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SOME RESULTS ON CERTAIN CLASS OF ANALYTIC FUNCTIONS BASED ON DIFFERENTIAL SUBORDINATION

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Abstract. In the present paper we derive various useful properties and characteristics for certain class of analytic functions by using the techniques of differential subordination. Some interesting corollaries and applications of the results presented here are also discussed.

1. Introduction

Let $\mathcal{A}$ denote a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. For the functions f and g analytic in $\mathbb{U}$, we say f is subordinate to g in $\mathbb{U}$ and write $f \prec g$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)| < 1$, $z \in \mathbb{U}$ and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If f is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}_{\lambda,\alpha}(A, B)$, if and only if, it satisfies the following subordination condition:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \frac{1+Az}{1+Bz}, \tag{1.2}$$

where $z \in \mathbb{U}$, $\lambda > 0$, $\alpha > 0$, $-1 \le B < A \le 1$.

For simplicity, we put

$$\mathcal{B}_{\lambda,\alpha}(1-2\rho, -1) = \mathcal{B}(\lambda, \alpha, \rho),$$

where $\mathcal{B}(\lambda, \alpha, \rho)$ denotes the class of the functions $f \in \mathcal{A}$ which satisfy the inequality

$$\Re\Big((1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\Big) > \rho. \tag{1.3}$$

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Here $z \in \mathbb{U}$, $\lambda > 0$, $\alpha > 0$, $0 \leq \rho < 1$.

The powers in (1.2) and (1.3) are understood as principal values. It is obvious that the subclass $\mathcal{B}(1, \alpha, 0)$ is the subclass of Bezilevic functions [2]. The subclasses $\mathcal{B}(1, \alpha, \rho) \equiv \mathcal{B}(\alpha, \rho)$ $(0 \leq \rho < 1)$ and $\mathcal{B}(0, \alpha, \rho)$ $(\rho < 1)$, have been studied by Liu [5] and Singh [8], respectively. The subclass $\mathcal{B}(\lambda, 1, \rho)$ $(0 \leq \rho < 1)$ has been studied by Chichra [3] and Ding *et al.* [4].

In the present paper we give some sharp sufficient conditions for the function class $\mathcal{B}_{\lambda,\alpha}(A, B)$. For obtaining our main results we will use the method of differential subordinations. Namely if $\phi : \mathbb{C}^2 \to \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) is analytic in domain $\mathbb{D} \subset \mathbb{C}$, if $h(z)$ is univalent in $\mathbb{U}$, and if $p(z)$ is analytic in $\mathbb{U}$ with $(p(z),\ zp'(z)) \in \mathbb{D}$, then $p(z)$ is said to satisfy a first order differential subordination if

$$\phi(p(z),\ zp'(z)) \prec h(z). \tag{1.4}$$

The univalent function $q(z)$ is said to be a dominent of the differential subordination (1.4), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4). If $\tilde{q}(z)$ is a dominent of (1.4) and $\tilde{q}(z) \prec q(z)$ for all dominents of (1.4), then we say that $\tilde{q}(z)$ is the best dominent of the differential subordination (1.4). For the general theory of differential subordination, one may refer to the Miller and Mocanu [7].

In our present investigation we shall need the following lemmas from the theory of differential subordination:

Lemma 1.1 (Miller and Mocanu [7, p. 71])**.** *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$ and let the function $\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots$ be analytic in $\mathbb{U}$. If*

$$\phi(z) + \frac{1}{\gamma} z\phi'(z) \prec h(z) \quad (\Re(\gamma) \geq 0; \gamma \neq 0; z \in \mathbb{U}), \tag{1.5}$$

then

$$\phi(z) \prec \psi(z) := \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in \mathbb{U}) \tag{1.6}$$

and $\psi(z)$ is the best dominant.

Lemma 1.2 (See [7, p. 132])**.** *Let $q(z)$ be analytic and univalent in $\mathbb{U}$ and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\theta(\omega) \neq 0$ when $\omega \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z)\phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z)$$

and suppose that

(1) *$Q(z)$ is univalent and starlike in $\mathbb{U}$;*
(2) $\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{U}).$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

The generalized hypergeometric function ${}_pF_q$ is defined by (see, for example, [9])

(1.7)
$$\begin{aligned}
{}_pF_q &\equiv {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)\\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\\
&(z\in\mathbb{U};\ \alpha_j\in\mathbb{C}\ (j=1,2,\ldots,p),\ \beta_j\in\mathbb{C}/\{0,-1,-2,\ldots\}\ (j=1,\ldots,q);\\
&\quad p\le q+1;\ p,q\in\mathbb{N}_0),
\end{aligned}$$

where $(\alpha)_k$ is Pochammer symbol defined by

$$(\alpha)_0 = 1,\ \ (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1);\ \ k\in\mathbb{N}.$$

Each of the following identities (asserted by Lemma 1.3) below is well known [1, p. 556–558]:

Lemma 1.3. *For real and complex numbers a, b and c ($c\neq 0,-1,-2,\ldots$)*

(1.8)
$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_2F_1(a,b;c;z)\ \ (\Re(c)>\Re(b)>0),$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right), \tag{1.9}$$

$$ {}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}, \tag{1.10}$$

$$ {}_2F_1\left(1,1;3;\frac{az}{1+az}\right) = \frac{2(1+az)}{az}\left(1-\frac{ln(1+az)}{az}\right)\quad (a\neq 0). \tag{1.11}$$

2. Main results

Our first main result is given by the following theorem:

Theorem 2.1. *If $f(z)\in\mathcal{B}_{\lambda,\alpha}(A,B)$, then*

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec \mathcal{X}(z) \prec \frac{1+Az}{1+Bz}\qquad (z\in\mathbb{U}), \tag{2.1}$$

where

$$\mathcal{X}(z) = \begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\alpha}{\lambda}+1;\frac{Bz}{1+Bz}\right) & (B\neq 0),\\ 1+\frac{\alpha}{\lambda+\alpha}Az & (B=0)\end{cases}$$

and $\mathcal{X}(z)$ is the best dominant of (2.1). *Also*

$$\Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} > \mathcal{X}(-1). \tag{2.2}$$

The result (2.2) *is sharp.*

Proof. Let $f(z) \in \mathcal{B}_{\lambda,\alpha}(A,B)$ and assume that

$$\left(\frac{f(z)}{z}\right)^{\alpha} = p(z). \tag{2.3}$$

We may express the function $p(z)$ as $p(z) = 1 + \alpha a_2 z + \cdots$, which is analytic in $\mathbb{U}$ with $p(0) = 1$. Differentiation of (2.3) and some computation gives us

$$\begin{aligned}&(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\\ &= p(z) + \frac{\lambda}{\alpha} z p'(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).\end{aligned}$$

Now applying Lemma 1.1, we obtain (2.1). Moreover by Lemma 1.3, we have

$$\begin{aligned}p(z) &\prec \frac{\alpha}{\lambda} z^{-\alpha/\lambda} \int_0^z t^{\alpha/\lambda-1} \frac{1+At}{1+Bt} dt\\ &= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\dfrac{\alpha}{\lambda}+1;\dfrac{Bz}{1+Bz}\right) & (B \neq 0),\\ 1 + \dfrac{\alpha}{\lambda+\alpha} Az & (B = 0)\end{cases}\\ &= \mathcal{X}(z).\end{aligned} \tag{2.4}$$

This shows that $\mathcal{X}(z)$ is best dominent of (2.1). Next to prove (2.2), we observe that the subordination relation (2.4) is equivalent to

$$\left(\frac{f(z)}{z}\right)^{\alpha} = \frac{\alpha}{\lambda}\int_0^1 u^{\alpha/\lambda-1} \frac{1+Au\,\omega(z)}{1+Bu\,\omega(z)} du,$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with $\omega(0) = 1$ and $|\omega(z)| < 1$ in $\mathbb{U}$. Hence

$$\begin{aligned}\Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} &= \frac{\alpha}{\lambda}\int_0^1 u^{\alpha/\lambda-1}\Re\left(\frac{1+Au\,\omega(z)}{1+Bu\,\omega(z)}\right) du\\ &> \frac{\alpha}{\lambda}\int_0^1 u^{\alpha/\lambda-1}\frac{1-Au}{1-Bu} du\\ &= \mathcal{X}(-1).\end{aligned}$$

The sharpness of the result (2.2) can be established by considering the function $\mathcal{X}(z)$ defined by (2.4). It is sufficient to show that

$$\inf_{|z|<1}\{\Re(\mathcal{X}(z))\} = \mathcal{X}(-1). \tag{2.5}$$

We observe from (2.4) that for $|z| \le r$ $(0 < r < 1)$,

$$\begin{aligned}\Re\{\mathcal{X}(z)\} &\geq \frac{\alpha}{\lambda}\int_0^1 u^{\alpha/\lambda-1}\Re\left(\frac{1-Aur}{1-Bur}\right) du = \mathcal{X}(-r)\\ &\to \mathcal{X}(-1) \quad \text{as} \quad r \to 1^-,\end{aligned}$$

which establishes (2.5) and this completes the proof of Theorem 2.1. □

Corollary 2.1. *Let $B \neq 0$ and $\alpha \geq 1$. If $f(z) \in \mathcal{B}_{\lambda,\alpha}(A, B)$, then*

$$\Re\left(\frac{f(z)}{z}\right) > \left[\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1; \frac{\alpha}{\lambda}+1; \frac{B}{B-1}\right)\right]^{1/\alpha}. \tag{2.6}$$

Proof. Using the elementary inequality $\Re(\omega^{1/m}) \geq (\Re(\omega))^{1/m}$ for $\Re(\omega) > 0$ and $m \geq 1$ in Theorem 2.1, we get the desired result. □

Letting $A = 1 - 2\rho$, $0 \leq \rho < 1$, $B = -1$, $\lambda = 2$ and $\alpha = 1$, in Corollary 2.1 and using (1.10), we obtain:

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies the inequality*

$$\Re\left(-\frac{f(z)}{z} + 2f'(z)\right) > \rho \qquad (0 \leq \rho < 1; z \in \mathbb{U}),$$

then

$$\Re\left(\frac{f(z)}{z}\right) > \rho + (1-\rho)(\pi/2 - 1).$$

The result is sharp.

Theorem 2.2. *If $f(z) \in \mathcal{B}_{\lambda,\alpha}(A^*, B)$ $(\alpha \geq 1; -1 \leq B < A^* \leq 1;\ B \neq 0)$, then*

$$\Re\left(\frac{f(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}), \tag{2.7}$$

where A^* is given by

$$A^* = \frac{B\ {}_2F_1\left(1, 1; \frac{\alpha}{\lambda}+1; \frac{B}{B-1}\right)}{{}_2F_1\left(1, 1; \frac{\alpha}{\lambda}+1; \frac{B}{B-1}\right) + (B-1)}. \tag{2.8}$$

The result (2.7) *is sharp.*

Proof. In view of Corollary 2.1, if

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \frac{1+A^*z}{1+Bz} \quad (z \in \mathbb{U}), \tag{2.9}$$

then

$$\Re\left(\frac{f(z)}{z}\right) > \left[\frac{A^*}{B} + \left(1 - \frac{A^*}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1; \frac{\alpha}{\lambda}+1; \frac{B}{B-1}\right)\right]^{1/\alpha}. \tag{2.10}$$

On substituting the value of A^* given by (2.8) in the right-hand side of the above inequality (2.10), we get

$$\Re\left(\frac{f(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}),$$

which proves Theorem 2.2. □

Remark 2.1. For $\lambda = 1$ and $\alpha = 2$, we note that Theorem 2.2 in view of (1.11) yields the assertion which we express as follows:

If

$$f'(z)\frac{f(z)}{z} \prec \frac{1+A_1 z}{1+Bz} \qquad (z \in \mathbb{U};\ -1 \le B < A_1 \le 1;\ B \neq 0), \tag{2.11}$$

where A_1 is given by

$$A_1 = \frac{2B[B+\ln(1-B)]}{B^2+2B+2\ln(1-B)}, \tag{2.12}$$

then

$$\Re\left(\frac{f(z)}{z}\right) > 0$$

and hence $f(z)$ is univalent in $\mathbb{U}$. Further on choosing $B = -1$ in (2.12) and using the principal of subordination, we arrive at the following assertion:

If

$$\Re\left(f'(z)\frac{f(z)}{z}\right) > \frac{4\ln 2 - 3}{4\ln 2 - 2} \quad (z \in \mathbb{U}; f \in \mathcal{A}),$$

then

$$\Re\left(\frac{f(z)}{z}\right) > 0.$$

Theorem 2.3. *If $f(z) \in \mathcal{A}$ satisfies the following inequality:*

$$\Re\left(\left(\frac{f(z)}{z}\right)^{\alpha}\right) \ > \ \rho \quad (z \in \mathbb{U};\ \alpha > 0;\ 0 \le \rho < 1), \tag{2.13}$$

then

$$\Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right) > \rho \quad (|z| < R_1), \tag{2.14}$$

where R_1 is given by

$$R_1 = \frac{\sqrt{\lambda^2+\alpha^2}-\lambda}{\alpha} \quad (\alpha > 0;\ \lambda > 0). \tag{2.15}$$

The result is sharp.

Proof. Let $f(z) \in \mathcal{A}$ satisfy the inequality (2.13), therefore

$$\left(\frac{f(z)}{z}\right)^{\alpha} = \rho + (1-\rho)\omega(z), \tag{2.16}$$

where $\omega(z) = 1 + a_1 z + a_2 z^2 + \cdots$ is analytic and has positive real part in $\mathbb{U}$. Differentiating (2.16) with respect to z, we get

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} - \rho = (1-\rho)\left(\omega(z) + \frac{\lambda}{\alpha}\ z\omega'(z)\right). \tag{2.17}$$

Now using the well-known estimate (cf. [6])

$$\frac{|z\omega'(z)|}{R\{\omega(z)\}} \leq \frac{2\gamma}{1-\gamma^2} \quad (|z| = \gamma < 1) \tag{2.18}$$

in (2.17), we deduce that

$$\begin{aligned} &\Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} - \rho\right) \\ &\geq (1-\rho)\Re\{\omega(z)\}\left(1 - \frac{2\lambda\gamma}{\alpha(1-\gamma^2)}\right). \end{aligned} \tag{2.19}$$

It follows easily that the right hand side of (2.19) is positive when $|z| < R_1$ where R_1 is given by (2.15), which implies that $f(z) \in \mathcal{B}(\lambda, \alpha, \rho)$ for $|z| < R_1$.

To show that the bound R_1 is sharp, we consider the function $f \in \mathcal{A}$ defined by

$$\left(\frac{f(z)}{z}\right)^{\alpha} = \rho + (1-\rho)\frac{1-z}{1+z} \quad (z \in \mathbb{U}),$$

equivalently

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} - \rho = (1-\rho)\left(\frac{\alpha(1-z^2)+2\lambda z}{\alpha(1-z^2)}\right) = 0$$

for $z = R_1$, which complete the proof of Theorem 2.3. □

Remark 2.2. We observe from Theorem 2.3 that

(**i**) If $f(z) \in \mathcal{A}$ satisfy the following inequality

$$\Re\left(\frac{f(z)}{z}\right) > \rho \qquad (z \in \mathbb{U};\ 0 \leq \rho < 1),$$

then

$$\Re\left(f'(z)\right) > \rho \qquad (|z| < \sqrt{2} - 1).$$

The result is sharp.

(**ii**) If $f(z) \in \mathcal{A}$ satisfy the following inequality

$$\Re\left(\frac{f(z)}{z}\right)^2 > \rho \qquad (z \in \mathbb{U};\ 0 \leq \rho < 1),$$

then

$$\Re\left(f'(z)\frac{f(z)}{z}\right) > \rho \qquad (|z| < \frac{\sqrt{5}-1}{2}).$$

The result is sharp.

(**iii**) If $f(z) \in \mathcal{A}$ satisfy the following inequality

$$\Re\left(\frac{f(z)}{z}\right) > \rho \qquad (z \in \mathbb{U}; 0 \leq \rho < 1),$$

then

$$\Re\left(f'(z) + \frac{f(z)}{z}\right) > 2\rho \qquad (|z| < \frac{\sqrt{5}-1}{2}).$$

The result is sharp.

Theorem 2.4. *Let* $f(z) \in \mathcal{B}_{\lambda,\alpha}(A,B)$ *and define* $I(f) : \mathcal{A} \to \mathcal{A}$ *such that*

$$I(f) = F(z) = \left[\frac{1}{\lambda} z^{\alpha - 1/\lambda} \int_0^z t^{1/\lambda - 1 - \alpha} \, (f(t))^\alpha \, dt\right]^{1/\alpha} \qquad (z \in \mathbb{U}). \tag{2.20}$$

Then

$$(1-\alpha\lambda)\left(\frac{F(z)}{z}\right)^\alpha + \alpha\lambda F'(z)\left(\frac{F(z)}{z}\right)^{\alpha-1} \prec \mathcal{X}(z) \prec \frac{1+Az}{1+Bz}, \tag{2.21}$$

where $\mathcal{X}(z)$ *is defined by* (2.4) *and is best dominant of* (2.11)*. Also*

$$\Re\left((1-\alpha\lambda)\left(\frac{F(z)}{z}\right)^\alpha + \alpha\lambda F'(z)\left(\frac{F(z)}{z}\right)^{\alpha-1}\right) > \mathcal{X}(-1). \tag{2.22}$$

The result (2.22) *is sharp.*

Proof. Let $f(z) \in \mathcal{B}_{\lambda,\alpha}(A,B)$. Differentiating (2.20), we have

$$(1-\alpha\lambda)\left(\frac{F(z)}{z}\right)^\alpha + \alpha\lambda F'(z)\left(\frac{F(z)}{z}\right)^{\alpha-1} = \left(\frac{f(z)}{z}\right)^\alpha.$$

Now, using Theorem 2.1, we obtain the required result. $\square$

Theorem 2.5. *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ *and* $-1 < A < 1$ *and*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\alpha + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec h(z), \tag{2.23}$$

where

$$h(z) = \frac{1+Az}{1-z} + \frac{\lambda(A+1)z}{\alpha(1-z)^2}. \tag{2.24}$$

Then

$$\left(\frac{f(z)}{z}\right)^\alpha \prec \frac{1+Az}{1-z}. \tag{2.25}$$

Proof. Define the function $p(z)$ by (2.3) and following Theorem 2.1, we see that subordination (2.23) becomes

$$p(z) + \frac{\lambda}{\alpha} z p'(z) \prec h(z). \tag{2.26}$$

Now choose

$$q(z) = \frac{1+Az}{1-z}, \qquad \theta(w) = w \qquad \text{and} \qquad \phi(w) = \frac{\lambda}{\alpha}.$$

Here $q(z)$ is analytic and univalent in $\mathbb{U}$ with $q(0) = 1$. Also $\theta(w)$ and $\phi(w)$ are analytic with $\theta(w) \neq 0$ in $\mathbb{C}/\{0\}$. We see that

$$Q(z) = zq'(z)\phi[q(z)] = \frac{\lambda(A+1)z}{\alpha(1-z)^2}$$

is univalent and starlike in $\mathbb{U}$, because

$$\Re\left(\frac{zQ'(z)}{Q(z)}\right) = \Re\left(\frac{1+z}{1-z}\right) = \Re\left(1 + 2\frac{z}{1-z}\right) > 0.$$

Further, we have

$$h(z) = \theta[q(z)] + Q(z) = \frac{1+Az}{1-z} + \frac{\lambda(A+1)z}{\alpha(1-z)^2} \quad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right) = \frac{\alpha}{\lambda} + \Re\left(\frac{zQ'(z)}{Q(z)}\right) > 0. \tag{2.27}$$

The inequality (2.27) shows that the function $Q(z)$ is close to convex and univalent in $\mathbb{U}$. Hence

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore by virtue of Lemma 1.2, we conclude that $p(z) \prec q(z)$, that is, the proof of Theorem 2.5 is complete. □

Remark 2.3. We observe from Theorem 2.5 that

(i) If $f(z) \in \mathcal{A}$ satisfies the following subordination condition

$$f'(z) \prec \frac{1+2Az-Az^2}{(1-z)^2} \qquad (z \in \mathbb{U};\ -1 < A < 1),$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1-z}.$$

(ii) If $f(z) \in \mathcal{A}$ satisfies the following subordination condition

$$f'(z)\frac{f(z)}{z} \prec \frac{(2-z)+Az(3-2z)}{2(1-z)^2} \qquad (z \in \mathbb{U};\ -1 < A < 1),$$

then

$$\left(\frac{f(z)}{z}\right)^2 \prec \frac{1+Az}{1-z}.$$

(iii) If $f(z) \in \mathcal{A}$ satisfies the following subordination condition

$$\frac{1}{2}\left(f'(z) + \frac{f(z)}{z}\right) \prec \frac{(2-z)+Az(3-2z)}{2(1-z)^2} \qquad (z \in \mathbb{U};\ -1 < A < 1),$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1-z}.$$

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