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NONEMPTY INTERSECTION THEOREMS AND SYSTEM OF GENERALIZED VECTOR EQUILIBRIUM PROBLEMS IN *FC*-SPACES

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Abstract. By using some existence theorems of maximal elements for a family of set-valued mappings involving a better admissible set-valued mapping under noncompact setting of FC-spaces, we present some nonempty intersection theorems for a family $\{G_i\}_{i\in I}$ in product FC-spaces. Then, as applications, some new existence theorems of equilibrium for a system of generalized vector equilibrium problems are proved in product FC-spaces. Our results improve and generalize some recent results.

1. Introduction and preliminaries

The vector variational inequality problem (in short, $VVIP$) was first introduced by Giannessi [17] in finite dimensional Euclidean spaces. Since then, the $VVIP$ has been generalized and applied by many authors in various directions. Inspired and motivated by applications of the $VVIP$ with set-valued mappings, various generalized vector variational inequality problem (in short, $GVVIP$) and generalized vector equilibrium problem (in short, $GVEP$) have become important developed directions of the classical $VVIP$ (see, for example, [1, 3, 4, 6, 15, 22] and the book edited by F. Giannessi [18], and the references therein).

Recently, Ding [12] introduced and studied a system of generalized vector equilibrium problems in the product space of G-convex spaces. Let X be a topological space and I be any index set. Let $\{D_i\}_{i\in I}$, $\{E_i\}_{i\in I}$, $\{Y_i\}_{i\in I}$ and $\{Z_i\}_{i\in I}$ be four families of topological spaces, and let $Y=\prod_{i\in I}Y_i$. For each $i\in I$, let $T_i:X\to 2^{D_i}, S_i:X\to 2^{E_i}, C_i:X\to 2^{Z_i}$ and $\eta_i:D_i\times E_i\times Y_i\to 2^{Z_i}$ be set-valued mappings. A system of generalized vector equilibrium problems

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(in short, $SGVEP$) is to find $\hat{x} \in X$ such that for each $i \in I$,

(1) $\forall y_i \in Y_i, \exists \hat{d}_i \in T_i(\hat{x}), \hat{e}_i \in S_i(\hat{x})$ such that $\eta_i(\hat{d}_i, \hat{e}_i, y_i) \not\subseteq C_i(\hat{x})$.

Motivated and inspired by the above research works, we shall present some nonempty intersection theorems for a family $\{G_i\}_{i\in I}$ in a product FC-spaces, which are generalizations of G-convex spaces, by applying an existence theorems of maximal elements obtained by He and Zhang [20]. As applications, several new existence theorems of equilibrium points for the $SGVEP$ (1) are established in product FC-space. Our results improve and generalize the corresponding results in [2, 12, 14, 19, 23].

Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X\rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. Let Δ_n denote the standard n-dimensional simplex with the vertices $\{e_0, \ldots, e_n\}$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we shall denote by $\triangle_J$ the convex hull of the vertices $\{e_j : j \in J\}$. For topological spaces X and Y, a subset A of X is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of X, $A \bigcap K$ is open (resp., closed) in K. The compact closure of A and the compact interior of A (see [5]) are defined respectively by

$$cclA = \bigcap\{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\},$$

$$cintA = \bigcup\{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}.$$

A set-valued mapping $T : X \to 2^Y$ is said to be transfer compactly closed valued on X (see [5]) if for each $x \in X$ and $y \notin T(x)$, there exists $x' \in X$ such that $y \notin cclT(x')$. T is said to be transfer compactly open valued on X if for each $x \in X$ and $y \in T(x)$, there exists $x' \in X$ such that $y \in cintT(x')$.

First, we recall the following concepts (see [13, 20]).

Definition 1.1. (X, φ_N) is said to be an FC-space if X is a topological space and for each $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$, where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset D of X is said to be an FC-subspace of X if for each $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N \bigcap D$, $\varphi_N(\Delta_k) \subset D$, where $\Delta_k = co(\{e_{i_j} : j = 0, \ldots, k\})$.

Definition 1.2. An FC-space (X, φ_N) is said to be a CFC-space if for each $N \in \langle Y\rangle$, there exists a compact FC-subspace L_N of Y containing N.

Now, we give some lemmas which will be useful in the proof of our main results (see [13, 20]).

Lemma 1.1. *Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space, $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then (Y, φ_N) is also an FC-space.*

The following notion was introduced by He and Zhang [20]. Let X be a topological space and I be any index set. For each $i \in I$, let $(Y_i, \varphi_{N_i})_{i\in I}$ be an FC-space and $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in

Lemma 1.1. Let $F \in \mathcal{B}(Y, X)$ and $A_i : X \to 2^{Y_i}, i \in I$ be set-valued mappings. For each $i \in I$,

(1) $A_i : X \to 2^{Y_i}$ is said to be a generalized $G_{\mathcal{B}}$-mapping if
  (a) for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, \dots, y_{i_k}\} \subset N$,

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} cint A_i^{-1}(\pi_i(y_{i_j})) \right) = \emptyset,$$

  where π_i is the projection of Y onto Y_i and $\Delta_k = co(\{e_{i_j} : j = 0, \dots, k\})$;
  (b) $A_i^{-1}(y_i) = \{x \in X : y_i \in A_i(x)\}$ is transfer compactly open in Y_i for each $y_i \in Y_i$.

(2) $A_{x,i} : X \to 2^{Y_i}$ is said to be a generalized $G_{\mathcal{B}}$-majorant of A_i at $x \in X$ if $A_{x,i}$ is a generalized $G_{\mathcal{B}}$-mapping and there exists an open neighborhood $N(x)$ of x in X such that $A_i(z) \subset A_{x,i}(z)$ for all $z \in N(x)$.

(3) A_i is said to be a generalized $G_{\mathcal{B}}$-majorized if for each $x \in X$ with $A_i(x) \neq \emptyset$, there exists a generalized $G_{\mathcal{B}}$-majorant $A_{x,i}$ of A_i at x, and for any $N \in \langle \{x \in X : A_i(x) \neq \emptyset\} \rangle$, the mapping $\bigcap_{x \in N} A_{x,i}^{-1}$ is transfer compactly open in Y_i.

(4) $\{A_i\}_{i \in I}$ is said to be a family of generalized $G_{\mathcal{B}}$-mappings (resp., $G_{\mathcal{B}}$-majorant mappings) if for each $i \in I, A_i : X \to 2^{Y_i}$ is a generalized $G_{\mathcal{B}}$-mapping (resp., $G_{\mathcal{B}}$-majorant mapping).

Lemma 1.2. *Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space and $Y = \prod_{i \in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 1.1. Let $F \in \mathcal{B}(Y, X)$ and for each $i \in I$, $A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathcal{B}}$-mapping such that the following condition holds*:

(P) *For each $i \in I$ and $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X \backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cint A_i(x) \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Lemma 1.3. *Let X be a topological space, and let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an CFC-space, and let $Y = \prod_{i \in I} Y_i$. Let $F \in \mathcal{B}(Y, X)$ be a compact mapping such that the following two conditions hold for each $i \in I$*:

(i) *$A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathcal{B}}$-majorized mapping*;
(ii) $\bigcup_{i \in I} \{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i \in I} cint\{x \in X : A_i(x) \neq \emptyset\}$.

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

2. Nonempty intersection theorems

Definition 2.1. Let D, E, Y and Z be nonempty sets and X be a topological space. Let $F : D \times E \times Y \to 2^Z$ and $C : X \to 2^Z$ be two set-valued mappings. $F(d, e, y)$ is said to be transfer compactly upper semicontinuous in x with respect to C if for any nonempty compact subset K of X and any $x \in K$, $\{(d, e, y) \in D \times E \times Y : F(d, e, y) \subseteq C(x)\} \neq \emptyset$ implies that there exist a relatively open neighborhood $N(x)$ of x in K and $(d', e', y') \in D \times E \times Y$ such that $F(d', e', y') \subseteq C(z)$ for all $z \in N(x)$.

We first give the following nonempty intersection theorem which is in fact equivalent to Lemma 1.2.

Theorem 2.1. *Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space and $Y = \prod_{i \in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 1.1. Let $F \in \mathcal{B}(Y, X)$ and $G_i : Y_i \to 2^X$ be such that the following conditions hold:*

(i) *For each $y_i \in Y_i$, $G_i(y_i)$ is transfer compactly closed in Y_i.*

(ii) *For each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and $\{y_{r_0}, \dots, y_{r_k}\} \subset N$,*

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} cint(X \backslash G_i(\pi_i(y_{r_j}))) \right) = \emptyset,$$

where π_i is the projection of Y onto Y_i and $\Delta_k = co(\{e_{r_j} : j = 0, \dots, k\})$.

(iii) *For each $i \in I$ and $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X \backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cint(Y_i \backslash G_i^{-1}(x)) \neq \emptyset$.*

Then we have

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{y_i \in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Proof. Lemma 1.2 $\Rightarrow$ Theorem 2.1. For each $i \in I$, define a set-valued mapping $A_i : X \to 2^{Y_i}$ by

$$A_i(x) = Y_i \backslash G_i^{-1}(x), \quad \forall x \in X.$$

Then for each $y_i \in Y_i$, we have

$$\begin{aligned} A_i^{-1}(y_i) &= \{x \in X : y_i \in A_i(x)\} = \{x \in X : y_i \in Y_i \backslash G_i^{-1}(x)\} \\ &= \{x \in X : x \notin G_i(y_i)\} = X \backslash G_i(y_i). \end{aligned}$$

Hence $A_i^{-1}(y_i)$ is transfer compactly open in X by condition (i). The condition (ii) of Theorem 2.1 implies that for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and $\{y_{r_0}, \dots, y_{r_k}\} \subset N$,

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} cintA_i^{-1}(\pi_i(y_{r_j})) \right) = \emptyset,$$

where π_i is the projection of Y onto Y_i and $\Delta_k = co(\{e_{r_j} : j = 0, \ldots, k\})$. So for each $i \in I$, $A_i : X \to 2^{Y_i}$ is a generalized $G_{\mathcal{B}}$-mapping. The condition (iii) of Theorem 2.1 implies that the condition (P) of Lemma 1.2 holds. Then by Lemma 1.2, there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$. Therefore, we have $A_i(\hat{x}) = Y_i \backslash G_i^{-1}(\hat{x}) = \emptyset$, and then $Y_i = G_i^{-1}(\hat{x})$ for $i \in I$. This implies that

$$\hat{x} \in \bigcap_{i \in I} \bigcap_{y_i \in Y_i} G_i(y_i).$$

Hence we obtain

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{y_i \in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Theorem 2.1 $\Rightarrow$ *Lemma 1.2.* For each $i \in I$, define a set-valued mapping $G_i : Y_i \to 2^X$ by

$$G_i(y_i) = X \backslash A_i^{-1}(y_i), \ \forall y_i \in Y_i.$$

Since for each $i \in I$, $A_i : X \to 2^{Y_i}$ is a generalized $G_{\mathcal{B}}$-mapping, conditions (i) and (ii) of Theorem 2.1 are satisfied. Condition (P) of Lemma 1.2 implies that condition (iii) of Theorem 2.1 holds. Consequently, all conditions of Theorem 2.1 are satisfied, and then we have

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{y_i \in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Taking any $\hat{x} \in K \bigcap (\bigcap_{i \in I} \bigcap_{y_i \in Y_i} G_i(y_i)) \neq \emptyset$, we have $\hat{x} \in K$ and

$$\hat{x} \in G_i(y_i) = X \backslash A_i^{-1}(y_i) \quad \text{for } y_i \in Y_i, i \in I,$$

which implies

$$y_i \notin A_i(\hat{x}) \quad \text{for } y_i \in Y_i, i \in I.$$

So $A_i(\hat{x}) = \emptyset$ for all $i \in I$. This completes the proof. □

Remark 2.1. Since Theorem 2.1 is an equivalent form of Lemma 1.2, Theorem 2.1 generalizes Theorem 2 and Theorem 4 of Park and Kim [23]. Moreover, Theorem 2.1 improves and generalizes Theorem 2.1 of Ding [12] in the following ways: (a) The setting spaces are generalized from "G-convex spaces" to "FC-spaces" without linear structure; (b) The assumption for mapping G_i is generalized from "compactly closed" to "transfer compactly closed"; (c) Conditions (ii) and (iii) of Theorem 2.1 are weaker than conditions (ii) and (iii) of Theorem 2.1 in [12].

Especially, if $X = Y$ in Theorem 2.1, we have the following result, which improves Theorem 3.2 in Ding [14].

Corollary 2.1. *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be an FC-space and $X = \prod_{i \in I} X_i$ such that (X, φ_N) is an FC-space defined as in*

Lemma 1.1. Let K be a nonempty compact subset of X and $G_i : X_i \to 2^X$ be such that the following conditions hold:

(i) *For each $x_i \in X_i$, $G_i(x_i)$ is transfer compactly closed in X_i.*

(ii) *For each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $\{x_{r_0}, \dots, x_{r_k}\} \subset N$,*

$$\varphi_N(\Delta_k) \bigcap \left(\bigcap_{j=0}^{k} cint(X \backslash G_i(\pi_i(x_{r_j}))) \right) = \emptyset,$$

where π_i is the projection of X onto X_i and $\Delta_k = co(\{e_{r_j} : j = 0, \dots, k\})$.

(iii) *For each $i \in I$ and $N_i \in \langle X_i \rangle$, there exists a compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cint(X_i \backslash G_i^{-1}(x)) \neq \emptyset$.*

Then we have

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{x_i \in X_i} G_i(x_i) \right) \neq \emptyset.$$

Proof. For each $i \in I$, let $Y_i = X_i, X = Y = \prod_{i \in I} X_i$ and F be the identity mapping on X. Then all the conditions of Theorem 2.1 are satisfied and the proof is completed. □

If X_i in Corollary 2.1 is a compact FC-space for each I, we have the following result with simpler form.

Corollary 2.2. *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and $X = \prod_{i \in I} X_i$ such that (X, φ_N) is an FC-space defined as in Lemma 1.1. Let $G_i : X_i \to 2^X$ be such that the following conditions hold for each $i \in I$*:

(i) *for each $x_i \in X_i$, $G_i(x_i)$ is transfer compactly closed in X_i;*

(ii) *for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $\{x_{r_0}, \dots, x_{r_k}\} \subset N$, $\varphi_N(\Delta_k) \bigcap (\bigcap_{j=0}^{k} cint(X \backslash G_i(\pi_i(x_{r_j})))) = \emptyset$, where π_i is the projection of X onto X_i and $\Delta_k = co(\{e_{r_j} : j = 0, \dots, k\})$.*

Then we have

$$\bigcap_{i \in I} \bigcap_{x_i \in X_i} G_i(x_i) \neq \emptyset.$$

Proof. For each $i \in I$ and $N_i \in \langle X_i \rangle$, let $L_{N_i} = X_i$ and let $K = \prod_{i \in I} X_i$. Then $K = X$ is compact. Clearly the condition (iii) of Corollary 2.1 is satisfied trivially. The conclusion of Corollary 2.2 holds from Corollary 2.1. □

By Corollary 2.1, we also have the following result, which improves Theorem 3.3 of Ding [14].

Corollary 2.3. *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be an FC-space and $X = \prod_{i \in I} X_i$ such that (X, φ_N) is an FC-space defined as in*

Lemma 1.1, and let K be a nonempty compact subset of X. Let $G_i : X_i \to 2^X$ be such that the following conditions hold for each $i \in I$:

(i) *For each $x_i \in X_i$, $G_i(x_i)$ is transfer compactly closed in X_i.*
(ii) *For each $x \in X, X_i \backslash G_i^{-1}(x)$ is an FC-subspace of X_i.*
(iii) *For each $x \in X, \pi_i(x) \in G_i^{-1}(x)$.*
(iv) *For each $i \in I$ and $N_i \in \langle X_i \rangle$, there exists a compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cint(X_i \backslash G_i^{-1}(x)) \neq \emptyset$.*

Then we have

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{x_i \in X_i} G_i(x_i) \right) \neq \emptyset.$$

Proof. We first show that conditions (ii) and (iii) imply conditions (ii) of Corollary 2.1 holds. If it is false, then there exist $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and $M = \{x_{r_0}, \ldots, x_{r_k}\} \subset N$ such that

$$\varphi_N(\Delta_k) \bigcap \left(\bigcap_{j=0}^{k} cint(X \backslash G_i(\pi_i(x_{r_j}))) \right) \neq \emptyset.$$

Hence there exists $\hat{x} \in \varphi_N(\Delta_k) = \prod_{i \in I} \varphi_{N_i}(\Delta_k)$ where $N_i = \pi_i(N)$ such that $\hat{x} \notin G_i(\pi_i(x))$ for all $x \in M$. So we have $\pi_i(M) \subseteq X_i \backslash G_i^{-1}(\hat{x})$. Since $X_i \backslash G_i^{-1}(\hat{x})$ is an FC-subspace of X_i by (ii), we have $\pi_i(\varphi_N(\Delta_k)) \subset X_i \backslash G_i^{-1}(\hat{x})$. So $\pi_i(\hat{x}) \in X_i \backslash G_i^{-1}(\hat{x})$ and then $\pi_i(\hat{x}) \notin G_i^{-1}(\hat{x})$, which contradicts the condition (iii). This show that the conditions (ii) and (iii) imply the condition (ii) of Corollary 2.1 holds. Corollary 2.3 follows from Corollary 2.1. □

As a direct consequence of Corollary 2.3, we have the following result, which generalizes and improves Theorem 2.1 of Guillerme [19].

Corollary 2.4. *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and $X = \prod_{i \in I} X_i$ such that (X, φ_N) is an FC-space defined as in Lemma 1.1. Let $G_i : X_i \to 2^X$ be such that the following conditions hold for each $i \in I$:*

(iii) *For each $x_i \in X_i$, $G_i(x_i)$ is transfer compactly closed in X_i.*
(iii) *For each $x \in X, X_i \backslash G_i^{-1}(x)$ is an FC-subspace of X_i.*
(iii) *For each $x \in X, \pi_i(x) \in G_i^{-1}(x)$.*

Then we have

$$\bigcap_{i \in I} \bigcap_{x_i \in X_i} G_i(x_i) \neq \emptyset.$$

3. System of generalized vector equilibrium problems

By using Theorem 2.1, we establish some new equilibrium existence results for $SGVEP$ (1) in this section.

Theorem 3.1. *Let X be a topological space, K be a nonempty compact subset of X, I be any index set, (Y_i, φ_{N_i}) be a family of FC-spaces, $\{D_i\}_{i\in I}$, $\{E_i\}_{i\in I}$ and $\{Z_i\}_{i\in I}$ be the families of topological spaces, and $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 1.1. Let $F \in \mathcal{B}(Y, X)$, and for each $i \in I$, $T_i : X \to 2^{D_i}, S_i : X \to 2^{E_i}, C_i : X \to 2^{Z_i}$ and $\eta_i : D_i \times E_i \times Y_i \to 2^{Z_i}$ be set valued mappings such that the following conditions hold for each $i \in I$:*

- (i) *$\eta_i(d_i, e_i, y_i)$ is transfer compactly upper semicontinuous in x with respect to C_i.*
- (ii) *For each $N = \{y_0, \dots, y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, \dots, y_{r_k}\} \subset N$, and for each $x \in F(\varphi_N(\Delta_k))$, there exists $d_i \in T_i(x), e_i \in S_i(x)$ and $y \in N_1$ such that $\eta_i(d_i, e_i, \pi_i(y)) \nsubseteq C_i(x)$.*
- (iii) *For each $N_i \in \langle Y_i\rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i, and for each $x \in X\backslash K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying*

$$y_i \notin ccl\{u_i \in Y_i : \exists d_i \in T_i(x), e_i \in S_i(x) \text{ such that } \eta_i(d_i, e_i, u_i) \nsubseteq C_i(x)\}.$$

Then there exists $\hat{x} \in K$ satisfying that, for each $i \in I$ and $y_i \in Y_i$, there exist $\hat{d}_i \in T_i(\hat{x})$ and $\hat{e}_i \in S_i(\hat{x})$ such that $\eta_i(\hat{d}_i, \hat{e}_i, y_i) \nsubseteq C_i(\hat{x})$.

Proof. For each $i \in I$, defined set-valued mappings $T_i, H_i : Y_i \to 2^X$ by

$$\begin{aligned} T_i(y_i) &= \{x \in X : \exists d_i \in T_i(x) \text{ and } e_i \in S_i(x) \text{ such that } \eta_i(d_i, e_i, y_i) \nsubseteq C_i(x)\}, \\ H_i(y_i) &= X\backslash T_i(y_i) \\ &= \{x \in X : \exists d_i \in T_i(x) \text{ and } e_i \in S_i(x) \text{ such that } \eta_i(d_i, e_i, y_i) \subseteq C_i(x)\}. \end{aligned}$$

Notice that for each $i \in I, \eta_i(d_i, e_i, y_i)$ is transfer compactly upper semicontinuous in x with respect to C_i. Then for any nonempty compact subset K of X, if $x \in H_i(y_i) \bigcap K$, we have $x \in K$ and there exist $d_i \in T_i(x)$ and $e_i \in S_i(x)$ such that $\eta_i(d_i, e_i, y_i) \subseteq C_i(x)$, i.e.,

$$\{(d_i, e_i, y_i) \in D_i \times E_i \times Y_i : \eta_i(d_i, e_i, y_i) \subseteq C_i(x)\} \neq \emptyset.$$

By condition (i), there exists a relatively open neighborhood $N(x)$ of x in K and $(d'_i, e'_i, y'_i) \in D_i \times E_i \times Y_i$ such that $\eta_i(d'_i, e'_i, y'_i) \subseteq C_i(z)$ for all $z \in N(x)$. Then we have

$$\begin{aligned} x \in N(x) &\subseteq int_K(K \textstyle\bigcap \{z \in X : \eta_i(d'_i, e'_i, y'_i) \subseteq C_i(z)\}) \\ &= int_K(H_i(y'_i) \textstyle\bigcap K) = cint(H_i(y'_i)) \textstyle\bigcap K. \end{aligned}$$

This implies that H_i is transfer compactly open in Y_i. Thus for each $i \in I$ and $y_i \in Y_i$, $T_i(y_i)$ is transfer compactly closed in Y_i. So condition (i) of Theorem 2.1 is satisfied.

From the condition (ii) it follows that for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and $N_1 = \{y_{r_0}, \dots, y_{r_k}\} \subset N$, we have

$$F(\varphi_N(\Delta_k)) \subseteq \bigcup_{y \in N_1} T_i(\pi_i(y)) \subset \bigcup_{y \in N_1} cclT_i(\pi_i(y)),$$

and then

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{y \in N_1} (X \backslash cclT_i(\pi_i(y))) \right)$$

$$= F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{y \in N_1} cint(X \backslash T_i(\pi_i(y))) \right) = \emptyset.$$

This yields that condition (ii) of Theorem 2.1 is satisfied.

For each $i \in I$ and $x \in X$, we have

$$T_i^{-1}(x) = \{y_i \in Y_i : \exists d_i \in T_i(x), e_i \in S_i(x) \text{ such that } \eta_i(d_i, e_i, y_i) \not\subseteq C_i(x)\},$$

and hence

$$Y_i \backslash T_i^{-1}(x) = \{y_i \in Y_i : \eta_i(d_i, e_i, y_i) \subseteq C_i(x), \forall d_i \in T_i(x), e_i \in S_i(x)\}.$$

By condition (iii), for each $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i, and for each $x \in X \backslash K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying

$$y_i \in Y_i \backslash cclT_i^{-1}(x) = cint(Y_i \backslash T_i^{-1}(x)).$$

So we have

$$L_{N_i} \bigcap cint(Y_i \backslash T_i^{-1}(x)) \neq \emptyset.$$

Then condition (iii) of Theorem 2.1 is satisfied. Now by Theorem 2.1 we have

$$K \bigcap \left(\bigcap_{i \in I} \bigcap_{y_i \in Y_i} T_i(y_i) \right) \neq \emptyset.$$

Taking any $\hat{x} \in K \bigcap (\bigcap_{i \in I} \bigcap_{y_i \in Y_i} T_i(y_i))$, we obtain that $\hat{x} \in K$, and for each $i \in I$ and $y_i \in Y_i$ there exist $\hat{d}_i \in T_i(\hat{x})$ and $\hat{e}_i \in S_i(\hat{x})$ such that $\eta_i(\hat{d}_i, \hat{e}_i, y_i) \not\subseteq C_i(\hat{x})$. This means that $\hat{x}$ is an equilibrium point of the $SGVEP$ (1). The proof is complete. □

Corollary 3.1. *Let I be any index set, $(X_i, \varphi_{N_i'})_{i \in I}$ and $(Y_i, \varphi_{N_i})_{i \in I}$ be two families of FC-spaces. Let $\{Z_i\}_{i \in I}$ be a family of topological spaces and $X = \prod_{i \in I} X_i, Y = \prod_{i \in I} Y_i$ such that $(X, \varphi_{N'}), (Y, \varphi_N)$ are FC-spaces defined as in Lemma 1.1. Let $F : Y \to X$ be a continuous single-valued mapping and for each $i \in I$, $C_i : X \to 2^{Z_i}$, $f_i : X \times Y_i \to 2^{Z_i}$ be set valued mappings such that the following conditions holds*:

(i) *For each $i \in I, f_i(x, y_i)$ is transfer compactly upper semicontinuous in x with respect to C_i.*

(ii) *For each* $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ *and* $N_1 = \{y_{r_0}, \ldots, y_{r_k}\} \subset N$*, and for each* $x \in F(\varphi_N(\Delta_k))$*, there exists* $y \in N_1$ *such that* $f_i(x, \pi_i(y)) \not\subseteq C_i(x)$*.*

(iii) *There exists a nonempty compact subset* K_i *of* X_i*. Let* $K = \prod_{i\in I} K_i$*. For each* $N_i \in \langle Y_i\rangle$*, there exists a compact* FC*-subspace* L_{N_i} *of* Y_i *containing* N_i*, and for each* $x \in X\backslash K$*, there exist* $i \in I$ *and* $y_i \in L_{N_i}$ *satisfying*

$$y_i \notin ccl\{u_i \in Y_i : f_i(x, u_i) \not\subseteq C_i(x)\}.$$

Then there exists $\hat{x} \in K$ *such that for each* $i \in I$ *and* $y_i \in Y_i$ *we have* $f_i(\hat{x}, y_i) \not\subseteq C_i(\hat{x})$*.*

Proof. Clearly $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X. For each $i \in I$, let $D_i = X_i$ and $E_i = X^i = \prod_{j\in I, j\neq i} X_j$. Write $x = (x_i, x^i)$ and $f_i(x, yi) = \eta_i(x_i, x^i, yi)$ for all $x \in X$ and $y_i \in Y_i$. Let $T_i(x) = \pi_i(x) = x_i$ and $S_i(x) = \pi^i(x) = x^i$ for each $i \in I$ and $x \in X$ where π_i and π^i are the projection mappings from X onto X_i and X^i, respectively. It is easy to check that all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists $\hat{x} \in K$ such that for each $i \in I$,

$$f_i(\hat{x}, y_i) = \eta_i(\hat{x_i}, \hat{x}^i, yi) \not\subseteq C_i(\hat{x}),\ \forall y_i \in Y_i.$$

This completes the proof. □

Remark 3.1. (I) Theorem 3.1 generalizes Theorem 3.1 in [12] in several aspects: (a) The setting spaces are generalized from "family of G-convex spaces" to "family of FC-space" without linear structure; (b) Conditions (i), (ii) and (iii) of Theorem 3.1 in [12] are replaced by condition (i) of Theorem 3.1; (c) conditions (ii) and (iii) of Theorem 3.1 are weaker than conditions (iv) and (v) of Theorem 3.1 in [12].

(II) Corollary 3.1 generalizes Theorem 3.2 in [12] in the aspects similar as those of Theorem 3.1 to Theorem 3.1 in [12]. Moreover, Corollary 3.1 improves Theorem 2.1 and Theorem 2.2 in [2] from topological vector spaces to FC-space under weaker assumptions.

When I be a singleton in Corollary 3.1, we obtain the following result.

Corollary 3.2. *Let* I *be any index set,* $(X, \varphi_{N'})$ *and* (Y, φ_N) *be two* FC*-spaces and* Z *be a topological space,* $F : Y \to X$ *be a continuous single-valued mapping,* $C : X \to 2^Z$ *and* $f : X \times Y \to 2^Z$ *be set-valued mappings such that*

(i) *For each* $f(x, y)$ *is transfer compactly upper semicontinuous in* x *with respect to* C*.*

(ii) *For each* $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$*,* $N_1 = \{y_{r_0}, \ldots, y_{r_k}\} \subset N$ *and* $x \in F(\varphi_N(\Delta_k))$*, there exists* $y \in N_1$ *such that* $f(x, y) \not\subseteq C(x)$*.*

(iii) *There exists a nonempty compact subset* K *of* X*. For each* $N \in \langle Y\rangle$*, there exists a compact* FC*-subspace* L_N *of* Y *containing* N*, and for* *each $x \in X\backslash K$, there exists $y \in L_N$ satisfying*

$$y \notin ccl\{u \in Y : f(x,u) \nsubseteq C(x)\}.$$

Then there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \nsubseteq C(\hat{x})$ for $y \in Y$.

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