Bull. Korean Math. Soc. **50** (2013), No. 1, pp. 97–103
http://dx.doi.org/10.4134/BKMS.2013.50.1.097

RIGIDITY THEOREMS IN THE HYPERBOLIC SPACE

Henrique Fernandes de Lima

Abstract. As a suitable application of the well known generalized maximum principle of Omori-Yau, we obtain rigidity results concerning to a complete hypersurface immersed with bounded mean curvature in the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. In our approach, we explore the existence of a natural duality between $\mathbb{H}^{n+1}$ and the half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models the so-called steady state space.

1. Introduction

In this paper, we are interested in the study of complete non-compact hypersurfaces immersed with bounded mean curvature in the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. Before giving details on our work, we present a brief outline of the main results related to our ones.

In [1], L. J. Alías and M. Dajczer studied complete surfaces properly immersed in $\mathbb{H}^3$ which are contained between two horospheres, obtaining a Bernstein-type result for the case of constant mean curvature $-1 \leq H \leq 1$.

The author and A. Caminha have studied in [3] complete vertical graphs of constant mean curvature in $\mathbb{H}^{n+1}$. Under appropriate restriction on the growth of the height function, they obtained necessary conditions for the existence of such a graph. Furthermore, for complete surfaces of nonnegative Gaussian curvature, they obtained a Bernstein-type theorem in $\mathbb{H}^3$.

More recently, by applying a technique of S. T. Yau [13], the author jointly with F. E. C. Camargo and A. Caminha [2] have also obtained Bernstein-type results in $\mathbb{H}^{n+1}$.

Here, under an appropriated restriction on the normal angle of the hypersurface (that is, the angle between the Gauss map of the hypersurface and the unitary vector field which determines on $\mathbb{H}^{n+1}$ a codimension one foliation by horospheres; see Section 3), we obtain rigidity theorems concerning to a complete hypersurface immersed with bounded mean curvature in $\mathbb{H}^{n+1}$. In our approach, we explore the existence of a natural duality between $\mathbb{H}^{n+1}$ and the

Received May 4, 2011.
2010 *Mathematics Subject Classification.* Primary 53C42; Secondary 53B30, 53C50.
Key words and phrases. hyperbolic space, complete hypersurfaces, mean curvature, Gauss map.
The author was partially supported by FAPESQ/CNPq/PPP, Brazil.

half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models the so-called *steady state space* (cf. Sections 2 and 3).

We prove the following (cf. Theorem 3.3; see also Corollaries 3.5 and 3.6):

Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form A. Suppose that the (not necessarily constant) mean curvature H of Σ^n is such that $0 \leq H \leq 1$. If Σ^n is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle θ satisfies $\cos\theta \geq \sup_\Sigma H$, then Σ^n is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathcal{H}^{n+1}$.

We want to point out that our restriction on the normal angle of the hypersurface is motivated by a gradient estimate due to R. López and S. Montiel [6] (for more details, see Remark 3.4).

Furthermore, by applying a classical result due to A. Huber [4] concerned with parabolic surfaces, we also prove the following (cf. Theorem 3.7):

Let $\psi : \Sigma^2 \to \mathbb{H}^3$ be a complete surface of nonnegative Gaussian curvature and with (not necessarily constant) mean curvature $0 \leq H \leq 1$. If the normal angle θ of Σ^2 satisfies $\cos\theta \geq H$, then Σ^2 is a horosphere and the image of its Lorentz Gauss map is exactly a plane of $\mathcal{H}^3$.

2. The steady state space $\mathcal{H}^{n+1}$

In order to study the geometry of the Gauss map of a hypersurface immersed in the hyperbolic space, we need some preliminaries of Lorentz geometry.

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2}$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{S}_1^{n+1} = \left\{p \in L^{n+2}; \langle p, p\rangle = 1\right\}.$$

The induced metric from $\langle , \rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Moreover, if $p \in \mathbb{S}_1^{n+1}$, we can put

$$T_p\left(\mathbb{S}_1^{n+1}\right) = \left\{v \in \mathbb{L}^{n+2}; \langle v, p\rangle = 0\right\}.$$

Let $a \in \mathbb{L}^{n+2}$ be a non-zero null vector in the past half of the null cone (with vertex in the origin), that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle > 0\right\}$$

is the so-called *steady state* space. Observe that $\mathcal{H}^{n+1}$ is extendible and, so, non-compact, being only half a de Sitter space. Its boundary, as a subset of $\mathbb{S}_1^{n+1}$, is the null hypersurface

$$\left\{x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle = 0\right\},$$

whose topology is that of $\mathbb{R} \times \mathbb{S}^{n-1}$ (cf. [7]).

Now, we shall consider in $\mathcal{H}^{n+1}$ the timelike field

$$\mathcal{K} = \langle x, a \rangle x - a.$$

We easily see that

$$\overline{\nabla}_V \mathcal{K} = \langle x, a \rangle V \text{ for all } V \in \mathfrak{X}(\mathcal{H}^{n+1}),$$

that is, $\mathcal{K}$ is closed and conformal field on $\mathcal{H}^{n+1}$ (cf. [5], Section 2). Then, from Proposition 1 of [9], we have that the n-dimensional distribution $\mathcal{D}$ defined on $\mathcal{H}^{n+1}$ by

$$p \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(p) = \left\{ v \in T_p\mathcal{H}^{n+1}; \langle \mathcal{K}(p), v \rangle = 0 \right\}$$

determines a codimension one spacelike foliation $\mathcal{F}(\mathcal{K})$ which is oriented by $\mathcal{K}$. Moreover (cf. [10], Example 1), the leaves of $\mathcal{F}(\mathcal{K})$ are hyperplanes

$$\mathcal{L}_\rho = \left\{ x \in \mathbb{S}_1^{n+1}; \langle x, a \rangle = \rho \right\}, \quad \rho > 0,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$ isometric to the Euclidean space $\mathbb{R}^n$, and having constant mean curvature 1 with respect to the unit past-directed normal fields

$$\eta_\rho(x) = x - \frac{1}{\rho} a, \quad x \in \mathcal{L}_\rho.$$

3. Rigidity results in $\mathbb{H}^{n+1}$

In this section, instead of the more commonly used half-space model for the $(n+1)$-dimensional hyperbolic space, we consider the *warped product model*

$$\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n.$$

It can easily be seen that the fibers $M_{t_0} = \{t_0\} \times \mathbb{R}^n$ of the warped product model are precisely the horospheres of $\mathbb{H}^{n+1}$. Moreover, these have constant mean curvature 1 if we take the orientation given by the unit normal vector field $N = -\partial_t$ (cf. [8], Example 3 of Section 4).

Another useful model for $\mathbb{H}^{n+1}$ is the so-called *Lorentz model*, obtained by furnishing the hyperquadric

$$\{p \in \mathbb{L}^{n+2};\ \langle p, p \rangle = -1,\ p_{n+2} > 0\}$$

with the (Riemannian) metric induced by the Lorentz metric of $\mathbb{L}^{n+2}$. In this setting, if $a \in \mathbb{L}^{n+2}$ denotes a fixed null vector as in the beginning of the previous section, a typical horosphere is

$$L_\tau = \{p \in \mathbb{H}^{n+1};\ \langle p, a \rangle = \tau\},$$

where τ is a positive real number. A straightforward computation shows that

$$\xi_p = -p - \frac{1}{\tau} a \in \mathcal{H}^{n+1}$$

is a unit normal vector field along L_τ, with respect to which L_τ has mean curvature 1 (cf. [6], Section 3).

In the context of the Lorentz model of $\mathbb{H}^{n+1}$, we say that a hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ is *under a horosphere* L_τ when $\langle \psi, a\rangle \leq \tau$. In this case, if we consider the warped model of $\mathbb{H}^{n+1}$, we easily see that the height function $h = \pi_{\mathbb{R}} \circ \psi$ of Σ^n is bounded from above.

Now, we present our analytical framework.

Lemma 3.1 ([3], Proposition 3.2)**.** *Let $\psi : \Sigma^n \to \mathbb{R} \times_f M^n$ be a hypersurface immersed into a Riemannian warped product $\mathbb{R} \times_f M^n$, with Gauss map N. Then, by denoting $h = \pi_I \circ \psi$ the height function of Σ^n, we have*

$$\Delta h = (\ln f)'(h)(n - |\nabla h|^2) + nH\langle N, \partial_t\rangle.$$

We also will need the well known generalized *Maximum Principle* due to H. Omori and S. T. Yau [11, 12].

Lemma 3.2. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from above on Σ^n. Then there is a sequence of points $\{p_k\}$ in Σ^n such that*

$$\lim_{k\to\infty} u(p_k) = \sup_{\Sigma} u, \quad \lim_{k\to\infty} |\nabla u(p_k)| = 0 \ \text{ and } \ \lim_{k\to\infty} \Delta u(p_k) \leq 0.$$

In what follows, we will consider an isometry Φ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$ which carries $(\partial_t)_q$ to $\Phi_*(\partial_t) = \xi_{\Phi(q)}$ (such isometry is given in [1]). In this setting, it is natural to consider the *Lorentz Gauss map* of Σ with respect to N as given by

$$\begin{array}{ccc} \Sigma^n & \to & \mathcal{H}^{n+1} \\ p & \mapsto & -\Phi_*(N_p). \end{array}$$

Given a hypersurface Σ^n in $\mathbb{H}^{n+1}$ whose Gauss map satisfies $\langle N, \partial_t\rangle < 0$, we define the *normal angle* θ of Σ^n as being the smooth function $\theta : \Sigma^n \to [0, \frac{\pi}{2}]$ given by

$$0 \leq \cos\theta = -\langle N, \partial_t\rangle \leq 1.$$

Now, we can state and prove our main result.

Theorem 3.3. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form A. Suppose that the mean curvature H of Σ^n is such that $0 \leq H \leq 1$. If Σ^n is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle θ satisfies $\cos\theta \geq \sup_\Sigma H$, then Σ^n is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathcal{H}^{n+1}$.*

Proof. Initially, let us consider $X \in \mathfrak{X}(\Sigma)$ with $|X| = 1$. It follows from Gauss equation that

$$\mathrm{Ric}_\Sigma(X) = 1 - n + nH\langle AX, X\rangle - \langle AX, AX\rangle,$$

where Ric_Σ stands for the Ricci curvature of Σ^n. Hence,

$$\mathrm{Ric}_\Sigma \geq 1 - n - nH|A| - |A|^2.$$

Thus, since H and A are supposed to be bounded, we conclude that Ric_Σ is bounded from below on Σ^n.

Now, from Lemma 3.1, we have that

$$\Delta h = n\,(1 + H\langle N, \partial_t\rangle) - |\nabla h|^2.$$

On the other hand, since Σ^n is supposed to be under a horosphere of $\mathbb{H}^{n+1}$ and its Ricci curvature is bounded from below, we are in position to apply Lemma 3.2 to the function h, obtaining a sequence $\{p_k\}$ in Σ^n such that

$$\lim_{k\to\infty} h(p_k) = \sup_\Sigma h, \quad \lim_{k\to\infty} |\nabla h(p_k)| = 0 \text{ and } \lim_{k\to\infty} \Delta h(p_k) \le 0.$$

Consequently, since the functions H and $\langle N, \partial_t\rangle$ are bounded on Σ^n, we get a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that

$$0 \ge \lim_{j\to\infty} \Delta h(p_{k_j}) \ge n\left(1 - \lim_{j\to\infty} H(p_{k_j})\right) \ge 0.$$

Then, $\lim_{j\to\infty} H(p_{k_j}) = 1$, and $\sup_\Sigma H = 1$. Thus, since we are supposing that the normal angle θ of Σ^n satisfies $\cos\theta \ge \sup_\Sigma H$, we get that $\langle N, \partial_t\rangle = -\cos\theta = -1$ on Σ^n and, hence, Σ^n is a horosphere. Moreover, by considering an isometry Φ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$, we get

$$\langle N, a\rangle = \langle -\partial_t, a\rangle = \langle -\xi_\Phi, a\rangle = \langle \psi, a\rangle$$

and, therefore, we conclude that $N(\Sigma)$ is exactly a hyperplane of $\mathcal{H}^{n+1}$. □

Remark 3.4. Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a immersion from a compact manifold Σ^n with mean convex boundary $\partial\Sigma$ contained into a horosphere L_τ, for some $\tau > 0$. Suppose that ψ has constant mean curvature $0 \le H \le 1$. From the gradient estimate (19) of [6], taking into account our choice of the orientation N of Σ^n, we get

$$\langle N, a\rangle \ge H\tau.$$

Consequently, by supposing that Σ^n is under the horosphere L_τ, we conclude that its normal angle θ satisfies

$$\cos\theta = -\langle N, \partial_t\rangle = \frac{1}{\langle \psi, a\rangle}\langle N, a\rangle \ge \frac{1}{\tau}\langle N, a\rangle \ge H.$$

Since for a hypersurface Σ^n immersed in $\mathbb{H}^{n+1}$ we have that

$$|A|^2 = n^2H^2 - n(n-1)(R+1),$$

where R denotes de scalar curvature of Σ^n, we get:

Corollary 3.5. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with scalar curvature R bounded from below. Suppose that the mean curvature H of Σ^n is such that $0 \le H \le 1$. If Σ^n is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle θ satisfies $\cos\theta \ge \sup_\Sigma H$, then Σ^n is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathcal{H}^{n+1}$.*

By using once more the existence of a natural duality between $\mathbb{H}^{n+1}$ and $\mathcal{H}^{n+1}$, we obtain the following consequence of Theorem 3.3.

Corollary 3.6. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form A. Suppose that the mean curvature H of Σ^n is such that $0 \leq H \leq 1$. If Σ^n is under a horosphere L_τ and the image of its Lorentz Gauss map $N(\Sigma)$ is contained in the closure of the interior domain enclosed by a hyperplane $\mathcal{L}_\rho$ of $\mathcal{H}^{n+1}$, with $\frac{\rho}{\tau} \geq \sup_\Sigma H$, then Σ^n is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathcal{H}^{n+1}$.*

Proof. By considering again an isometry Φ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$, we get

$$\langle N, \partial_t \rangle = \langle N, -\psi - \frac{1}{\langle \psi, a \rangle} a \rangle = -\frac{1}{\langle \psi, a \rangle} \langle N, a \rangle.$$

Consequently, since we are supposing that Σ^n is under the horosphere L_τ and that its Lorentz Gauss map $N(\Sigma)$ is contained in the closure of the interior domain enclosed by the hyperplane $\mathcal{L}_\rho$,

$$\cos\theta = -\langle N, \partial_t \rangle \geq \frac{\rho}{\tau}.$$

Therefore, our hypothesis on the image of the Lorentz Gauss map of Σ^n amounts to

$$\cos\theta \geq \sup_\Sigma H$$

and, hence, the result follows from Theorem 3.3. □

In the 3-dimensional case, we obtain the following rigidity result concerning to complete surfaces of nonnegative Gaussian curvature.

Theorem 3.7. *Let $\psi : \Sigma^2 \to \mathbb{H}^3$ be a complete surface of nonnegative Gaussian curvature and with mean curvature $0 \leq H \leq 1$. If the normal angle θ of Σ^2 satisfies $\cos\theta \geq H$, then Σ^2 is a horosphere and the image of its Lorentz Gauss map is exactly a plane of $\mathcal{H}^3$.*

Proof. By applying Lemma 3.1, we get

$$\begin{aligned} \Delta e^{-h} &= e^{-h}\left(|\nabla h|^2 - \Delta h\right) \\ &= 2e^{-h}\left(|\nabla h|^2 - 1 - H\langle N, \partial_t \rangle\right). \end{aligned}$$

On the other hand, since $h = \pi_{\mathbb{R}|\Sigma}$, one has

$$\begin{aligned} \nabla h &= \nabla(\pi_{\mathbb{R}|\Sigma}) = (\overline{\nabla}\pi_\mathbb{R})^\top = \partial_t^\top \\ &= \partial_t - \langle N, \partial_t \rangle N, \end{aligned}$$

where $\overline{\nabla}$ denotes the gradient with respect to the metric of $\mathbb{H}^3$, and $(\,)^\top$ the tangential component of a vector field in $\mathfrak{X}(\mathbb{H}^3)$ along Σ^2. Consequently, $|\nabla h|^2 = 1 - \cos^2\theta$. Thus,

$$\Delta e^{-h} = 2e^{-h}\cos\theta\,(H - \cos\theta)$$

and, hence, our hypothesis on the normal angle θ of Σ^2 guarantees that the function e^{-h} is a superharmonic positive function on Σ. However, a classical result due to A. Huber [4] assures that complete surfaces of nonnegative Gaussian curvature must be parabolic. Therefore, h is constant on Σ^2, that is, Σ^2 is a horosphere of $\mathbb{H}^3$ and the image of its Lorentz Gauss map $N(\Sigma)$ is exactly a plane of $\mathcal{H}^3$. $\square$

References

[1] L. J. Alías and M. Dajczer, *Uniqueness of constant mean curvature surfaces properly immersed in a slab*, Comment. Math. Helv. **81** (2006), no. 3, 653–663.

[2] F. E. C. Camargo, A. Caminha, and H. F. de Lima, *Bernstein-type Theorems in Semi-Riemannian Warped Products*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1841–1850.

[3] A. Caminha and H. F. de Lima, *Complete vertical graphs with constant mean curvature in semi-Riemannian warped products*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 1, 91–105.

[4] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[5] H. F. de Lima, *Spacelike hypersurfaces with constant higher order mean curvature in de Sitter space*, J. Geom. Phys. **57** (2007), no. 3, 967–975.

[6] R. López and S. Montiel, *Existence of constant mean curvature graphs in hyperbolic space*, Calc. Var. Partial Differential Equations **8** (1999), no. 2, 177–190.

[7] S. Montiel, *Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces*, J. Math. Soc. Japan **55** (2003), no. 4, 915–938.

[8] ———, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), no. 2, 711–748.

[9] ———, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. **314** (1999), no. 3, 529–553.

[10] ———, *An integral inequality for compact spacelike hypersurfaces in De Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), no. 4, 909–917.

[11] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[12] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[13] ———, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), no. 7, 659–670.

Departamento de Matemática e Estatística
Universidade Federal de Campina Grande
Campina Grande, Paraíba, Brazil
E-mail address: henrique.delima@pq.cnpq.br