxiv:1102.5732v1.

[6] T. Itoh, *On Veronese manifolds*, J. Math. Soc. Japan **27** (1975), no. 3, 497–506.

[7] ———, *Addendum to my paper "On Veronese manifolds"*, J. Math. Soc. Japan **30** (1978), no. 1, 73–74.

[8] B. Lawson, *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 179–185.

[9] A. M. Li and J. M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. (Basel) **58** (1992), no. 6, 582–594.

[10] Z. Lu, *Proof of the normal scalar curvature conjecture*, arXiv:0711.3510v1.

[11] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

[12] W. D. Song, *On minimal submanifolds in a locally symmetric space*, Chinese Ann. Math. Ser. A **19** (1998), 693–698.

[13] S. M. Wei and H. W. Xu, *Scalar curvature of minimal hypersurfaces in a sphere*, Math. Res. Lett. **14** (2007), no. 3, 423–432.

[14] H. W. Xu, *On closed minimal submanifolds in pinched Riemannian manifolds*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1743–1751.

[15] H. C. Yang and Q. M. Cheng, *Chern's conjecture on minimal hypersurfaces*, Math. Z. **227** (1998), no. 3, 377–390.

[16] S. T. Yau, *Submanifolds with constant mean curvature I, II*, Amer. J. Math. **96**, **97** (1974, 1975), 346–366, 76–100.

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