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# LOCAL AND GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS TO A POLYTROPIC FILTRATION SYSTEM WITH NONLINEAR MEMORY AND NONLINEAR BOUNDARY CONDITIONS

Jian Wang, Meng-long Su, and Zhong-Bo Fang

Abstract. This paper deals with the behavior of positive solutions to the following nonlocal polytropic filtration system

$$\begin{cases} u_t=(|(u^{m_1})_x|^{p_1-1}(u^{m_1})_x)_x+u^{l_{11}}\int_0^a v^{l_{12}}(\xi,t)\mathrm{d}\xi, (x,t) \text{ in } [0,a]\times(0,T), \\ v_t=(|(v^{m_2})_x|^{p_2-1}(v^{m_2})_x)_x+v^{l_{22}}\int_0^a u^{l_{21}}(\xi,t)\mathrm{d}\xi, (x,t) \text{ in } [0,a]\times(0,T) \end{cases}$$

with nonlinear boundary conditions $u_x|_{x=0}=0$, $u_x|_{x=a}=u^{q_{11}}v^{q_{12}}|_{x=a}$, $v_x|_{x=0}=0$, $v_x|_{x=a}=u^{q_{21}}v^{q_{22}}|_{x=a}$ and the initial data $(u_0,\ v_0)$, where $m_1,m_2\geq 1$, $p_1,p_2>1$, $l_{11}$, $l_{12}$, $l_{21}$, $l_{22}$, $q_{11}$, $q_{12}$, $q_{21}$, $q_{22}>0$. Under appropriate hypotheses, the authors establish local theory of the solutions by a regularization method and prove that the solution either exists globally or blows up in finite time by using a comparison principle.

## 1. Introduction and main results

In this paper, we consider the following system:

(1.1)
$$\begin{cases} u_t=(|(u^{m_1})_x|^{p_1-1}(u^{m_1})_x)_x+u^{l_{11}}\displaystyle\int_0^a v^{l_{12}}(\xi,t)\mathrm{d}\xi, & (x,t)\in[0,a]\times(0,T), \\ v_t=(|(v^{m_2})_x|^{p_2-1}(v^{m_2})_x)_x+v^{l_{22}}\displaystyle\int_0^a u^{l_{21}}(\xi,t)\mathrm{d}\xi, & (x,t)\in[0,a]\times(0,T), \\ u_x|_{x=0}=0, \quad u_x|_{x=a}=u^{q_{11}}v^{q_{12}}|_{x=a}, & t\in(0,T), \\ v_x|_{x=0}=0, \quad v_x|_{x=a}=u^{q_{21}}v^{q_{22}}|_{x=a}, & t\in(0,T), \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in[0,a], \end{cases}$$

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where $T > 0$, $l_{11}$, $l_{12}$, $l_{21}$, $l_{22}$, $q_{11}$, $q_{12}$, $q_{21}$, $q_{22} > 0$, $m_1, m_2 \geq 1$, $p_1, p_2 > 1$. Let $Q_T = [0, a] \times (0, T]$, $T > 0$, $z_+ = \max\{z, 0\}$. Throughout this paper we assume that:

(i) $u_0(x), v_0(x) \in C^{2+\alpha}([0, a])$ for some $0 < \alpha < 1$, $u_0(x), v_0(x) \geq \delta > 0$;
(ii) $(|u_{0x}^{m_1}|^{p_1-1} u_{0x}^{m_1})_x, (|v_{0x}^{m_2}|^{p_2-1} v_{0x}^{m_2})_x \in L^2([0, a])$ on $[0, a]$;
(iii) $u_0(x), v_0(x)$ satisfy the compatibility conditions:

$$(A) \qquad \begin{aligned} &u_{0x}(0) = 0, \quad u_{0x}(a) = u_0^{q_{11}} v_0^{q_{12}}(a), \\ &v_{0x}(0) = 0, \quad v_{0x}(a) = u_0^{q_{21}} v_0^{q_{22}}(a). \end{aligned}$$

Problems of this form arise in mathematical models such as modeling gas or fluid flow through a porous medium and completely turbulent flow and for the spread of certain biological populations (see [3, 5, 15] and the references therein). In the non-Newtonian fluids theory, the pair $(p_1, p_2)$ is a characteristic quantity of medium. Media with $(p_1, p_2) > (2, 2)$, which means $p_1 > 2$, $p_2 > 2$, are called dilatant fluids and those with $(p_1, p_2) < (2, 2)$ are called pseudo-plastics. If $(p_1, p_2) = (2, 2)$, they are called Newtonian fluids. When $(p_1, p_2) = (2, 2)$ and $(m_1, m_2) = (1, 1)$ the connection with the flow in porous media is by now classical. When $(m_1, m_2) \geq (1, 1)$ and $(p_1, p_2) > (2, 2)$, the system models the non-stationary, polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence on the velocity of the displacement under polytropic conditions (non-Newtonian elastic filtration); it has been intensively studied (see [16, 17, 21] and references therein). The nonlocal growth terms present a more realistic model of a population [6, 10, 14, 18]. The nonlinear boundary conditions in (1.1) can be physically interpreted as a nonlinear radiation law (see [9]).

In recent years, many authors have studied the global existence or blow-up of solutions to some parabolic problems with nonlinear boundary conditions. In [1], G. Acosta and J. D. Rossi considered the global existence of solutions to the following problem:

$$\begin{cases} u_t = \Delta u + f(u, v), v_t = \Delta v + g(u, v), & \text{in } \Omega \times (0, T), \\ \frac{\partial u}{\partial \eta} = h(u, v), \quad \frac{\partial v}{\partial \eta} = s(u, v), & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & \text{in } \Omega, \end{cases}$$

which can be viewed as a heat conduction problem with nonlinear diffusivity, source and a nonlinear radiation law coupling on the boundary of the material body.

In [7], Y. Chen considered the following semilinear parabolic systems with nonlocal source:

$$\begin{aligned} &u_t = \Delta u + \int_\Omega v^p(\xi, t) \mathrm{d}\xi, \quad v_t = \Delta v + \int_\Omega u^p(\xi, t) \mathrm{d}\xi, \quad (x, t) \in \Omega \times (0, T), \\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\ &u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), x \in \Omega. \end{aligned}$$

She obtained some blowup criteria and a blowup rate.

Recently, in [8], L. Du studied the following problem

$$u_t = \Delta u^m + u^{p_1}\int_\Omega v^{q_1}(\xi,t)\mathrm{d}\xi,\ v_t = \Delta v^n + v^{p_2}\int_\Omega u^{q_2}(\xi,t)\mathrm{d}\xi,\ (x,t)\in\Omega\times(0,T),$$
$$u(x,t) = v(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T),$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), x\in\Omega.$$

And he also get the criteria for solution exists globally or blows up in finite time. Moreover, if $p_1 = 0$ or $p_1 > m$; $p_2 = 0$ or $p_2 > n$; $q_1 > n$, $q_2 > m$ and satisfy $q_2 > p_1 - 1$, $q_1 > p_2 - 1$, he also get the blow-up rates under the monotone assumption for initial data.

In [20], X. Wu investigate the behavior of positive solutions to the following system of evolution $p$-Laplace equations coupled via nonlocal sources with nonlinear boundary conditions and the initial data

$$(1.2)\quad \begin{aligned} &u_t = (|u_x|^{p_1-1}u_x)_x + \int_0^a v^{m_1}(\xi,t)\mathrm{d}\xi, && (x,t) \text{ in } [0,a]\times(0,T),\\ &v_t = (|v_x|^{p_2-1}v_x)_x + \int_0^a u^{m_2}(\xi,t)\mathrm{d}\xi, && (x,t) \text{ in } [0,a]\times(0,T)\\ &u_x|_{x=0} = 0, \quad u_x|_{x=a} = u^{q_{11}}v^{q_{12}}|_{x=a}, && t\in(0,T),\\ &v_x|_{x=0} = 0, \quad v_x|_{x=a} = u^{q_{21}}v^{q_{22}}|_{x=a}, && t\in(0,T),\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x\in[0,a]. \end{aligned}$$

Under appropriate hypotheses, the authors first prove a local existence result by a regularization method. Then the authors discuss the global existence and blow-up of positive weak solutions by using a comparison principle. And F. Li [13] considered the problem (1.2) with homogeneous Dirichlet boundary conditions and obtained some necessary and sufficient conditions on the global existence of the positive solutions.

In [19], the following problem has been intensively studied by S. Wang:

$$(1.3)\quad \begin{aligned} &(|u|^{r_1-1}u)_t = (|u_x|^{p_1-1}u_x)_x, && 0<x<1, t>0,\\ &(|v|^{r_2-1}v)_t = (|v_x|^{p_2-1}v_x)_x, && 0<x<1, t>0,\\ &u_x|_{x=0} = 0, \quad u_x|_{x=1} = \lambda u^{l_{11}}v^{l_{12}}|_{x=1}, && t>0,\\ &v_x|_{x=0} = 0, \quad v_x|_{x=1} = \lambda u^{l_{21}}v^{l_{22}}|_{x=1}, && t>0,\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && 0\le x\le 1. \end{aligned}$$

The author proved a local existence result and obtained some necessary and sufficient conditions on the global existence of all positive (weak) solutions.

Recently, J. Zhou, in [23], considered problem (1.1) with homogeneous Dirichlet boundary conditions and obtained some necessary and sufficient conditions on the global existence of the positive solutions. More results for parabolic equations with nonlinear boundary conditions can be found in [4, 12, 22] and the references therein.

However, to the author's best knowledge, there is little literature on the study of the global existence and blow-up properties for the system (1.1). This paper extends their results of the references cited above essentially to non-Newton polytropic filtration system (1.1). Therefore, this paper is also an extension of the above results. Due to the nonlinear diffusion terms and doubly degeneration for $u = 0$, $|\nabla u| = 0$ or $v = 0$, $|\nabla v| = 0$, we have some new difficulties to be overcome. Noticing that the system (1.1) includes the Newtonian filtration system ($(p_1, p_2) = (2, 2)$) and the non-Newtonian filtration system ($(m_1, m_2) = (1, 1)$) formally, so the method for it should be synthetic. In fact, we can use the methods for the above two systems to deal with it. First under appropriate hypotheses, we established local theory of the solutions. The method we used is the so-called 'test function method' and some modifications and adaptations of ideas from [19] and [20]. Our proof is based on argument by the different method of regularization, which involves considering the regularized problem firstly and making a priori estimates for the nonnegative approximate solutions by carefully choosing special test functions and a scaling argument, then obtaining the results based on the estimates by a standard limiting process. Then we investigate the global existence or blow-up properties of weak solutions to the problem (1.1) depending on the relations among the parameters $m_1, m_2, p_1, p_2, l_{11}, l_{12}, l_{21}, l_{22}, q_{12}, q_{12}, q_{21}, q_{22}$. Note that (1.1) has nonlinear and nonlocal sources $u^{l_{11}} \int_0^a v^{l_{12}}(\xi, t)\mathrm{d}\xi$, $v^{l_{22}} \int_0^a u^{l_{21}}(\xi, t)\mathrm{d}\xi$ and nonlinear boundary sources $u^{q_{11}}v^{q_{12}}, u^{q_{21}}v^{q_{22}}$, which make the behavior of the solution different from that for that of homogeneous Neumann or Dirichlet boundary value problems. However, it is difficult to use the same methods as that in [23] to get the desired result. To overcome these difficulties, we used some modification of the technique in [19] so that we can handle the nonlinearities. Roughly speaking, the proof consists of several steps. First, we establish the comparison principle for system (1.1) by choosing suitable test function and Gronwall's inequality. Then, we use some functions to control the nonlocal sources and prove, with the technique in [19], that the control for the nonlocal sources is suitable. Finally we also need to consider the effect of these nonlinear terms in the proof of the global existence (blow-up) property of solutions to (1.1). In this paper we will give some necessary and sufficient conditions on the global existence of positive weak solutions to (1.1). These results extend the results of [19, 20] to the general case with nonlocal sources and nonlinear boundary sources.

The main results of this paper are the following theorems:

**Theorem 1.1.** *All positive weak solutions of* (1.1) *exist globally if and only if*

$$l_{11} \leq 1,\ l_{22} \leq 1,\ q_{11} \leq \frac{1}{p_1} + 1 - m_1,\ q_{22} \leq \frac{1}{p_2} + 1 - m_2,$$
$$l_{12}l_{21} \leq (1 - l_{11})(1 - l_{22}),\ l_{12}q_{21} \leq [1 - l_{11}][\frac{1}{p_2} + 1 - m_2 - q_{22}],$$

$$l_{21}q_{12} \le [\frac{1}{p_1} + 1 - m_1 - q_{11}][1 - l_{22}] \quad \textit{and}$$
$$q_{12}q_{21} \le (\frac{1}{p_1} + 1 - m_1 - q_{11})(\frac{1}{p_2} + 1 - m_2 - q_{22}).$$

**Note.** The system of inequalities in Theorem 1.1 is very large, so we give an example. Let $p_1 = p_2 = \frac{3}{2}$, $m_1 = m_2 = \frac{7}{6}$, $q_{11} = q_{22} = \frac{1}{3}$, $l_{11} = l_{22} = \frac{1}{2}$, $l_{12} = l_{21} = \frac{1}{4}$, $q_{21} = q_{12} = \frac{1}{8}$. Then we can obviously prove that the above inequalities hold.

**Theorem 1.2.** *All positive weak solutions of* (1.1) *blow up in finite time if one of the following inequalities holds*:
$(B)$

(i) $l_{11} > 1$ *or* $l_{22} > 1$ *or* $q_{11} > \frac{1}{p_1} + 1 - m_1$ *or* $q_{22} > \frac{1}{p_2} + 1 - m_2$;

(ii) $q_{11} \le \frac{1}{p_1} + 1 - m_1, q_{22} \le \frac{1}{p_2} + 1 - m_2$ *and*
$$q_{12}q_{21} > (\frac{1}{p_1} + 1 - m_1 - q_{11})(\frac{1}{p_2} + 1 - m_2 - q_{22});$$

(iii) $l_{11} \le 1, l_{22} \le 1$ *and* $l_{12}l_{21} > (1 - l_{11})(1 - l_{22})$;

(iv) $l_{11} \le 1, q_{22} \le \frac{1}{p_2} + 1 - m_2$ *and* $l_{12}q_{21} > [1 - l_{11}][\frac{1}{p_2} + 1 - m_2 - q_{22}]$;

(v) $l_{22} \le 1, q_{11} \le \frac{1}{p_1} + 1 - m_1$ *and* $l_{21}q_{12} > [\frac{1}{p_1} + 1 - m_1 - q_{11}][1 - l_{22}]$.

The outline of this paper is as follows: In the next section, we will give the proof of a weak comparison principle and a simple fact without proof. In Section 3, we will prove the local existence results by a regularization method. In Section 4, we will discuss the global existence and blow-up property of solutions to (1.1) by constructing various upper and lower solutions.

## 2. Preliminaries

In this paper, we use the following definition of the weak solutions.

**Definition 2.1.** A pair of functions $(u, v) \in C(\overline{Q}_T) \times C(\overline{Q}_T)$ is called a supersolution (subsolution) of problem (1.1) in $Q_T$ if all of the following hold:

(i) $u, v \in L^\infty(0, T; W^{1,\infty}(0, a)) \cap W^{1,2}(0, T; L^2(0, a))$, $(u(x, 0), v(x, 0)) \ge (\le)(u_0(x), v_0(x))$;

(ii) For any nonnegative functions $\varphi_1(x, t), \varphi_2(x, t) \in L^1(0, T; W^{1,2}(0, a)) \cap L^2(Q_T)$,

(2.1)
$$\iint_{Q_T} u_t(x,t)\varphi_1(x,t)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} |(u^{m_1})_x(x,t)|^{p_1-1}(u^{m_1})_x(x,t)\varphi_{1x}(x,t)\mathrm{d}x\mathrm{d}t$$

$$\geq (\leq) \int_0^T u^{q_{11}} v^{q_{12}}(a,t)\varphi_1(a,t)\mathrm{d}t + \iint_{Q_T} (u^{l_{11}} \int_0^a v^{l_{12}}(\xi,t)\mathrm{d}\xi)\varphi_1(x,t)\mathrm{d}x\mathrm{d}t,$$

(2.2)

$$\iint_{Q_T} v_t(x,t)\varphi_2(x,t)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} |(v^{m_2})_x(x,t)|^{p_2-1}(v^{m_2})_x(x,t)\varphi_{2x}(x,t)\mathrm{d}x\mathrm{d}t$$

$$\geq (\leq) \int_0^T u^{q_{21}} v^{q_{22}}(a,t)\varphi_2(a,t)\mathrm{d}t + \iint_{Q_T} (v^{l_{22}} \int_0^a u^{l_{21}}(\xi,t)\mathrm{d}\xi)\varphi_2(x,t)\mathrm{d}x\mathrm{d}t.$$

$(u,v)$ is called a weak solution of (1.1) if it is both a supersolution and a subsolution.

**Definition 2.2.** We say the solution $(u,v)$ of the problem (1.1) blows up in finite time if there exists a positive constant $T < +\infty$ such that

$$\lim_{t\to T^-} \sup_{x\in[0,a]} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty}) = +\infty.$$

We say the solution $(u,v)$ exists globally if

$$\sup_{t\in(0,+\infty)} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty}) < +\infty.$$

We first give a weak comparison principle.

**Proposition 2.1** (Comparison principle)**.** *Let $(u,v)$ be a weak solution of (1.1), $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ a subsolution and a supersolution of (1.1) in $Q_T$, respectively, with nonlinear boundary flux $\underline{\lambda}\underline{u}^{q_{11}}\underline{v}^{q_{12}}$, $\underline{\lambda}\underline{u}^{q_{21}}\underline{v}^{q_{22}}$, $\overline{\lambda}\overline{u}^{q_{11}}\overline{v}^{q_{12}}$, $\overline{\lambda}\overline{u}^{q_{21}}\overline{v}^{q_{22}}$ and nonlocal terms $\underline{u}^{l_{11}}\int_0^a \underline{v}^{l_{12}}(\xi,t)d\xi$, $\underline{v}^{l_{21}}\int_0^a \underline{u}^{l_{22}}(\xi,t)d\xi$, $\overline{u}^{l_{11}}\int_0^a \overline{v}^{l_{12}}(\xi,t)d\xi$, $\overline{v}^{l_{21}}\int_0^a \overline{u}^{l_{22}}(\xi,t)d\xi$ where $0<\underline{\lambda}<1<\overline{\lambda}$. Then $(\underline{u},\underline{v}) \leq (u,v) \leq (\overline{u},\overline{v})$ in $\overline{Q}_T$, if $(\underline{u}_0,\underline{v}_0) \leq (\overline{u}_0,\overline{v}_0)$ and there exists a positive constant $\delta$, such that either*

$$\int_0^a \underline{v}^{l_{12}}dx \geq \delta, \quad \int_0^a \underline{u}^{l_{22}}dx \geq \delta, \quad \underline{u},\underline{v} \geq \delta \tag{2.3}$$

*or*

$$\int_0^a \overline{v}^{l_{12}}dx \geq \delta, \quad \int_0^a \overline{u}^{l_{22}}dx \geq \delta, \quad \overline{u},\overline{v} \geq \delta \tag{2.4}$$

*hold.*

*Proof.* Similarly to the proof of Proposition 2.1 in [19]. For small $\delta > 0$, let

$$H_\delta(z) = \min(1, \max(\frac{z}{\delta}, 0))$$

and set $\varphi_1 = H_\delta(\underline{u}^{m_1} - u^{m_1})$, then according to the definitions of solution and lower solution we have

(2.5)
$$\begin{aligned}
&\iint_{Q_t} \{(\underline{u} - u)_t H_\delta(\underline{u}^{m_1} - u^{m_1})\\
&+ (H_\delta(\underline{u}^{m_1} - u^{m_1}))_x[|(\underline{u}^{m_1})_x|^{p_1-1}(\underline{u}^{m_1})_x - |(u^{m_1})_x|^{p_1-1}(u^{m_1})_x]\}\mathrm{d}x\mathrm{d}t\\
\le &\int_0^t H_\delta(\underline{u}^{m_1} - u^{m_1})[(\underline{\lambda u}^{q_{11}}\underline{v}^{q_{12}})^{p_1} - (\lambda u^{q_{21}}v^{q_{22}})^{p_2}]|_{x=a}\mathrm{d}t\\
&+ \iint_{Q_t} H_\delta(\underline{u}^{m_1} - u^{m_1})\left(\underline{u}^{l_{11}}\int_0^a \underline{v}^{l_{12}} - u^{l_{11}}\int_0^a v^{l_{12}}\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Define
$$\chi(x) = \begin{cases} 1, & x \ge 0,\\ 0, & x < 0.\end{cases}$$

As in [2], by letting $\delta \to 0$, noticing
$$\iint_{Q_t} (H_\delta(\underline{u}^{m_1} - u^{m_1}))_x[|(\underline{u}^{m_1})_x|^{p_1-1}(\underline{u}^{m_1})_x - |(u^{m_1})_x|^{p_1-1}(u^{m_1})_x]\mathrm{d}x\mathrm{d}t \ge 0$$
and $\chi[\underline{u}^{m_1} > u^{m_1}] = \chi[\underline{u} > u]$, we get

$$\begin{aligned}
&\iint_{Q_t} (\underline{u} - u)_t\chi[\underline{u} > u]\mathrm{d}x\mathrm{d}t\\
\le &\int_0^t f_1(x,t)\chi[\underline{u} > u]|_{x=a}\mathrm{d}t\\
&+ \iint_{Q_t} \chi[\underline{u} > u]\underline{\lambda u}^{l_{11}}\left(\int_0^a G(x,t)(\underline{v}(x,t) - v(x,t))\mathrm{d}x\right)\mathrm{d}x\mathrm{d}t\\
&+ \iint_{Q_t} \chi[\underline{u} > u]\Phi(x,t)(\underline{u} - u)(\int_0^a v^{l_{12}}(\xi,t)\mathrm{d}\xi)\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{2.6}$$

where
$$f_1(x,t) = \underline{v}^{p_1q_{12}}[(\underline{\lambda u}^{q_{11}})^{p_1} - (u^{q_{11}})^{p_1}] + \lambda^{p_1}u^{p_1q_{11}}p_1q_{12}\theta_1^{p_1q_{12}-1}(\underline{v} - v),$$
$$G(x,t) = \int_0^1 l_{12}[\xi\underline{v} + (1-\xi)v]^{l_{12}-1}\mathrm{d}\xi,\ \Phi(x,t) = \int_0^1 l_{11}[\xi\underline{u} + (1-\xi)u]^{l_{11}-1}\mathrm{d}\xi$$
for some $\theta_1 > 0$ lying between $\underline{v}(a,t)$ and $v(a,t)$.

Since $(0,0) < (\delta,\delta) \le (\underline{u}(x,0), \underline{v}(x,0)) \le (u(x,0), v(x,0)), 0 \le x \le a, \underline{\lambda} < 1$, by the continuity of $\underline{u}, \underline{v}, u, v$, there exists a time $\tau > 0$ such that $f_1(a,t) \le 0$ for all $t \in [0,\tau]$. Since $(\underline{u}, \underline{v})$ and $(u,v)$ are bounded in $Q_t$, it follows from $m_1, m_2, p_1, p_2 > 1$, $l_{12}, l_{11} \ge 1$ that $G(x,t), \Phi(x,t)$ are bounded nonnegative functions. Now, if $l_{12}, l_{11} < 1$, we have $G(x,t) \le \delta^{l_{12}-1}, \Phi(x,t) \le \delta^{l_{11}-1}$ by the assumptions (2.3) or (2.4). It follows that

$$\begin{aligned}&\int_0^a (\underline{u}-u)_+ \mathrm{d}x \\ &\le c_1 \iint_{Q_t} (\underline{u}-u)_+ \mathrm{d}x\mathrm{d}t + c_2 \iint_{Q_t} G(x,t)[\underline{v}(x,t)-v(x,t)]_+ \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.7}$$

where $\omega_+ = \max\{\omega, 0\}$ and $c_1, c_2$ are bounded constants. Similarly, we can prove

$$\begin{aligned}&\int_0^a (\underline{v}-v)_+ \mathrm{d}x \\ &\le c_3 \iint_{Q_t} (\underline{v}-v)_+ \mathrm{d}x\mathrm{d}t + c_4 \iint_{Q_t} F(x,t)[\underline{u}(x,t)-u(x,t)]_+ \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.8}$$

where $c_3, c_4$ are bounded constants, and

$$F(x,t) = \int_0^1 l_{21}[\xi\underline{u} + (1-\xi)u]^{l_{21}-1} \mathrm{d}\xi$$

is a bounded nonnegative function. Now (2.7), (2.8) combined with the Gronwall's inequality show that $(\underline{u}, \underline{v}) \le (u, v)$ on $\overline{Q}_\tau$.

Define $\tau^* = \sup\{\tau \in [0,T] : (\underline{u}(x,t), \underline{v}(x,t)) \le (u(x,t), v(x,t))$ for all $(x,t) \in \overline{Q}_\tau\}$. We claim that $\tau^* = T$. Otherwise, from the continuity of $\underline{u}, \underline{v}, u, v$ there exists an $\varepsilon > 0$, such that $\tau^* + \varepsilon < T$, $f_1(a,t), g_1(a,t) \le 0$ for all $t \in [0, \tau^* + \varepsilon]$. By (2.7), (2.8) and Gronwall's inequality we have $(\underline{u}, \underline{v}) \le (u, v)$ on $\overline{Q}_{\tau^*+\varepsilon}$, which contradicts the definition of $\tau^*$. Hence, $(\underline{u}, \underline{v}) \le (u, v)$ on $\overline{Q}_T$.

Similarly, we can prove that $(u, v) \le (\overline{u}, \overline{v})$ on $\overline{Q}_T$. This completes the proof of Proposition 2.1. □

At the end of this section, we describe a simple fact without proof.

**Fact 2.1.** *Suppose that positive constants $A_i, B_i, C_i, D_i$, $i = 1, 2$ satisfy $A_1/C_1 \le D_1/B_1$, $A_2/C_2 \le D_2/B_2$ and that either $A_2/C_2 \in [A_1/C_1, D_1/B_1]$ or $A_1/C_1 \in [A_2/C_2, D_2/B_2]$ holds. Then there exist positive constants $K$ and $L$ such that*

$$\max(A_1/C_1, A_2/C_2) \le K/L \le \min(D_1/B_1, D_2/B_2).$$

## 3. Local existence

In this section, we study local existence of solutions to problem (1.1).

**Theorem 3.1.** *Assume* (A). *Then there exists a time $T > 0$ such that* (1.1) *has a unique weak solution $(u, v)$ on $\overline{Q}_T$ satisfying $(u, v) \ge (\delta, \delta) > (0, 0)$ for some positive constant $\delta$.*

The proof of this theorem basically follows line by line the proof of Theorem 1 in [19]. However the nonlocal term causes some difficulties, we will give the outline of the proof by pointing out the differences. Consider the following approximating problems for problem (1.1):

(3.1)
$$\begin{cases} u_{\varepsilon t} = [(|(u_\varepsilon^{m_1})_x|^2+\varepsilon)^{\frac{p_1-1}{2}}\Phi_1(u_\varepsilon)u_{\varepsilon x}]_x + F_1(u_\varepsilon, v_\varepsilon), & (x,t)\in(0,a)\times(0,T),\\ v_{\varepsilon t} = [(|(v_\varepsilon^{m_2})_x|^2+\varepsilon)^{\frac{p_2-1}{2}}\Phi_2(v_\varepsilon)v_{\varepsilon x}]_x + F_2(u_\varepsilon, v_\varepsilon), & (x,t)\in(0,a)\times(0,T),\\ u_{\varepsilon x}|_{x=0}=0, \quad u_{\varepsilon x}|_{x=a} = G_1(u_\varepsilon(a,t), v_\varepsilon(a,t)), & t\in(0,T),\\ v_{\varepsilon x}|_{x=0}=0, \quad v_{\varepsilon x}|_{x=a} = G_2(u_\varepsilon(a,t), v_\varepsilon(a,t)), & t\in(0,T),\\ u_\varepsilon(x,0)=u_0(x), v_\varepsilon(x,0)=v_0(x), & x\in[0,a]. \end{cases}$$

We need to control the nonlocal term by applying the technique developed in [20]. Choose the bounded functions $\Phi_i(w)$, $F_i(w,z)$, $G_i(w,z)\in C^\infty(R)$ such that: $\Phi_i(w)=m_iw^{m_i-1}$, $F_1(w,z)=w^{l_{11}}\int_0^a z^{l_{12}}\mathrm{d}\xi$, $F_2(w,z)=z^{l_{22}}\int_0^a w^{l_{21}}\mathrm{d}\xi$, $G_1(w,z)=w^{q_{11}}z^{q_{12}}$, $G_2(w,z)=w^{q_{21}}z^{q_{22}}$ for $\delta\le w,z\le M+1$, where $M=\|u_0(x)\|_\infty+\|v_0(x)\|_\infty$. And we assume that there exist positive constants $l$ and $L$ such that

$$0<l\le\Phi_1(w),\Phi_2(w),F_1(w,z),F_2(w,z),G_1(w,z),G_2(w,z)\le L<+\infty,$$

$$\frac{\partial G_1(w,z)}{\partial z},\frac{\partial G_2(w,z)}{\partial w}\ge 0$$

for any $w,z\in R$.

First, we claim that there exist a small constant $\tau_1>0$ and a positive constant $C$ independent of $\varepsilon$ such that

$$\|(u_\varepsilon^{m_1})_x\|_\infty, \|(v_\varepsilon^{m_2})_x\|_\infty\le C \text{ on } \overline{Q}_{\tau_1}.$$

*Proof.* Choose bounded functions: $a_{i\varepsilon}(z)\in C^\infty(R)$, $0<\rho_\varepsilon\le a'_{i\varepsilon}(z)\le\rho_\varepsilon^{-1}$ on $R$ for some $0<\rho_\varepsilon<1$ and

$$a_{i\varepsilon}(z)=(|z|^2+\varepsilon)^{\frac{p_i-1}{2}}z \quad \text{for } |z|\le K+L+1, \quad i=1,2,$$

where $K=\|(u_0^{m_1})_x(x)\|_\infty+\|(v_0^{m_2})_x(x)\|_\infty$. Then consider the following problem:

(3.2)
$$\begin{cases} u_{\varepsilon t} = [a_{1\varepsilon}((u_\varepsilon^{m_1})_x)]_x + F_1(u_\varepsilon, v_\varepsilon), & (x,t)\in[0,a]\times(0,T),\\ v_{\varepsilon t} = [a_{2\varepsilon}((v_\varepsilon^{m_2})_x)]_x + F_2(u_\varepsilon, v_\varepsilon), & (x,t)\in[0,a]\times(0,T),\\ u_{\varepsilon x}|_{x=0}=0, \quad u_{\varepsilon x}|_{x=a} = G_1(u_\varepsilon(a,t), v_\varepsilon(a,t)), & t\in(0,T),\\ v_{\varepsilon x}|_{x=0}=0, \quad v_{\varepsilon x}|_{x=a} = G_2(u_\varepsilon(a,t), v_\varepsilon(a,t)), & t\in(0,T),\\ u_\varepsilon(x,0)=u_0(x), v_\varepsilon(x,0)=v_0(x), & x\in[0,a]. \end{cases}$$

For (3.2), standard parabolic theory (see [11]) shows that there is a pair of unique smooth solutions $(u_\varepsilon, v_\varepsilon)$ in the class $H^{2+\beta,1+\beta/2}(\overline{Q}_T)$ for some $\beta\in$

$(0,1)$. Obviously, comparison principle holds for (3.2). Therefore,

$$u_\varepsilon(x,t), v_\varepsilon(x,t) \geq \delta > 0, \quad u_{\varepsilon t}(x,t), v_{\varepsilon t}(x,t) \geq 0, \ (x,t) \in \overline{Q}_T$$

and for some constant $c \in R$,

$$0 < c \leq (F_1)'_1(w,z), (F_1)'_2(w,z), (F_2)'_1(w,z), (F_2)'_2(w,z) \leq c^{-1}. \tag{3.3}$$

In fact, let $w = (u_\varepsilon^{m_1})_x$, $z = (v_\varepsilon^{m_2})_x$, then $(w,z)$ satisfies

$$\begin{cases} \Phi_1(u_\varepsilon)w_t - \Phi_1^2(u_\varepsilon)a'_{1\varepsilon}((u_\varepsilon^{m_1})_x)w_{xx} - \Phi_1^2(u_\varepsilon)a''_{1\varepsilon}((u_\varepsilon^{m_1})_x)(w_x)^2 \\ - [\Phi'_1(u_\varepsilon)u_{\varepsilon t} + \Phi_1(u_\varepsilon)(F_1)'_1(u_\varepsilon, v_\varepsilon)]w - \Phi_1^2(u_\varepsilon)(F_1)'_2(u_\varepsilon, v_\varepsilon)v_{\varepsilon x} = 0 \\ ((x,t) \in [0,a] \times (0,T)), \\ \Phi_2(v_\varepsilon)z_t - \Phi_2^2(v_\varepsilon)a'_{2\varepsilon}((v_\varepsilon^{m_2})_x)z_{xx} - \Phi_2^2(v_\varepsilon)a''_{2\varepsilon}((v_\varepsilon^{m_2})_x)(z_x)^2 \\ - [\Phi'_2(v_\varepsilon)v_{\varepsilon t} + \Phi_2(v_\varepsilon)(F_2)'_2(u_\varepsilon, v_\varepsilon)]w - \Phi_2^2(v_\varepsilon)(F_2)'_1(u_\varepsilon, v_\varepsilon)u_{\varepsilon x} = 0 \\ ((x,t) \in [0,a] \times (0,T)), \\ w|_{x=0} = 0, \ w|_{x=a} = G_1(u_\varepsilon, v_\varepsilon) \quad (t \in (0,T)), \\ z|_{x=0} = 0, \ z|_{x=a} = G_2(u_\varepsilon, v_\varepsilon) \quad (t \in (0,T)), \\ w(x,0) = (u_0^{m_1})_x(x), \ z(x,0) = (v_0^{m_2})_x(x) \quad (x \in [0,a]). \end{cases}$$

Similarly to the proof of Proposition 3.1 in [19], the maximum principle yields that there exists a small constant $\tau_1 > 0$ such that

$$\|w\|_\infty \leq \max\{\|(u_0^{m_1})_x\|_\infty, \|(v_0^{m_2})_x\|_\infty, L\},$$
$$\|z\|_\infty \leq \max\{\|(u_0^{m_1})_x\|_\infty, \|(v_0^{m_2})_x\|_\infty, L\}.$$

Therefore we have

$$\|(u_\varepsilon^{m_1})_x\|_\infty, \|(v_\varepsilon^{m_2})_x\|_\infty \leq K + L + 1 \quad \text{on } \overline{Q}_{\tau_1}.$$

Thus $(u_\varepsilon, v_\varepsilon)$ is a solution of (3.1) in $\overline{Q}_{\tau_1}$. Setting $C = K + L + 1$, we draw our conclusion. □

Next, we prove the following energy estimates.

**Proposition 3.2.** *There exists a $\tau_2 \in (0,T)$ such that*

$$u_\varepsilon(x,t), v_\varepsilon(x,t) \leq M + 1 \ \textit{on } \overline{Q}_{\tau_2}.$$

*Proof.* We may assume that $T \in [0,1)$. Let $h \geq M$. Multiplying (3.1) by $(u_\varepsilon - h)_+$ and noting that $0 < l \leq G_1, F_1 \leq L$, it is easy to see that

$$\int_0^a (u_\varepsilon(\cdot,t) - h)_+^2 \mathrm{d}x + \iint_{Q_T} |u_{\varepsilon x}|^{p_1 - 1}[(u_\varepsilon - h)_+^2]_x \mathrm{d}x\mathrm{d}t$$
$$\leq c \int_0^T (u_\varepsilon - h)_+^2|_{x=a} \mathrm{d}t + c \iint_{Q_T} (u_\varepsilon - h)_+^2 \mathrm{d}x\mathrm{d}t$$

for some positive constant $c$ independent of $\varepsilon$. Then similarly to the proof of Proposition 3.1 in [19], there exists a $\tau_2 > 0$, independent of $\varepsilon$, such that

$$u_\varepsilon \leq M + 1 \quad \text{on } \overline{Q}_{\tau_2}.$$

Similarly, we have

$$v_\varepsilon \le M+1 \quad \text{on } \overline{Q}_{\tau_2}.$$

This completes the proof of Proposition 3.2. □

**Proposition 3.3.** *There exists a constant $C>0$, independent of $\varepsilon$, such that*

$$\iint_{Q_T} (u_{\varepsilon t}^2 + v_{\varepsilon t}^2) dx dt \le C < +\infty,$$

*where $T=\min\{\tau_1,\tau_2\}$.*

*Proof.* Differentiate the first equation of (3.1) with respect to $t$ and multiply both sides of the equation by $u_{\varepsilon t}$, integrate over $Q_T$ to get

$$\begin{aligned} &\frac{1}{2}\int_0^a u_{\varepsilon t}^2(x,t)\mathrm{d}x + \iint_{Q_T} ((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}-1}[p_1(u_\varepsilon^{m_1})_x^2+\varepsilon] m u_\varepsilon^{m_1-1} u_{\varepsilon xt}^2 \mathrm{d}x\mathrm{d}t \\ =& \frac{1}{2}\int_0^a u_{\varepsilon t}^2(x,0)\mathrm{d}x \\ &- \iint_{Q_T} ((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}-1}[p_1(u_\varepsilon^{m_1})_x^2+\varepsilon] m u_\varepsilon^{m_1-1} u_{\varepsilon tx} u_{\varepsilon x} u_{\varepsilon t}\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T [((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}}(u_\varepsilon^{m_1})_x]_t u_{\varepsilon t}(a,t)\mathrm{d}t \\ &+ \iint_{Q_T} [(F_1)'_1(u_\varepsilon,v_\varepsilon)u_{\varepsilon t}^2 + (F_1)'_2(u_\varepsilon,v_\varepsilon) v_{\varepsilon t} u_{\varepsilon t}]\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Using Hölder's inequality, we have

(3.4)

$$\begin{aligned} &\int_0^a u_{\varepsilon t}^2(x,t)\mathrm{d}x + \iint_{Q_T} ((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}-1}[p_1(u_\varepsilon^{m_1})_x^2+\varepsilon] m u_\varepsilon^{m_1-1} u_{\varepsilon xt}^2 \mathrm{d}x\mathrm{d}t \\ =& m_1(m_1-1)^2 \iint_{Q_T} ((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}-1}[p_1(u_\varepsilon^{m_1})_x^2+\varepsilon] u_\varepsilon^{m_1-3}[u_{\varepsilon x}u_{\varepsilon t}]^2\mathrm{d}x\mathrm{d}t \\ &+ \int_0^a u_{\varepsilon t}^2(x,0)\mathrm{d}x + 2\int_0^T [((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}}(u_\varepsilon^{m_1})_x]_t u_{\varepsilon t}(a,t)\mathrm{d}t \\ &+ 2\iint_{Q_T} [(F_1)'_1(u_\varepsilon,v_\varepsilon)u_{\varepsilon t}^2 + (F_1)'_2(u_\varepsilon,v_\varepsilon) v_{\varepsilon t} u_{\varepsilon t}]\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Using $u_\varepsilon(x,t), v_\varepsilon(x,t) \ge \delta$ and the boundary conditions in (3.1), we know that there exists $x_0 \in [0,a)$ such that

$$u_{\varepsilon x}(x,t) \ge \frac{\delta^{q_{11}q_{12}}}{2}, \quad v_{\varepsilon x}(x,t) \ge \frac{\delta^{q_{21}q_{22}}}{2}, \quad (x,t)\in[x_0,a]\times[0,T].$$

Hence, we have

(3.5)

$$\iint_{Q_T} ((u_\varepsilon^{m_1})_x^2+\varepsilon)^{\frac{p_1-1}{2}-1}[p_1(u_\varepsilon^{m_1})_x^2+\varepsilon] m u_\varepsilon^{m_1-1} u_{\varepsilon xt}^2 \mathrm{d}x\mathrm{d}t \ge c\int_0^T \int_{x_0}^a u_{\varepsilon xt}^2\mathrm{d}x\mathrm{d}t$$

for some positive constant $c$ independent of $\varepsilon$.

Then, (3.4), together with (3.5), gives

$$\begin{aligned}\int_0^a u_{\varepsilon t}^2(x,t)\mathrm{d}x + c\int_0^T\int_{x_0}^a u_{\varepsilon xt}^2\mathrm{d}x\mathrm{d}t \\ = c\iint_{Q_T}(u_{\varepsilon t})^2\mathrm{d}x\mathrm{d}t + \int_0^a u_{\varepsilon t}^2(x,0)\mathrm{d}x + c\int_0^T[u_{\varepsilon t}^2+v_{\varepsilon t}^2](a,t)\mathrm{d}t \\ + 2\iint_{Q_T}[(F_1)'_1(u_\varepsilon,v_\varepsilon)u_{\varepsilon t}^2 + (F_1)'_2(u_\varepsilon,v_\varepsilon)v_{\varepsilon t}u_{\varepsilon t}]\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.6}$$

with the help of the boundary conditions and Young's inequality.

Similarly, we have

$$\begin{aligned}\int_0^a v_{\varepsilon t}^2(x,t)\mathrm{d}x + c\int_0^T\int_{x_0}^a v_{\varepsilon xt}^2\mathrm{d}x\mathrm{d}t \\ = c\iint_{Q_T}(v_{\varepsilon t})^2\mathrm{d}x\mathrm{d}t + \int_0^a v_{\varepsilon t}^2(x,0)\mathrm{d}x + c\int_0^T[u_{\varepsilon t}^2+v_{\varepsilon t}^2](a,t)\mathrm{d}t \\ + 2\iint_{Q_T}[(F_2)'_1(u_\varepsilon,v_\varepsilon)u_{\varepsilon t}v_{\varepsilon t} + (F_2)'_2(u_\varepsilon,v_\varepsilon)v_{\varepsilon t}^2]\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.7}$$

Using Sobolev's inequalities, we have

$$\begin{aligned} c\int_0^T[u_{\varepsilon t}^2+v_{\varepsilon t}^2](a,t)\mathrm{d}t \\ \le \tau_1\int_0^T\int_{x_0}^a(u_{\varepsilon xt}^2+v_{\varepsilon xt}^2)\mathrm{d}x\mathrm{d}t + c(\tau_1)\int_0^T\int_{x_0}^a(u_{\varepsilon t}^2+v_{\varepsilon t}^2)\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.8}$$

for any positive constant $\tau_1$ independent of $\varepsilon$. Noticing (3.3) and using Young's inequality again, we obtain

$$\begin{aligned}\iint_{Q_T}[(F_1)'_1u_{\varepsilon t}^2 + (F_1)'_2v_{\varepsilon t}u_{\varepsilon t}]\mathrm{d}x\mathrm{d}t \le c\iint_{Q_T}(u_{\varepsilon t}^2+v_{\varepsilon t}^2)\mathrm{d}x\mathrm{d}t, \\ \iint_{Q_T}[(F_2)'_2v_{\varepsilon t}^2 + (F_2)'_1u_{\varepsilon t}v_{\varepsilon t}]\mathrm{d}x\mathrm{d}t \le c\iint_{Q_T}(u_{\varepsilon t}^2+v_{\varepsilon t}^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.9}$$

Combing (3.6)-(3.9), we have

$$\int_0^a(u_{\varepsilon t}^2+v_{\varepsilon t}^2)(x,t)\mathrm{d}x \le \int_0^a(u_{\varepsilon t}^2+v_{\varepsilon t}^2)(x,0)\mathrm{d}x + C\int\int_{Q_T}(u_{\varepsilon t}^2+v_{\varepsilon t}^2)\mathrm{d}x\mathrm{d}t.$$

By the Gronwall's Lemma, we obtain the desired results.

Therefore, by compactness arguments and the standard monotonicity argument, it follows that (up to extraction of a subsequence):

$$\begin{aligned}&(u_\varepsilon,v_\varepsilon)\to(u,v) \quad \text{a.e. for } (x,t)\in Q_T,\\ &(u_{\varepsilon x},v_{\varepsilon x})\to(u_x,v_x) \quad \text{weakly star in } L^\infty(Q_T),\\ &(u_{\varepsilon t},v_{\varepsilon t})\to(u_t,v_t) \quad \text{weakly in } L^2(Q_T),\end{aligned}$$

$$a_{1\varepsilon}((u_\varepsilon^{m_1})_x) \to w_1 \quad \text{weakly star in } L^\infty(Q_T),$$
$$a_{2\varepsilon}((v_\varepsilon^{m_2})_x) \to w_2 \quad \text{weakly star in } L^\infty(Q_T).$$

We show that $w_1 = |(u^{m_1})_x|^{p_1-1}(u^{m_1})_x$, $w_2 = |(v^{m_2})_x|^{p_2-1}(v^{m_2})_x$.

From Proposition 3.2, we have

$$\lim_{n\to\infty} \int\int_{Q_T} \psi|(u_\varepsilon^{m_1})_x|^{p_1-1}(u_\varepsilon^{m_1})_x(u_\varepsilon^{m_1} - u^{m_1})_x \mathrm{d}x\mathrm{d}t = 0,$$

where $\psi \in C_0^{1,1}(Q_T), \psi \geq 0$.

Then similarly to the proof of Theorem 1 in [17], we have

$$\int\int_{Q_T} \psi(w_1 - |(u^{m_1})_x|^{p_1-1}(u^{m_1})_x)\mathrm{d}x\mathrm{d}t = 0.$$

Thus $w_1 = |(u^{m_1})_x|^{p_1-1}(u^{m_1})_x$. Similarly, $w_2 = |(v^{m_2})_x|^{p_2-1}(v^{m_2})_x$.

The proof of Theorem 3.1 is completed by a standard limiting process.

The uniqueness of the solution is obvious. In fact, assume that $(u_1, v_1)$, $(u_2, v_2)$ are two nonnegative solutions of (1.1), using Proposition 2.1 repeatedly, we can get $u_1 = u_2$, $v_1 = v_2$ a.e. in $Q_T$. □

## 4. Proof of the theorems

In this section we will discuss the global existence of solutions to problem (1.1).

We will divide the proof of Theorem 1.1 and Theorem 1.2 into six lemmas. Throughout this section we denote

$$\tau_i = \frac{p_i}{p_i + 1}, \quad i = 1, 2,$$

and choose $\overline{\lambda}, \underline{\lambda}$ satisfying $\overline{\lambda} > 1 > \underline{\lambda} > 0$.

**Lemma 4.1.** *For $l_{11} > 1$ or $l_{22} > 1$ or $q_{11} > \frac{1}{p_1} + 1 - m_1$ or $q_{22} > \frac{1}{p_2} + 1 - m_2$, the solution $(u, v)$ of (1.1) blows up in finite time.*

*Proof.* Without loss of generality, assume $l_{11} > 1$. Consider the single equation

$$\begin{aligned}
&z_t = (|(z^{m_1})_x|^{p_1-1}(z^{m_1})_x)_x + a\delta^{l_{12}}z^{l_{11}}, && (x,t) \in [0,a] \times (0,T),\\
&z_x|_{x=0} = 0, \quad z_x|_{x=a} = \underline{\lambda}\delta^{q_{12}}z^{q_{11}}|_{x=a}, && t > 0,\\
&z(x,0) = u_0(x), && x \in [0,a].
\end{aligned}$$

We know from [18] that $z$ blows up in finite time. Since $v \geq \delta$ by the comparison principle, thus $(z, \delta)$ is a subsolution of (1.1) and $(u, v)$ blows up in finite time if $l_{11} > 1$. □

**Lemma 4.2.** *For $q_{11} \leq \frac{1}{p_1} + 1 - m_1$, $q_{22} \leq \frac{1}{p_2} + 1 - m_2$ and $q_{12}q_{21} > (\frac{1}{p_1} + 1 - m_1 - q_{11})(\frac{1}{p_2} + 1 - m_2 - q_{22})$, the solution $(u, v)$ of (1.1) blows up in finite time.*

*Proof.* Notice that the solutions of (1.3) are just subsolutions of (1.1). The blow-up result for the solutions of (1.3) (see [19]) yields the blow-up of solutions to (1.1). □

**Lemma 4.3.** *For* $l_{11} \leq 1$, $l_{22} \leq 1$ *and* $l_{12}l_{21} > (1 - l_{11})(1 - l_{22})$, *the solution* $(u, v)$ *of* (1.1) *blows up in finite time.*

*Proof.* We choose $k_i > 0, i = 1, 2$ such that

$$-k_1 l_{11} - k_2 l_{12} + k_1 + 1 \leq 0,$$
$$-k_1 l_{21} - k_2 l_{22} + k_2 + 1 \leq 0.$$

Denote $w_i = (c - bt)^{-k_i}$ and

$$c = \max\{\delta^{-\frac{1}{k_1}}, \delta^{-\frac{1}{k_2}}\}, b = \min\{\frac{ac^{-k_1 l_{11} - k_2 l_{12} + k_1 + 1}}{k_1}, \frac{ac^{-k_1 l_{21} - k_2 l_{22} + k_2 + 1}}{k_2}\}.$$

A routine calculation yields:

$$w_{1t} = k_1 b(c - bt)^{-k_1 - 1} \leq a(c - bt)^{-k_1 l_{11} - k_2 l_{12}} = w_1^{l_{11}} \int_0^a w_2^{l_{12}}(\xi, t)\mathrm{d}\xi.$$

$$w_1(x, 0) \leq c^{-k_1} \leq \min u_0(x) \leq u_0(x).$$

Similarly, we have

$$w_{2t} \leq w_2^{l_{22}} \int_0^a w_1^{l_{21}}(\xi, t)\mathrm{d}\xi, \quad w_2(x, 0) \leq v_0(x).$$

Hence, by the comparison principle we have that $(u, v) \geq (w_1, w_2)$. Therefore, $(u, v)$ blows up in finite time. The proof is completed. □

**Lemma 4.4.** *For* $l_{11} \leq 1$, $q_{22} \leq \frac{1}{p_2} + 1 - m_2$ *and* $l_{12}q_{21} > [1 - l_{11}][\frac{1}{p_2} + 1 - m_2 - q_{22}]$, *the solution* $(u, v)$ *of* (1.1) *blows up in finite time.*

*Proof.* It is easy to prove that by Fact 2.1 there exist positive constants $k_1, k_2 > 0, \beta_1 > m_1, \beta_2 > m_2$ satisfying

$$k_1 \frac{\beta_1(1 - l_{11})}{m_1} + 1 - k_2 \frac{\beta_2 l_{12}}{m_2} = 0,$$
$$k_2\beta_2 p_2 - k_2 p_2 - k_2 \frac{\beta_2}{m_2} - 1 \geq 0,$$
$$k_1 \frac{\beta_1}{m_1} - k_1 - k_1 \frac{\beta_1 q_{11}}{m_1} - k_2 \frac{\beta_2}{m_2} q_{12} \leq 0,$$
$$k_2 \frac{\beta_2}{m_2} - k_2 - k_2 \frac{\beta_2 q_{22}}{m_2} - k_1 \frac{\beta_1 q_{21}}{m_1} \leq 0.$$

Set

$$w_1 = [d(1 + x^{\frac{1}{\tau_1}}) + (c - bt)^{-k_1}]^{\beta_1/m_1} = [S_1]^{\beta_1/m_1},$$
$$w_2 = [d(1 + x^{\frac{1}{\tau_2}}) + (c - bt)^{-k_2}]^{\beta_2/m_2} = [S_2]^{\beta_2/m_2},$$

where $b, c, d > 0$ satisfy

$$\begin{aligned}
&c \geq \max\{2^{\frac{1}{k_1}}\delta^{-\frac{m_1}{k_1\beta_1}}, 2^{\frac{1}{k_2}}\delta^{-\frac{m_2}{k_2\beta_2}}\},\\
&d \leq \min\{\frac{1}{1+a^{\frac{1}{\tau_1}}}c^{-k_1}, \frac{1}{1+a^{\frac{1}{\tau_2}}}c^{-k_2},\\
&\qquad \underline{\lambda}(\beta_1/m_1\frac{1}{\tau_1}a^{\frac{1}{p_1}}2^{\beta_1/m_1-1})^{-1}c^{k_1\beta_1/m_1-k_1-k_1\beta_1q_{11}/m_1-k_2\beta_2q_{12}/m_2},\\
&\qquad \underline{\lambda}(\frac{\beta_2}{m_2}\frac{1}{\tau_2}a^{\frac{1}{p_2}}2^{\frac{\beta_2}{m_2}-1})^{-1}c^{k_2\beta_2/m_2-k_2-k_2\beta_2q_{22}/m_2-k_1\beta_1q_{21}/m_1}\},\\
&b \leq \min\{\underline{\lambda}a(k_1\beta_1/m_1)^{-1}2^{1-\beta_1},\\
&\quad (\frac{\beta_2}{m_2}d\frac{1}{\tau_2})^{p_2}(k_2\frac{\beta_2}{m_2}2^{\frac{\beta_2}{m_2}-1})^{-1}c^{-(k_2\beta_2p_2-k_2p_2-k_2\frac{\beta_2}{m_2}-1)}\}.
\end{aligned}$$

Computing directly, we obtain

$$\begin{aligned}
w_{1t} &= \frac{\beta_1}{m_1}[S_1]^{\frac{\beta_1}{m_1}-1}k_1b(c-bt)^{-k_1-1},\\
(w_1^{m_1})_x &= \beta_1[S_1]^{\beta_1-1}d\frac{1}{\tau_1}x^{\frac{1}{m_1p_1}},
\end{aligned}$$

$$\begin{aligned}
\Big(\big((w_1^{m_1})_x\big)^{p_1}\Big)_x + w_1^{l_{11}}\int_0^a w_2^{l_{12}}(\xi,t)\mathrm{d}\xi &> [S_1]^{\frac{\beta_1 l_{11}}{m_1}}\int_0^a [S_2]^{\frac{\beta_2 l_{12}}{m_2}}(\xi,t)\mathrm{d}\xi\\
&\geq a(c-bt)^{-k_1(\frac{\beta_1}{m_1}-1)-k_1-1}\\
&\geq \frac{\beta_1}{m_1}k_1b2^{\frac{\beta_1}{m_1}-1}(c-bt)^{-k_1(\frac{\beta_1}{m_1}-1)-k_1-1}\\
&= w_{1t}
\end{aligned}$$

and

$$\begin{aligned}
\Big(\big((w_2^{m_2})_x\big)^{p_2}\Big)_x + \underline{\lambda}\int_0^a w_1^{n_2} &\geq (\beta_2 d\frac{1}{\tau_2})^{p_2}\{[S_2]^{p_2(\beta_2-1)}x\}_x\\
&\geq (\beta_2 d\frac{1}{\tau_2})^{p_2}[S_2]^{p_2(\beta_2-1)}\\
&\geq \frac{\beta_2}{m_2}k_2b2^{\frac{\beta_2}{m_2}-1}(c-bt)^{-k_2(\frac{\beta_2}{m_2}-1)-k_2-1}\\
&\geq w_{2t}.
\end{aligned}$$

Then we have

$$\begin{aligned}
w_{1t} &\leq \Big(\big((w_1^{m_1})_x\big)^{p_1}\Big)_x + w_1^{l_{11}}\int_0^a w_2^{l_{12}}(\xi,t)\mathrm{d}\xi,\\
w_{2t} &\leq \Big(\big((w_2^{m_2})_x\big)^{p_2}\Big)_x + w_2^{l_{22}}\int_0^a w_1^{l_{21}}(\xi,t)\mathrm{d}\xi.
\end{aligned} \tag{4.1}$$

Noting that on the boundary

$$w_{1x}(a,t) = \frac{\beta_1}{m_1} d \frac{1}{\tau_1} a^{\frac{1}{p_1}} [S_1]^{\beta_1/m_1-1} \leq \frac{\beta_1}{m_1} d \frac{1}{\tau_1} 2^{\beta_1/m_1-1} a^{\frac{1}{p_1}} (c-bt)^{-k_1(\beta_1/m_1-1)}$$
$$\leq \underline{\lambda} w_1^{q_{11}} w_2^{q_{12}}.$$

Similarly, we have

$$w_{2x}(a,t) \leq \underline{\lambda} w_1^{q_{21}} w_2^{q_{22}}. \tag{4.2}$$

Under the assumptions of $a, b, k_i$ and $\beta_i$, we have that for $x \in [0,1]$,

$$w_1(x,0) \leq (d(1+a^{\frac{1}{\tau_1}}) + c^{-k_1})^{\beta_1/m_1} \leq \delta \leq u_0(x), \tag{4.3}$$

$$w_2(x,0) \leq (d(1+a^{\frac{1}{\tau_2}}) + c^{-k_2})^{\beta_2/m_2} \leq \delta \leq v_0(x). \tag{4.4}$$

From (4.3)-(4.7) and the comparison principle, it follows that $(u,v) \geq (w_1, w_2)$. This shows that $(u,v)$ blows up in finite time. □

**Lemma 4.5.** *For $l_{22} \leq 1$, $q_{11} \leq \frac{1}{p_1} + 1 - m_1$ and $l_{21}q_{12} > [\frac{1}{p_1} + 1 - m_1 - q_{11}][1 - l_{22}]$, the solution $(u,v)$ of (1.1) blows up in finite time.*

*Proof.* The proof is similar to that of Lemma 4.4. □

**Lemma 4.6.** *For $l_{11} \leq 1$, $l_{22} \leq 1$, $q_{11} \leq \frac{1}{p_1} + 1 - m_1$, $q_{22} \leq \frac{1}{p_2} + 1 - m_2$, $l_{12}l_{21} \leq (1-l_{11})(1-l_{22})$, $l_{12}q_{21} \leq [1-l_{11}][\frac{1}{p_2} + 1 - m_2 - q_{22}]$, $l_{21}q_{12} \leq [\frac{1}{p_1} + 1 - m_1 - q_{11}][1 - l_{22}]$ and $q_{12}q_{21} \leq (\frac{1}{p_1} + 1 - m_1 - q_{11})(\frac{1}{p_2} + 1 - m_2 - q_{22})$, the solution $(u,v)$ of (1.1) exists globally.*

*Proof.* First, it is easy to prove that by Fact 2.1 and $(B)$ there exist $l_1, l_2 > 1$ such that

$$\frac{p_1(\frac{1}{p_1} + 1 - m_1 - q_{11})}{m_1p_1 - 1} l_1 - \frac{q_{12}p_2}{m_2p_2 - 1} l_2 \geq 0,$$
$$\frac{p_2(\frac{1}{p_2} + 1 - m_2 - q_{22})}{m_2p_2 - 1} l_2 - \frac{q_{21}p_1}{m_1p_1 - 1} l_1 \geq 0,$$
$$\frac{p_1(1-l_{11})}{m_1p_1 - 1} l_1 - \frac{p_2 l_{12}}{m_2p_2 - 1} l_2 \geq 0,$$
$$\frac{p_2(1-l_{22})}{m_2p_2 - 1} l_2 - \frac{n_2p_1}{m_1p_1 - 1} l_1 \geq 0. \tag{4.5}$$

Denote $w_i = [d_i(1 + x^{\frac{1}{\tau_i}}) + e^{ll_i(t+1)}]^{\frac{p_i}{m_ip_i-1}} = [S_i]^{\frac{p_i}{m_ip_i-1}}, i = 1,2$, where $l_i > 0$ satisfy (4.5) and

$$d_1 = \frac{m_1p_1 - 1}{p_1} \tau_1 \overline{\lambda} a^{-\frac{1}{p_1}} 2^{\frac{p_2q_{12}}{m_2p_2-1}}, \quad d_2 = \frac{m_2p_2 - 1}{p_2} \tau_2 \overline{\lambda} a^{-\frac{1}{p_2}} 2^{\frac{p_1q_{21}}{m_1p_1-1}},$$
$$l \geq \max\{\ln(d_1(1+a^{\frac{1}{\tau_1}}))/l_1, \ln(d_2(1+a^{\frac{1}{\tau_2}}))/l_2,$$
$$(\frac{p_1}{m_1p_1-1})^{p_1-1} \frac{4m_1^{p_1}d_1^{p_1}}{l_1\tau_1^{p_1}}(1 + \frac{d_1p_1a^{\frac{1}{\tau_1}}}{(m_1p_1-1)\tau_1}),$$

$$(\frac{p_2}{m_2p_2-1})^{p_2-1}\frac{4m_2^{p_2}d_2^{p_2}}{l_2\tau_2^{p_2}}(1+\frac{d_2p_2a^{\frac{1}{\tau_2}}}{(m_2p_2-1)\tau_2}),$$
$$a\frac{m_1p_1-1}{p_1l_1}2^{2-\frac{p_1(1-l_{11})l_1}{m_1p_1-1}+\frac{p_2l_{12}l_2}{m_2p_2-1}}, a\frac{m_2p_2-1}{p_2l_2}2^{2-\frac{p_2(1-l_{22})l_2}{m_2p_2-1}+\frac{p_1l_{21}l_1}{m_1p_1-1}},$$
$$\frac{m_1p_1-1}{l_1p_1}\ln(\max u_0(x)), \frac{m_2p_2-1}{l_2p_2}\ln(\max v_0(x))\}.$$

By the choices of $d_1, d_2$ and $l$ we have

$$S_1 \le 2e^{ll_1(t+1)}, \quad S_2 \le 2e^{ll_2(t+1)}. \tag{4.6}$$

By direct computations and (4.6) we have, for $(x,t) \in (0,a)\times[0,+\infty)$,

$$w_{1t} = \frac{p_1ll_1}{m_1p_1-1}S_1^{\frac{p_1}{m_1p_1-1}-1}e^{ll_1(t+1)},$$
$$(w_1^{m_1})_x = \frac{m_1p_1d_1}{(m_1p_1-1)\tau_1}S_1^{\frac{m_1p_1}{m_1p_1-1}-1}x^{\frac{1}{\tau_1}-1},$$
$$\begin{aligned}\Big(((w_1^{m_1})_x)^{p_1}\Big)_x &= (\frac{m_1p_1d_1}{(m_1p_1-1)\tau_1})^{p_1}([S_1]^{\frac{p_1}{m_1p_1-1}}x)_x\\
&= (\frac{m_1p_1d_1}{(m_1p_1-1)\tau_1})^{p_1}\big([S_1]^{\frac{p_1}{m_1p_1-1}}+\frac{p_1d_1}{(m_1p_1-1)\tau_1}S_1^{\frac{p_1}{m_1p_1-1}-1}x^{\frac{1}{\tau_1}}\big)\\
&= (\frac{m_1p_1d_1}{(m_1p_1-1)\tau_1})^{p_1}S_1^{\frac{p_1}{m_1p_1-1}-1}\big(S_1+\frac{p_1d_1}{(m_1p_1-1)\tau_1}x^{\frac{1}{\tau_1}}\big)\\
&\le (\frac{m_1p_1d_1}{(m_1p_1-1)\tau_1})^{p_1}S_1^{\frac{p_1}{m_1p_1-1}-1}2e^{ll_1(t+1)}\big(1+\frac{p_1d_1}{(m_1p_1-1)\tau_1}\big)\\
&\le \frac{1}{2}\frac{p_1ll_1}{m_1p_1-1}S_1^{\frac{p_1}{m_1p_1-1}-1}e^{ll_1(t+1)}\\
&= \frac{1}{2}w_{1t},\end{aligned}$$

$$\begin{aligned}w_1^{l_{11}}\int_0^a w_2^{l_{12}}(\xi,t)\mathrm{d}\xi &= [S_1]^{\frac{p_1l_{11}}{m_1p_1-1}}\int_0^a[S_2]^{\frac{p_2l_{12}}{m_2p_2-1}}\mathrm{d}\xi\\
&\le [S_1]^{\frac{p_1}{m_1p_1-1}-1}2ae^{ll_1(t+1)}[2e^{l(t+1)}]^{-\frac{p_1(1-l_{11})l_1}{m_1p_1-1}+\frac{p_2l_{12}l_2}{m_2p_2-1}}\\
&\le \frac{1}{2}\frac{p_1ll_1}{m_1p_1-1}S_1^{\frac{p_1}{m_1p_1-1}-1}e^{ll_1(t+1)}\\
&= \frac{1}{2}w_{1t},\end{aligned}$$

i.e.,

$$w_{1t} \ge \Big(((w_1^{m_1})_x)^{p_1}\Big)_x + w_1^{l_{11}}\int_0^a w_2^{l_{12}}(\xi,t)\mathrm{d}\xi. \tag{4.7}$$

Similarly, we have

$$w_{2t} \ge \Big(((w_2^{m_2})_x)^{p_2}\Big)_x + w_2^{l_{22}}\int_0^a w_1^{l_{21}}(\xi,t)\mathrm{d}\xi. \tag{4.8}$$

And

$$
\begin{aligned}
& w_{1x}|_{x=0} = 0, \quad w_{2x}|_{x=0} = 0, \\
& w_{1x} = \frac{p_1 d_1}{(m_1 p_1 - 1)\tau_1} S_1^{\frac{p_1}{m_1 p_1 - 1} - 1} a^{\frac{1}{\tau_1} - 1} \\
& \quad = \frac{p_1 d_1}{(m_1 p_1 - 1)\tau_1} a^{\frac{1}{\tau_1} - 1} S_1^{\frac{p_1 q_{11}}{m_1 p_1 - 1}} S_2^{\frac{p_2 q_{12}}{m_2 p_2 - 1}} S_1^{\frac{p_1 - p_1 q_{11}}{m_1 p_1 - 1} - 1} S_2^{-\frac{p_2 q_{12}}{m_2 p_2 - 1}} \\
& \quad \geq w_1^{q_{11}} w_2^{q_{12}} \frac{p_1 d_1 a^{\frac{1}{\tau_1} - 1}}{(m_1 p_1 - 1)\tau_1} 2^{-\frac{p_2 q_{12}}{m_2 p_2 - 1}} \\
& \qquad \exp\{[\frac{p_1(1 - q_{11} - m_1) + 1}{m_1 p_1 - 1} l_1 - \frac{p_2 q_{12}}{m_2 p_2 - 1} l_2] l(t+1)\} \\
& \quad \geq w_1^{q_{11}} w_2^{q_{12}} \frac{p_1 d_1}{(m_1 p_1 - 1)\tau_1} 2^{-\frac{p_2 q_{12}}{m_2 p_2 - 1}} a^{\frac{1}{\tau_1} - 1} \\
& \quad = \overline{\lambda} w_1^{q_{11}} w_2^{q_{12}}, \quad x = a, \ t > 0. \qquad (4.9)
\end{aligned}
$$

Similarly, we have

$$w_{2x} \geq \overline{\lambda} w_1^{q_{21}} w_2^{q_{22}}, \quad x = a, \ t > 0. \tag{4.10}$$

For $x \in [0, a]$, we have

(4.11)

$$w_1(x, 0) = [d_1(1 + x^{\frac{1}{\tau_1}}) + e^{ll_1}]^{\frac{p_1}{m_1 p_1 - 1}} \geq e^{ll_1 \frac{p_1}{m_1 p_1 - 1}} \geq \max u_0(x) \geq u_0(x),$$

(4.12)

$$w_2(x, 0) = [d_2(1 + x^{\frac{1}{\tau_2}}) + e^{ll_2}]^{\frac{p_2}{m_2 p_2 - 1}} \geq e^{ll_2 \frac{p_2}{m_2 p_2 - 1}} \geq \max v_0(x) \geq v_0(x).$$

From (4.7)-(4.11), we see that $(w_1, w_2)$ is a supersolution of (1.1) with $\overline{\lambda}$, which means that solutions of (1.1) exists globally. □

Combining Lemmas 4.1-4.6, we see that Theorem 1.1 and Theorem 1.2 are true.

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## References

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