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ON 2×2 STRONGLY CLEAN MATRICES

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Abstract. An element in a ring R is strongly clean provided that it is the sum of an idempotent and a unit that commutate. In this note, several necessary and sufficient conditions under which a 2×2 matrix over an integral domain is strongly clean are given. These show that strong cleanness over integral domains can be characterized by quadratic and Diophantine equations.

1. Introduction

An element in a ring R is strongly clean provided that it is the sum of an idempotent and a unit that commutate, which was firstly introduced by Nicholson in 1999 ([7]). It seems to be rather hard to determine 2×2 matrices over a commutative ring strongly clean. A ring R is local provided that it has only a maximal right ideal. Many authors extensively studied the strongly clean 2×2 matrices over a commutative local ring (cf. [2], [3], [4] and [6]). A commutative ring R is called an integral domain provided that R does not have any nonzero zero divisor. An element in a ring is said to be clean in the case that it is the sum of an idempotent and a unit. In [5], Khurana and Lam explored the cleanness of the matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$, where $a, b \in \mathbb{Z}$. Further, the author extended Khurana and Lam's result to Dedekind domains (cf. [2, Corollary 16.3.7]). The strong cleanness over integral domains is less considered in the literature, while Rajeswari and Aziz obtained several criteria on the strong cleanness of 2×2 matrices over the ring $\mathbb{Z}$ of all integers (cf. [9] and [10]).

The main purpose of this note is to determine the strong cleanness of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a - d \in U(R)$ over a general integral domain R. We give the necessary and sufficient conditions under which such 2×2 matrices are strongly clean. For several kind of 2×2 matrices over $\mathbb{Z}$, we can derive more explicit characterizations than that of Rajeswari and Aziz's. We refer the reader to [8] for more results on strong cleanness.

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Throughout this paper, all rings are associative rings with an identity, $U(R)$ stands for the group of all invertible elements in a ring R, and $GL_2(R)$ denotes the 2-dimensional general linear group of R.

2. Quadratic equations

Lemma 2.1. *Let R be an integral domain, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Then A is strongly clean in $M_2(R)$ if and only if A is invertible, or $I_2 - A$ is invertible, or there exists a $u \in U(R)$ such that the system $(*)$ of equations*

$$(*) \qquad \begin{aligned} (a-d)x + cy + bz &= a - \det(A) + u; \\ yz &= x - x^2; \\ (a-d)y &= b(2x-1); \\ (a-d)z &= c(2x-1) \end{aligned}$$

is solvable.

Proof. Suppose that A is strongly clean and $A, I_2 - A \notin GL_2(R)$. In view of [1, Lemma 1.5], there exists $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \in M_2(R)$ such that $A - E \in GL_2(R)$ and $AE = EA$, where $yz = x - x^2$. It follows from $AE = EA$ that $(a-d)y = b(2x-1)$ and $(a-d)z = c(2x-1)$. One easily checks that

$$\begin{aligned} u := \det(A-E) &= (a-x)(d-1+x) - (b-y)(c-z) \\ &= \det(A) - a + (a-d)x + bz + cy \in U(R). \end{aligned}$$

Therefore $(a-d)x + cy + bz = a - \det(A) + u$, as desired.

Conversely, if either $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, then A is strongly clean; otherwise, there exists a $u \in U(R)$ such that the preceding system $(*)$ of equations is solvable. Set $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$. Then $E = E^2$ and $AE = EA$. Further, $\det(A-E) = \det(A) - a + (a-d)x + bz + cy = u \in U(R)$. Therefore $A - E \in GL_2(R)$, as required. $\square$

Let $A = (a_{ij}) \in M_2(R)$, $s_A = a_{11} - a_{22}$ and $t_A = \mathrm{tr}^2(A) - 4\det(A)$.

Lemma 2.2. *Let R be an integral domain, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. If $s_A \neq 0$ and $t_A = v^2$ for a $v \in U(R)$, then A is strongly clean in $M_2(R)$ if and only if A is invertible, or $I_2 - A$ is invertible, or the equation $x^2 - x + t_A^{-1}bc = 0$ has a root $t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$ for some $u \in U(R)$.*

Proof. Suppose that A is strongly clean in $M_2(R)$. If $A, I_2 - A \notin GL_2(R)$, it follows from Lemma 2.1 that there exists a $u \in U(R)$ such that the system $(*)$ of equations is solvable. Hence, $s_A^2 yz = bc(2x-1)^2$, and so $s_A^2(x - x^2) = 4bc(x^2 - x) + bc$. Thus, $(s_A^2 + 4bc)(x - x^2) = bc$, and so $t_A(x - x^2) = bc$. Further,

$$s_A^2 x + c(s_A y) + b(s_A z) = s_A(a - \det(A) + u).$$

As a result, $s_A^2 x + cb(2x-1) + bc(2x-1) = s_A(a - \det(A) + u)$, and so $(s_A^2 + 4bc)x = 2bc + s_A(a - \det(A) + u)$. Consequently, $t_A x = s_A(a - \det(A) + u) + 2bc$.

Thus, the equation $x^2-x+t_A^{-1}bc=0$ has a root $x=t_A^{-1}\big(s_A(a-\det(A)+u)+2bc\big)$ for some $u \in U(R)$.

Conversely, assume that there exists some $u \in U(R)$ such that the system of equations

$$t_A(x^2-x)+bc=0,\ t_Ax=s_A(a-\det(A)+u)+2bc$$

is solvable. As $t_A=s_A^2+4bc$, we get $1-4t_A^{-1}bc=s_A^2t_A^{-1}$; hence, $1-4(x-x^2)=s_A^2v^{-2}$. This implies that $(1-2x)^2=(s_Av^{-1})^2$. Since R is an integral domain, we get either $1-2x=s_Av^{-1}$ or $1-2x=-s_Av^{-1}$.

Suppose that $1-2x=s_Av^{-1}$. Then $-s_Abv^{-1}=b(2x-1)$. This implies that $(a-d)(-bv^{-1})=b(2x-1)$. Likewise, $(a-d)(-cv^{-1})=c(2x-1)$. Set $y=-bv^{-1}$ and $z=-cv^{-1}$. Then $(a-d)y=b(2x-1)$ and $(a-d)z=c(2x-1)$. Therefore we verify that

$$\begin{aligned} AE &= \begin{pmatrix} ax+bz & ay+b(1-x) \\ cx+dz & cy+d(1-x) \end{pmatrix} \\ &= \begin{pmatrix} ax+cy & bx+dy \\ az+c(1-x) & bz+d(1-x) \end{pmatrix} = EA. \end{aligned}$$

In addition, $yz=bcv^{-2}=t_A^{-1}bc=x-x^2$. Let $E=\left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right)$. Then $E=E^2\in M_2(R)$. One easily checks that

$$\begin{aligned} s_A\det(A-E) &= s_A\big((a-d)x+cy+bz-a+\det(A)\big) \\ &= s_A\big((a-d)x-2bcv^{-1}-a+\det(A)\big) \\ &= s_A^2x+2bc(2x-1)-s_A(a-\det(A)) \\ &= (s_A^2+4bc)x-2bc-s_A(a-\det(A)) \\ &= t_Ax-2bc-s_A(a-\det(A)) \\ &= s_Au. \end{aligned}$$

As $s_A\neq 0$, we get $\det(A-E)\in U(R)$; hence, $A-E\in GL_2(R)$. Therefore $A\in M_2(R)$ is strongly clean.

Suppose that $1-2x=-s_Av^{-1}$. Then $s_Abv^{-1}=b(2x-1)$. This implies that $(a-d)(bv^{-1})=b(2x-1)$. Likewise, $(a-d)(cv^{-1})=c(2x-1)$. Set $y=bv^{-1}$ and $z=cv^{-1}$. Then $(a-d)y=b(2x-1)$ and $(a-d)z=c(2x-1)$; hence, $AE=EA$. In addition, $yz=bcv^{-2}=t_A^{-1}bc=x-x^2$. Let $E=\left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right)$. Then $E=E^2\in M_2(R)$. As in the preceding discussion, $s_A\det(A-E)=s_Au$. As $s_A\neq 0$, $\det(A-E)\in U(R)$, and so $A-E\in GL_2(R)$. Therefore $A\in M_2(R)$ is strongly clean, as required. □

Lemma 2.3. *Let R be an integral domain, let $A, I_2-A\notin GL_2(R)$, and let $s_A\in U(R)$. If A is strongly clean in $M_2(R)$, then $t_A=v^2$ for some $v\in U(R)$.*

Proof. Since A and I_2-A are nonunits, in view of Lemma 2.1, there exist $x,y,z\in R$ such that $A=E+(A-E)$, where $E=E^2=\left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right), yz=x-x^2, EA=AE, A-E\in GL_2(R)$. As $EA=AE$, we get $s_Ay=a_{12}(2x-1)$

and $s_A z = a_{21}(2x-1)$. Hence, $s_A^2(x-x^2) = s_A^2 bc = a_{12}a_{21}(2x-1)^2$. That is, $(s_A^2 + 4a_{12}a_{21})(x-x^2) = a_{12}a_{21}$. Obviously, $t_A = s_A^2 + 4a_{12}a_{21}$. Therefore $s_A^2 = t_A - 4a_{12}a_{21} = t_A - 4t_A(x-x^2) = t_A(1-2x)^2$. As $s_A \in U(R)$, we deduce that $1-2x \in U(R)$. Therefore $t_A = \big(s_A(1-2x)^{-1}\big)^2$, as asserted. □

Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \text{ is odd}\}$. Then $\mathbb{Z}_{(2)}$ is an integral domain. Choose $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$, $p,\ q \in \mathbb{Z},\ p \neq \pm 1$. If $1+4pq$ is not a square of a prime, then $A \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. Clearly, $\mathbb{Z}_{(2)}$ is an integral domain with $s_A = 1 \in U(\mathbb{Z}_{(2)})$. As $t_A = 1+4pq$ is not a square of an invertible element, it follows from Lemma 2.3 that $A \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. For instance, $\begin{pmatrix} 8 & 7 \\ 3 & 7 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. This is the case for $p = 7, q = 3$.

Theorem 2.4. *Let R be an integral domain, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $s_A \in U(R)$, then A is strongly clean in $M_2(R)$ if and only if A is invertible, or $I_2 - A$ is invertible, or the equation $x^2 - x + t_A^{-1}bc = 0$ has a root $t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$ for some $u \in U(R)$.*

Proof. Suppose that A is strongly clean in $M_2(R)$. Since $s_A \in U(R)$, it follows from Lemma 2.3 that $t_A = v^2$ for some $v \in U(R)$. According to Lemma 2.2, we are done.

Conversely, assume that there exists a $u \in U(R)$ such that the system of equations $t_A(x^2-x)+bc = 0, t_A x = s_A(a - \det(A) + u) + 2bc$ is solvable. Set $y = s_A^{-1}b(2x-1)$ and $z = s_A^{-1}c(2x-1)$. Then $s_A^2 yz = bc(2x-1)^2 = 4bc(x^2-x) + bc = s_A^2(x-x^2) - t_A(x-x^2) + bc = s_A^2(x-x^2)$. We infer that $yz = x - x^2$. Let $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$. Then $E = E^2 \in M_2(R)$. One easily checks that

$$\begin{aligned}
\det(A-E) &= (a-d)x + cy + bz - a + \det(A) \\
&= (a-d)x + cs_A^{-1}b(2x-1) + bs_A^{-1}c(2x-1) - a + \det(A) \\
&= s_A^{-1}\big(s_A^2 x + 2bc(2x-1) - s_A(a - \det(A))\big) \\
&= s_A^{-1}\big((s_A^2 + 4bc)x - 2bc - s_A(a - \det(A))\big) \\
&= s_A^{-1}\big(t_A x - 2bc - s_A(a - \det(A))\big) \\
&= s_A^{-1}\big(s_A(a - \det(A) + u) - s_A(a - \det(A))\big) \\
&= u \\
&\in U(R).
\end{aligned}$$

Thus, $A - E \in GL_2(R)$. Moreover,

$$\begin{aligned}
AE &= \begin{pmatrix} ax+bz & ay+b(1-x) \\ cx+dz & cy+d(1-x) \end{pmatrix} \\
&= \begin{pmatrix} ax+cy & bx+dy \\ az+c(1-x) & bz+d(1-x) \end{pmatrix} = EA.
\end{aligned}$$

Therefore $A \in M_2(R)$ is strongly clean, as asserted. □

Let $A = \begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then $s_A = 1 \in U(\mathbb{Z}_{(2)})$ and $t_A = \frac{1}{9}$. Clearly, $x^2 - x + t_A^{-1}bc = x^2 - x - 2$, and so $x^2 - x + t_A^{-1}bc = 0$ has a root $2 \in \mathbb{Z}_{(2)}$. Choose $u = -\frac{1}{2} \in U(\mathbb{Z}_{(2)})$. Then $2 = t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$. In light of Theorem 2.4, we conclude that $\begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is strongly clean.

We use $J(R)$ to stand for the Jacobson radical of the ring R.

Corollary 2.5. *Let R be an integral domain, and let $p, q \in R$. If $p \in J(R)$, then the following are equivalent*:

(1) $\begin{pmatrix} p & p \\ q & p+1 \end{pmatrix}$ *is strongly clean.*
(2) *The equation* $x^2 - x + \frac{pq}{1+4pq} = 0$ *has a root in* $U(R)$.

Proof. Let $A = \begin{pmatrix} p & p \\ q & p+1 \end{pmatrix}$. Then $s_A = -1$ and $t_A = 1 + 4pq$. If A is clean, by virtue of Theorem 2.4, the equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root $x = t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$. One easily checks that $x = t_A^{-1}\big(-p + p(p + 1) - pq - u + 2pq\big) \in U(R)$, as desired.

Conversely, assume that the equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root x in $U(R)$. Choose $u = p^2 - \frac{x}{1+4pq} \in U(R)$. Then the equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root x, which can be written in the form $x = t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$. According to Theorem 2.4, A is strongly clean. □

Corollary 2.6. *Let R be an integral domain, and let $p, q \in J(R)$. Then the following are equivalent*:

(1) $\begin{pmatrix} 0 & p \\ 1 & 1+q \end{pmatrix}$ *is strongly clean.*
(2) *The equation* $x^2 - x + \frac{p}{(1+p)^2+4q} = 0$ *has a root in* $U(R)$.

Proof. Let $A = \begin{pmatrix} 0 & p \\ 1 & 1+q \end{pmatrix}$. Then $s_A = -1 - q \in U(R)$ and $t_A = (1 + q)^2 + 4p \in U(R)$. If $A \in M_2(R)$ is clean, by virtue of Theorem 2.4, the equation $x^2 - x + \frac{p}{(1+p)^2+4q} = 0$ has a root $x = t_A^{-1}\big(s_A(p + u) + 2p\big)$ for a $u \in U(R)$. Therefore $x \in U(R)$, as required.

Conversely, assume that the equation $x^2 - x + \frac{p}{(1+p)^2+4q} = 0$ has a root x in $U(R)$. Set $u = s_A^{-1}(t_A x - 2p) - p$. Then $x = t_A^{-1}\big(s_A(a - \det(A) + u) + 2bc\big)$ for some $u \in U(R)$. In light of Theorem 2.4, $A \in M_2(R)$ is strongly clean. □

Let R be a local ring, and let $A \in M_2(R)$. As is well known, A is strongly clean if and only if $A \in GL_2(R)$, $I_2 - A \in GL_2(R)$ or A is similar to $\begin{pmatrix} 0 & -\det(A) \\ 1 & \mathrm{tr}(A) \end{pmatrix}$ for some $p, q \in J(R)$ (cf. [2, Lemma 16.4.11]). Thus, we deduce the following: Let R be a local integral domain. Then every $A \in M_2(R)$ is strongly clean if and only if $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or the equation $x^2 - x + \frac{p}{(1+p)^2+4q} = 0$ has a root in $U(R)$, where $p = -\det(A)$ and $q = \mathrm{tr}(A) - 1$.

3. Diophantine forms

Let $A \in M_2(R)$. We say that A is strongly e-clean in case there exists an idempotent $E \in M_2(R)$ such that $A - E \in GL_2(R)$, $AE = EA$ and $\det E = e$. The aim of this section is to characterize strongly clean 2×2 matrices over an integral domain by means of a kind of Diophantine equations.

Proposition 3.1. *Let R be an integral domain, and let $A \in M_2(R)$. Then A is strongly 1-clean in $M_2(R)$ if and only if $\det(A) - \mathrm{tr}(A) + 1 \in U(R)$.*

Proof. It is easy to verify that

$$\begin{aligned} A \text{ is strongly 1-clean} &\Leftrightarrow \text{there exists a } U \in GL_2(R) \text{ such that } A = I_2 + U \\ &\Leftrightarrow \begin{vmatrix} a-1 & -b \\ -c & d-1 \end{vmatrix} \in U(R) \\ &\Leftrightarrow \det(A) - \mathrm{tr}(A) + 1 \in U(R), \end{aligned}$$

as desired. □

Lemma 3.2. *Let R be an integral domain. Then $E = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in M_2(R)$ is an idempotent if and only if it is one of the following forms:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}, yz = x - x^2, \text{ either } y \neq 0 \text{ or } z \neq 0.$$

Proof. Suppose $E = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in M_2(R)$ is an idempotent. Then

$$\begin{aligned} x^2 + yz &= x; \\ xy + yw &= y; \\ zx + wz &= z; \\ zy + w^2 &= w. \end{aligned}$$

If either $y \neq 0$ or $z \neq 0$, then $x + w = 1$, and so $w = 1 - x$. In addition, $yz = x - x^2$.

If $y = z = 0$, then $x = x^2$ and $w = w^2$. This implies that $x = 0, 1; w = 0, 1$. Thus, E must be one of the preceding forms, as required.

Conversely, one directly checks that each one of the preceding forms is an idempotent, and therefore we complete the proof. □

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$, and let $x, y \in R$. We say that (x, y) is A-reducible in the case that $y \neq 0, s_A y = b(2x - 1)$ and $t_A(x^2 - x) + bc = 0$.

Theorem 3.3. *Let R be an integral domain, and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$. If $s_A \in U(R)$, then A is strongly 0-clean in $M_2(R)$ if and only if*

(1) *A is invertible, or*
(2) *$a \in 1 + U(R), b = c = 0, d \in U(R)$, or*
(3) *$a \in U(R), b = c = 0, d \in 1 + U(R)$, or*

(4) *there exists a* $u \in U(R)$ *such that* $(-b)x^2 + s_A xy + cy^2 + bx + \big(\det(A) - a + u\big)y = 0$ *has an* A*-reducible root, or*
(5) *there exists a* $u \in U(R)$ *such that* $(-c)x^2 + s_A xy + by^2 + cx + \big(\det(A) - a + u\big)y = 0$ *has an* A^T*-reducible root.*

Proof. Suppose that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is 0-clean. Then there exists an idempotent $E \in M_2(R)$ such that $A - E \in GL_2(R)$ and $AE = EA$, where $\det E = 0$. Clearly, E is one of the matrix forms described in Lemma 3.2.

If $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then $A \in GL_2(R)$.

If $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then $a \in 1 + U(R), b = c = 0, d \in U(R)$.

If $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $a \in U(R), b = c = 0, d \in 1 + U(R)$.

If $E = \left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right), yz = x - x^2$, either $y \neq 0$ or $z \neq 0$, then $\left(\begin{smallmatrix} a-x & b-y \\ c-z & d-1+x \end{smallmatrix}\right) \in GL_2(R)$. Hence, $\left|\begin{smallmatrix} a-x & b-y \\ c-z & d-1+x \end{smallmatrix}\right| \in U(R)$. This implies that $u := -\big((ad - bc) + (a-d)x + bz + cy - a\big) \in U(R)$. If $y \neq 0$, then

$$(a-d)xy + b(x - x^2) + cy^2 + \big(\det(A) - a + u\big)y = 0.$$

That is, $(-b)x^2 + (a-d)xy + cy^2 + bx + \big(\det(A) - a + u\big) = 0$ has a root (x, y). As $AE = EA$, we get $s_A y = b(2x-1)$ and $s_A z = c(2x-1)$. Hence, $s_A yz = bc(2x-1)^2$, and so $t_A(x^2 - x) + bc = 0$. Thus, (x, y) is an A-reducible root. If $z \neq 0$, then $(ad - bc)z + (a-d)xz + bz^2 + cyz - az = uz$. Hence,

$$(-c)x^2 + (a-d)xz + bz^2 + cx + \big(\det(A) - a + u\big)z = 0$$

has a root (x, z). Obviously, $s_A = s_{A^T}$ and $t_A = t_{A^T}$. Thus, $s_{A^T} z = c(2x-1)$ and $t_{A^T}(x^2 - x) + cb = 0$. Therefore, (x, z) is an A^T-reducible root, as required.

Now we prove the converse. If $A \in GL_2(R)$, then A is strongly clean. If $a \in 1 + U(R), b = c = 0, d \in U(R)$, then $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a-1 & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ is strongly clean. If $a \in U(R), b = c = 0, d \in 1 + U(R)$, then $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a & b \\ c & d-1 \end{smallmatrix}\right) \in M_2(R)$ is strongly clean.

Suppose that there exists a $u \in U(R)$ such that $(-b)x^2 + (a-d)xy + cy^2 + bx + \big(\det(A) + u - a\big)y = 0$ has an A-reducible root (x, y). Then $s_A y = b(2x-1)$ and $t_A(x^2 - x) + bc = 0$. Choose $z = s^{-1}c(2x-1)$. Then $s_A^2 yz = bc(2x-1)^2 = s_A^2(x - x^2) - t_A(x - x^2) + bc = s_A^2(x - x^2)$. This infers that $yz = x - x^2$. Choose $E = \left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right)$. Then $E = E^2$. Obviously, $s_A y = b(2x-1)$ and $s_A z = c(2x-1)$. Hence,

$$\begin{aligned} AE &= \begin{pmatrix} ax + bz & ay + b(1-x) \\ cx + dz & cy + d(1-x) \end{pmatrix} \\ &= \begin{pmatrix} ax + cy & bx + dy \\ az + c(1-x) & bz + d(1-x) \end{pmatrix} = EA. \end{aligned}$$

It is easy to verify that

$$\det(A - E) = \begin{vmatrix} a - x & b - y \\ c - z & d - 1 + x \end{vmatrix}$$

$$= ad - a + ax - dx + x - x^2 - bc + bz + yc - yz$$
$$= (a-d)x + bz + yc + \det(A) - a.$$

This implies that

$$y\det(A-E) = \big((a-d)x + bz + yc + \det(A) - a\big)y$$
$$= (-b)x^2 + (a-d)xy + cy^2 + bx + \big(\det(A) - a\big)y = -uy.$$

As $y \neq 0$, we get $\det(A-E) = -u \in U(R)$. Therefore $A - E \in GL_2(R)$, as required.

Suppose that there exists a $u \in U(R)$ such that $(-c)x^2 + (a-d)xy + by^2 + cx + \big(\det(A) - a + u\big)y = 0$ has an A^T-reducible root (x, y). Then $s_{A^T}y = c(2x-1)$ and $t_{A^T}(x^2-x) + cb = 0$. Thus, $s_A y = c(2x-1)$ and $t_A(x^2-x) + bc = 0$. Choose $z = s_A^{-1}b(2x-1)$. Choose $E = \left(\begin{smallmatrix} x & z \\ y & 1-x \end{smallmatrix}\right)$. As in the proceeding discussion, we see that $E = E^2$ and $EA = AE$. Further,

$$\det(A-E) = \begin{vmatrix} a-x & b-z \\ c-y & d-1+x \end{vmatrix}$$
$$= ad - a + ax - dx + x - x^2 - bc + by + zc - yz$$
$$= (a-d)x + by + zc + \det(A) - a.$$

This implies that $y\det(A-E) = \big((a-d)x + by + zc + \det(A) - a\big)y = (-c)x^2 + (a-d)xy + b^2 + cx + \big(\det(A) - a\big)y = -uy$. As $y \neq 0$, we get $\det(A-E) = -u \in U$, and so $A - E \in GL_2(R)$.

In any case, $A \in M_2(R)$ is strongly clean, as asserted. $\square$

Corollary 3.4. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$. If $|a-d| = 1$, then A is strongly 0-clean in $M_2(\mathbb{Z})$ if and only if*

(1) $ad - bc = \pm 1$, *or*
(2) $a = 0, 2; b = c = 0; d = \pm 1$, *or*
(3) $a = \pm 1; b = c = 0; d = 0, 2$, *or*
(4) $A = \left(\begin{smallmatrix} 1 & 0 \\ ** & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ ** & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ ** & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ ** & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ ** & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ ** & -1 \end{smallmatrix}\right)$.
(5) $A = \left(\begin{smallmatrix} 1 & * \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & * \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & * \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & * \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & * \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & * \\ 0 & -1 \end{smallmatrix}\right)$.

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be 0-clean. Suppose that (1), (2) and (3) do not hold. As $s_A = \pm 1$, $t_A = 1$ from Lemma 2.3. That is, $1 + 4bc = 1$; hence, either $b = 0$ or $c = 0$.

I. If $b = 0$, then $s_A y = b(2x-1)$ implies that $y = 0$. Thus, by Theorem 3.3, there exists a $u \in U(\mathbb{Z})$ such that $(-c)x^2 + (a-d)xy + by^2 + cx + \big(\det(A) - a + u\big)y = 0$ has an A^T-reducible root. Hence, $y \neq 0, s_{A^T}y = c(2x-1)$ and $t_{A^T}(x^2-x) + cb = 0$. Obviously, $s_{A^T} = s_A$ and $t_{A^T} = t_A$. Thus, $s_A y = c(2x-1)$ and $x^2 - x = 0$. This implies that either $x = 0$ or $x = 1$.

(1) If $x = 0$, then $y = \pm c$ and $ad - a \pm 1 = 0$. As $s_A = \pm 1$, we deduce that

$$A = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ c & 0 \end{pmatrix}.$$

(2) If $x = 1$, then $y = \pm c$ and $ad - d \pm 1 = 0$. As $s_A = \pm 1$, we deduce that

$$A = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & -1 \end{pmatrix}.$$

II. If $c = 0$, then $s_{A^T} y = c(2x-1)$ implies that $y = 0$. Thus, by Theorem 3.3, there exists a $u \in U(\mathbb{Z})$ such that $(-b)x^2 + (a-d)xy + cy^2 + bx + \big(\det(A) - a + u\big)y = 0$ has an A-reducible root. Hence, $y \neq 0, s_A y = b(2x-1)$ and $t_A(x^2 - x) + bc = 0$. As $s_A = \pm 1$ and $t_A = 1$, we get $y = \pm b(2x-1)$ and $x^2 - x = 0$. This implies that either $x = 0$ or $x = 1$.

(1) If $x = 0$, then $y = \pm b$ and $ad - a \pm 1 = 0$. As $s_A = \pm 1$, we deduce that

$$A = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & b \\ 0 & 0 \end{pmatrix}.$$

(2) If $x = 1$, then $y = \pm b$ and $ad - d \pm 1 = 0$. As $s_A = \pm 1$, we deduce that

$$A = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & -1 \end{pmatrix},$$

as required.

The converse is obvious from the direct verifications. □

For instance, it follows from Proposition 3.1 and Corollary 3.4 that $\left(\begin{smallmatrix} 8 & 7 \\ 3 & 7 \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ is not strongly clean.

Example 3.5. Let $A = \left(\begin{smallmatrix} -2 & 3 \\ -4 & 5 \end{smallmatrix}\right) \in M_2(\mathbb{Z})$. Then A is strongly clean in $M_2(\mathbb{Z})$. This can be seen from the strongly clean expression $A = \left(\begin{smallmatrix} -3 & 3 \\ -4 & 4 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. But A is not one of the forms given in Corollary 3.4. In this case, $s_A = -7 \notin U(\mathbb{Z})$.

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