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A CHARACTERIZATION OF THE GROUP $\mathbb{A}_{22}$ BY NON-COMMUTING GRAPH

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Abstract. Let G be a finite non-abelian group. We define the non-commuting graph $\nabla(G)$ of G as follows: the vertex set of $\nabla(G)$ is $G-Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. In this paper we prove that if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{A}_{22})$, then $G \cong \mathbb{A}_{22}$, where $\mathbb{A}_{22}$ is the alternating group of degree 22.

1. Introduction

The study of relation between groups and graphs is one of the main research topics in group theory. For instance the prime graph $\Gamma(G)$ associated with a finite group G introduced by Gruenberg and Kegel is defined as follows: The vertex set of $\Gamma(G)$ is $\pi(G)$, the set of prime divisors of the order of G. Two distinct primes p and q, considered as vertices of $\Gamma(G)$, are adjacent by an edge if and only if G contains an element of order pq. The graph we will consider in this paper is denoted by $\nabla(G)$ and is called the non-commuting graph of G that has attracted the attention of many authors.

The vertex set of $\nabla(G)$ is $G - Z(G)$, where $Z(G)$ is the center of G and two distinct vertices x and y are adjacent whenever $xy \neq yx$. For a graph X, we denote the set of vertices and edges of X by $V(X)$ and $E(X)$ respectively. Two graphs X and Y are isomorphic and we denote it by $X \cong Y$, if there exists a bijective map $\phi : V(X) \longrightarrow V(Y)$ such that if x and y are adjacent in X, then $\phi(x)$ and $\phi(y)$ are adjacent in Y and vice versa. In [2] relation between some graph theoretical properties of $\nabla(G)$ and the group theoretic properties of the group G are studied. In particular the following two conjectures are raised.

Conjecture 1. *Let G and H be two arbitrary finite groups such that $\nabla(G) \cong \nabla(H)$. Then $|G| = |H|$.*

Conjecture 2. *Let S be a finite non-abelian simple group. If G is a group such that $\nabla(G) \cong \nabla(S)$, then $G \cong S$.*

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In [3] M. R. Darafsheh proved Conjecture 1 for any simple group. Also Conjecture 2 is known to hold for all simple groups with non connected prime graphs. In [6] L. Wang and W. Shi verified Conjecture 2 for the alternating group $\mathbb{A}_{10}$ that has connected prime graph. Also in [1], [4], [5], [7], [9] and [8] Conjecture 2 has been verified for the groups $SL(2,q)$, $\mathbb{A}_{16}$, $L_2(q)$, $L_4(8)$, $U_4(7)$, $L_4(9)$ and $S_4(q)$ respectively. But it is still unknown for many simple groups with connected prime graphs. In this paper we will prove Conjecture 2 for the alternating group $\mathbb{A}_{22}$ that has connected prime graph. We believe our method in this paper will help the interested researcher to characterize simple alternating groups such as $\mathbb{A}_{p+3}$ where p is an odd prime.

2. Preliminaries

In this section we list some basic and known results which will be used later. For the proof see [2], [3].

Lemma 2.1. *Let G be a finite group such that $\nabla(G) \cong \nabla(\mathbb{A}_{22})$. Then*

$$|G| = |\mathbb{A}_{22}|.$$

Lemma 2.2. *Let G and H be finite groups. If $\nabla(G) \cong \nabla(H)$, then*

$$|C_G(x) - Z(G)| = |C_H(\phi(x)) - Z(H)|$$

for all $1 \neq x \in G - Z(G)$, where ϕ is an isomorphism from $\nabla(G)$ to $\nabla(H)$.

By Lemmas 2.1 and 2.2 we have the following lemma.

Lemma 2.3. *Let G be a finite group such that $\nabla(G) \cong \nabla(\mathbb{A}_{22})$. Then*

$$|C_G(x)| = |C_{\mathbb{A}_{22}}(\phi(x))|$$

for all $1 \neq x \in G - Z(G)$, where ϕ is an isomorphism from $\nabla(G)$ to $\nabla(\mathbb{A}_{22})$.

3. Characterization of $\mathbb{A}_{22}$

Theorem. *Let $\mathbb{A}_{22}$ be the alternating group of degree 22. If G is a finite group with $\nabla(G) \cong \nabla(\mathbb{A}_{22})$, then $G \cong \mathbb{A}_{22}$.*

Proof. We divide the proof into the following seven lemmas. We use the following table in our proof. Note that by Lemma 2.1 we have $|G| = |\mathbb{A}_{22}|$, hence

$$\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19\}.$$

TABLE 1. Finite non-abelian simple groups S such that $\pi(S) \subseteq \{2,3,5,7,11,13,17,19\}$ except alternating ones.

S	$\lvert S\rvert$	$\lvert \mathrm{Out}(S)\rvert$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
$L_3(2^2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12
$L_2(2^3)$	$2^3 \cdot 3^2 \cdot 7$	3
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(7^2)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2
$U_3(2^2)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4
$G_2(2^2)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
$S_4(2^3)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$Sz(2^3)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3
$L_2(2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4
$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_3(3^2)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$L_2(3^3)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6
$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2

$L_2(5^2)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_4(2^2)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
$U_4(2^2)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$S_4(2^2)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$S_6(2^2)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$O_8^+(2^2)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
$L_2(2^4)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$L_3(2^4)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	6
$U_3(2^3)$	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	18
$U_4(2^3)$	$2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	6
$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	6
$L_4(7)$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^6 \cdot 19$	4
$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	2
$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	2
$U_3(19)$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 19^3$	2
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	2
F_5	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	2

Lemma 3.1. *Let* $p \in \{2, 5, 7, 11, 13, 17\}$ *and* $x \in \mathbb{A}_{22}$. *If* $p \cdot 19 \big| |C_{\mathbb{A}_{22}}(x)|$, *then either* $x = 1$ *or* $|C_{\mathbb{A}_{22}}(x)| = \frac{19! \times 3}{2}$ *and* x *is a 3-cycle.*

Proof. We know that for all $x \in S_{22}, |C_x| = \frac{22!}{1^{x_1} x_1! 2^{x_2} x_2! \cdots 22^{x_{22}} x_{22}!}$, where S_{22} is the symmetric group of degree 22, C_x is the conjugacy class of S_{22} containing x and x has x_1 1-cycles, x_2 2-cycles and so on, up to x_{22} 22-cycles, where $1x_1 + 2x_2 + \cdots + 22x_{22} = 22$. But $|C_x| = [S_{22} : C_{S_{22}}(x)]$ thus $|C_{S_{22}}(x)| = \frac{22!}{|C_x|} = 1^{x_1} x_1! 2^{x_2} x_2! \cdots 22^{x_{22}} x_{22}!$.

Assume that there exists $p \in \{2, 5, 7, 11, 13, 17\}$ and $x \in \mathbb{A}_{22}$ such that $p \cdot 19 \big| |C_{\mathbb{A}_{22}}(x)|$, where x has x_1 1-cycles, x_2 2-cycles and so on, up to x_{22} 22-cycles.

Thus $p\cdot 19\big|1^{x_1}x_1!2^{x_2}x_2!\cdots 22^{x_{22}}x_{22}!$ since $|C_{\mathbb{A}_{22}}(x)|\big||C_{S_{22}}(x)|$. It is easy to see that $x_{19}=1$ or $x_1\geq 19$. But if $x_{19}=1$, then x is a 19-cycle or x is the product of a 19-cycle and a 3-cycle, since x is an even permutation. In any case $|C_{\mathbb{A}_{22}}(x)|=19\cdot 3$. But for all $p\in\{2,5,7,11,13,17\}$, $p\nmid 19\cdot 3$ and this implies that $x_{19}=0$ and $x_1\neq 19$. Since x is an even permutation, $x_2=0$ and $x_1=22$ or $x_1=19$, $x_3=1$. It means that $x=1$ or $|C_{\mathbb{A}_{22}}(x)|=\frac{19!\times 3}{2}$ and x is a 3-cycle. □

Lemma 3.2. *If $\nabla(G)\cong\nabla(\mathbb{A}_{22})$, then $p\cdot 19\notin\pi_e(G)$ for all $p\in\{2,5,7,11,13,17\}$.*

Proof. Assume there exists $p\in\{2,5,7,11,13,17\}$ such that $p\cdot 19\in\pi_e(G)$. So there exists $x\in G-Z(G)$ such that $o(x)=|\langle x\rangle|=p\cdot 19$. It is obvious that $p\cdot 19\big||C_G(x)|$, since $\langle x\rangle\leq C_G(x)$. But $\nabla(G)\cong\nabla(\mathbb{A}_{22})$, thus by Lemma 2.3 $|C_G(x)|=|C_{\mathbb{A}_{22}}(\phi(x))|$ for all $x\in G-Z(G)$, where ϕ is an isomorphism from $\nabla(G)$ to $\nabla(\mathbb{A}_{22})$. Hence $p\cdot 19\big||C_{\mathbb{A}_{22}}(\phi(x))|$, which implies that $|C_{\mathbb{A}_{22}}(\phi(x))|=\frac{19!\times 3}{2}$ and $\phi(x)$ is a 3-cycle or $\phi(x)=1$ by Lemma 3.1. But clearly $\phi(x)\neq 1$, since $\phi(x)\notin Z(\mathbb{A}_{22})$ and so $|C_G(x)|=|C_{\mathbb{A}_{22}}(\phi(x))|=\frac{19!\times 3}{2}$. Let $C_G(x)=\{g_1,g_2,\ldots,g_{\frac{19!\times 3}{2}}\}$, where $g_1=1$. We have $x\in C_G(g_i)$ and $g_i\notin Z(G)$ for $i=2,3,\ldots,\frac{19!\times 3}{2}$ and since $o(x)=p\cdot 19$, by similar argument we obtain $|C_{\mathbb{A}_{22}}(\phi(g_i))|=\frac{19!\times 3}{2}$ and $\phi(g_i)$ is a 3-cycle for $i=2,3,\ldots,\frac{19!\times 3}{2}$. But the number of 3-cycles in $\mathbb{A}_{22}$ is equal to $\binom{22}{3}\times 2=\frac{22\times 21\times 20}{6}\times 2=3080$. Hence the number of distinct $\phi(g_i)$ must be less than or equal to 3080. But $\frac{19!\times 3}{2}-1>3080$ and so there exist $i,j=2,3,\ldots,\frac{19!\times 3}{2}$ such that $\phi(g_i)=\phi(g_j)$ which is a contradiction since ϕ is injective. □

Now we suppose that N is a minimal normal subgroup of G. Hence N is characteristically simple and so N is the direct product of isomorphic simple groups. Let $N\cong A\times A\times\cdots\times A$, where A is a simple group. In the sequel we will prove that $N\cong\mathbb{A}_{22}$.

Lemma 3.3. *Let N be a minimal normal subgroup of G. If N is non-abelian, then $\pi(N)=\pi(G)=\pi(\mathbb{A}_{22})$ and N is a simple subgroup of G.*

Proof. Since N is non-abelian, A is non-abelian too. Thus A is a non-abelian simple group and so $2\big||A|$ and since $|A|\big||N|$, we have $2\big||N|$. Also we know that $|N|\big||G|=|\mathbb{A}_{22}|=2^{18}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19$. Therefore N is the direct product of at most 18 isomorphic simple groups. Thus there exists $r\leq 18$ such that $|\mathrm{Aut}(N)|=|\mathrm{Aut}(A)|^r\cdot r!=|\mathrm{Out}(A)|^r\cdot|A|^r\cdot r!$, where A is a simple group such that $\pi(A)\subseteq\{2,3,5,7,11,13,17,19\}$, since $|A|\big||N|\big||G|$. Because $\pi(A)\subseteq\{2,3,5,7,11,13,17,19\}$, $|\mathrm{Out}(\mathbb{A}_n)|=2$ for all $n\geq 7$ and by Table 1 it follows that $19\nmid|\mathrm{Out}(A)|$. Now if $19\nmid|A|$, then $19\nmid|N|$ and $19\nmid|\mathrm{Aut}(N)|=|\mathrm{Aut}(A)|^r\cdot r!=|\mathrm{Out}(A)|^r\cdot|A|^r\cdot r!$ ($r\leq 18$) and since $\frac{|G|}{|C_G(N)|}=\frac{22!/2}{|C_G(N)|}\big||\mathrm{Aut}(N)|$, we have $19\big||C_G(N)|$. Thus there exists an element $x\in C_G(N)$ such that $o(x)=19$ and since $2\big||N|$ we can find an element $y\in N$ such that $o(y)=2$. Because $xy=yx$

it follows that $o(xy) = 2 \cdot 19$. This contradicts Lemma 3.2 and so $19\big||N|$ and $19 \in \pi(N) = \pi(A)$. But $19^2 \nmid |G|$ and $|N| = |A|^r||G|$. Hence $r = 1$ and N is a simple subgroup of G. By similar argument $2, 5, 7, 11, 13, 17 \in \pi(A) = \pi(N)$. Thus $\{2, 5, 7, 11, 13, 17, 19\} \subseteq \pi(A) = \pi(N)$. It is sufficient to prove that $3 \in \pi(N)$. If N is an alternating group, then $N \cong \mathbb{A}_{19}, \mathbb{A}_{20}, \mathbb{A}_{21}$ or $\mathbb{A}_{22}$, which in any case $3\big||N|$. If N is a simple group except alternating groups, then since $\{2, 5, 7, 11, 13, 17, 19\} \subseteq \pi(N) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19\}$ by Table 1, $3 \in \pi(N)$. □

Lemma 3.4. *Let N be a minimal normal subgroup of G. If N is non-abelian, then $N \cong \mathbb{A}_{22}$.*

Proof. By Lemma 3.3 and Table 1 it is easy to see that N is an alternating group. Thus $N \cong \mathbb{A}_{19}, \mathbb{A}_{20}, \mathbb{A}_{21}$ or $\mathbb{A}_{22}$. Suppose there exists $i = 19, 20, 21$ such that $N \cong \mathbb{A}_i$. Since $|\mathrm{Aut}(N)| = |\mathrm{Aut}(\mathbb{A}_i)| = i!$, we have $\frac{|G|}{|C_G(N)|} = \frac{11\times 21!}{|C_G(N)|}\big||\mathrm{Aut}(N)| = i!$. Thus there exists $k \in \mathbb{N}$ such that $i! = \frac{11\times 21!}{|C_G(N)|} \times k$. Therefore $|C_G(N)| = 11 \times \frac{21!}{i!} \times k$. Since $i \leq 21$, $\frac{21!}{i!} \in \mathbb{N}$ and so $11\big||C_G(N)|$. Thus there exists an element $x \in C_G(N)$ such that $o(x) = 11$ and since $19\big||N|$, we can find an element $y \in N$ such that $o(y) = 19$. But $xy = yx$ and so $o(xy) = 11 \cdot 19$. This is a contradiction by Lemma 3.2. □

Lemma 3.5. *If N is a minimal normal subgroup of G, then N is not isomorphic to $\mathbb{Z}_3^i$ for $i = 1, 2, \ldots, 9$ (direct product of i cyclic group $\mathbb{Z}_3$).*

Proof. We know that N is a union of conjugacy classes of G and the size of each conjugacy class of G and $\mathbb{A}_{22}$ are the same by Lemma 2.3. But it is obvious that $\frac{19!\times 3}{2}$ is the maximum centralizer order in $\mathbb{A}_{22}$ and G. Thus the minimum size of a conjugacy class in G is equal to $\frac{19!\times 20\times 21\times 22/2}{\frac{19!\times 3}{2}} = 3080$. Hence all normal subgroups except the trivial subgroup have more than 3080 elements. But $\mathbb{Z}_3^i = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ (i times) has at most 2187 elements for $i = 1, 2, \ldots, 7$ and so N is not isomorphic to $\mathbb{Z}_3^i$ for $i = 1, 2, \ldots, 7$. Clearly the second centralizer order in $\mathbb{A}_{22}$ and G is $\frac{18!\times 8}{2}$ (in fact the centralizer order of a permutation of type 2^2). Thus after 3080, minimum size of conjugacy class in G is equal to $\frac{18!\times 19\times 20\times 21\times 22/2}{18!\times 8/2} = 21945$. But $\mathbb{Z}_3^8$ and $\mathbb{Z}_3^9$ have at most 19683 elements. Therefore there exists $m \in N$ such that $|N| = 1 + 3080 \times m$. It means that $3080\big||N| - 1$ and since $3080 \nmid (|\mathbb{Z}_3^8| - 1), (|\ \mathbb{Z}_3^9| - 1)$, N is not isomorphic to $\mathbb{Z}_3^8$ and $\mathbb{Z}_3^9$. Hence N is not isomorphic to the direct product of $\mathbb{Z}_3$. □

Lemma 3.6. *If N is a minimal normal subgroup of G, then N is non-abelian.*

Proof. Suppose that N is a minimal normal and abelian subgroup of G. Thus there exists $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$ such that N is isomorphic to the direct product of $\mathbb{Z}_p$. By Lemma 3.5 we have $p \neq 3$. If $N \cong \mathbb{Z}_{19}$, then $|\mathrm{Aut}(N)| = 19 \cdot 18$. Thus $17 \nmid |\mathrm{Aut}(N)|$ and since $\frac{|G|}{|C_G(N)|} = \frac{22!/2}{|C_G(N)|}\big||\mathrm{Aut}(N)|$, we have $17\big||C_G(N)|$. So there exists $x \in C_G(N)$ such that $o(x) = 17$ and since $|N| = 19$

we can find an element $y \in N$ such that $o(y) = 19$. But $xy = yx$ and so $o(xy) = 17 \cdot 19$. This contradicts Lemma 3.2. Hence $p \neq 3, 19$. By similar argument we can see $p \neq 2, 5, 7, 11, 13, 17$ and the proof is completed. □

Lemma 3.7. *If N is a minimal normal subgroup of G, then N is non-abelian and $N \cong \mathbb{A}_{22}$.*

Proof. This follows immediately from Lemmas 3.6 and 3.4. □

We assume that N is a minimal normal subgroup of G. From Lemma 3.7, $N \cong \mathbb{A}_{22}$. But $|N| = |\mathbb{A}_{22}| = |G|$ and $N \leq G$. Hence $N = G$ and therefore $G \cong \mathbb{A}_{22}$. □

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