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ON ϕ-VON NEUMANN REGULAR RINGS

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Abstract. Let R be a commutative ring with $1 \neq 0$ and let $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal$\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. In this paper, we introduce the concepts of ϕ-torsion modules, ϕ-flat modules, and ϕ-von Neumann regular rings.

1. Introduction

Let R be a commutative ring with $1 \neq 0$ and $Nil(R)$ be its set of nilpotent elements. Recall from [15] and [3] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \backslash P$. Set $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring.

Throughout this paper, it is assumed that all rings are commutative and associative with identity $1 \neq 0$ and all modules are unitary. Recently, the authors in [1], [2], [14], and [18] generalized the concept of Prüfer, Bezout domains, Dedekind domains, Krull domains, Mori domains, and Strongly Mori domains to the context of rings that are in the class $\mathcal{H}$. Also, the authors in [4], [3], [5], [6], [7], and [9], investigated the following classes of rings: ϕ-CR, ϕ-PVR, and ϕ-$ZPUI$. Furthermore, in [11], the authors investigated going-down ϕ-rings. The authors in [8], [13] and [16], introduced the notion of nonnil-Noetherian rings (later called ϕ-Noetherian rings). This notion was extended to noncommutative rings in [19]. The authors in [10], stated many of the main results on ϕ-rings.

The classic homological algebra enlighten us that the categories of modules and homological dimensions are beneficial to characterize a ring from its external structure. For example, $w.gl.dim(R) = 0$, equivalently, every R-module is flat, if and only if R is a von Neumann regular ring. This naturally leads to the question: How to characterize a ring $R \in \mathcal{H}$ in terms of modules theoretic methods? To this end, we introduce a class of modules. Set $NN(R) = \{J \mid J$ is a nonnil ideal of ring $R\}$. Let M be an R-module. We

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define ϕ-tor$(M) = \{x \in M \mid Ix = 0$ for some $I \in NN(R)\}$. If ϕ-tor$(M) = M$, then M is called a ϕ-torsion module, and if ϕ-tor$(M) = 0$, then M is called a ϕ-torsion free module. In Section 2, we investigate some basic properties of ϕ-torsion modules and ϕ-torsion free modules.

In Section 3, we define a ϕ-flat module with the help of ϕ-torsion modules. An R-module M is called ϕ-flat, if $-\bigotimes_R M$ is exact for every exact R-sequence $0 \to A \to B \to C \to 0$, where C is a ϕ-torsion R-module. We show that an R-module M is ϕ-flat if and only if $0 \to I\bigotimes_R M \to R\bigotimes_R M$ is an exact sequence for all nonnil ideals I of R. This helps us generalize the theory of flat modules to the ϕ-flat modules.

Recall that a ring R is said to be von Neumann regular if every R-module is flat. A ring R is von Neumann regular, if and only if there is an element $x \in R$ such that $a = xa^2$ for each $a \in R$, if and only if every principal ideal I of R is generated by an idempotent, if and only if every finitely generated ideal I of R is generated by an idempotent. We define a ϕ-ring R to be a ϕ-von Neumann regular ring if every R-module is ϕ-flat. In the last section of this paper, we characterize ϕ-von Neumann regular rings and we give an example of a ϕ-von Neumann regular ring that is not a von Neumann regular ring.

2. On ϕ-torsion modules and ϕ-torsion free modules

Set $NN(R) = \{I \mid I$ is a nonnil ideal of ring $R\}$. Let M be an R-module. We define

$$\phi\text{-tor}(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in NN(R)\}.$$

If ϕ-tor$(M) = M$, then M is called a *ϕ-torsion module*, and if ϕ-tor$(M) = 0$, then M is called a *ϕ-torsion free module*. Clearly, submodules and quotient modules of ϕ-torsion modules are still ϕ-torsion; submodules of ϕ-torsion free modules are still ϕ-torsion free.

If $Nil(R)$ is a prime ideal, then ϕ-tor(M) is a submodule of M which is called the *total ϕ-torsion* submodule of M. Set $T = \phi$-tor(M). Then T is always ϕ-torsion and M/T is always ϕ-torsion free.

Example 2.1. Let R be a commutative ring. Then R/I is a ϕ-torsion R-module for any nonnil ideal I of R.

Every regular ideal is a nonnil ideal, thus every torsion R-module is ϕ-torsion R-module, and every ϕ-torsion free R-module is torsion free R-module. If R is a strong ϕ-ring, in the sense that each zero divisor is nilpotent, or a domain, then every ϕ-torsion R-module is torsion R-module, and every torsion free R-module is ϕ-torsion free R-modules.

The following results give us a criterion to ϕ-torsion module, and ϕ-torsion free module.

Theorem 2.2. *An R-module M is ϕ-torsion if and only if $ann_R(x)$ is a nonnil ideal for all x in M.*

Proof. M is ϕ-torsion if and only if for any $x \in M$, there is a nonnilpotent element $r \in R$ such that $rx = 0$. □

Theorem 2.3. *The following statements are equivalent for a module M:*
(1) *M is ϕ-torsion free.*
(2) $\mathrm{Hom}_R(R/J, M) = 0$ *for all $J \in NN(R)$.*
(3) $\mathrm{Hom}_R(B, M) = 0$ *for all $J \in NN(R)$ and all R/J-modules B.*

Proof. (1)⇒(2). Let $f \in \mathrm{Hom}_R(R/J, M)$ and write $x = f(\overline{1})$. Thus $Jx = 0$, whence $x = 0$. Consequently, $f = 0$.

(2)⇒(1). Let $x \in M$ and $J \in NN(R)$ with $Jx = 0$. Define $f : R/J \to M$ by $f(\overline{r}) = rx$. Then f is a well-defined homomorphism. As $\mathrm{Hom}_R(R/J, M) = 0$ we have $x = f(\overline{1}) = 0$.

(2)⇒(3). Let $F = \bigoplus(R/J)$ be a free R/J-module and let $f : F \to B$ be epimorphic. Then $0 \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(F, M)$ is an exact sequence. Since $\mathrm{Hom}_R(F, M) = \prod \mathrm{Hom}_R(R/J, M) = 0$, we have $\mathrm{Hom}_R(B, M) = 0$.

(3)⇒(2). Trivially. □

Theorem 2.4. *Let R be a commutative ring with prime nil ideal $Nil(R)$.*

(1) *A module M is ϕ-torsion if and only if* $\mathrm{Hom}_R(M, N) = 0$ *for any ϕ-torsion free module N.*

(2) *A module N is ϕ-torsion free if and only if* $\mathrm{Hom}_R(M, N) = 0$ *for any ϕ-torsion module M.*

Proof. (1) Let M be ϕ-torsion and let $f \in \mathrm{Hom}_R(M, N)$. Then $\mathrm{Im}(f)$ is a ϕ-torsion submodule of N. Since N is ϕ-torsion free, we have $f(M) = 0$, and hence $f = 0$.

Conversely, set $T = \phi\text{-tor}(M)$ and $N = M/T$. Then N is ϕ-torsion free. Thus the natural homomorphism $\pi : M \to N$ is the zero homomorphism since $\mathrm{Hom}_R(M, N) = 0$. Therefore $N = 0$, that is, $M = T$.

(2) Let N be ϕ-torsion free. By (1) we have $\mathrm{Hom}_R(M, N) = 0$ for any ϕ-torsion module M.

Conversely, let $M = \phi\text{-tor}(N)$. Then $\mathrm{Hom}_R(M, N) = 0$. Thus the inclusion homomorphism $M \to N$ is the zero homomorphism. Therefore $M = 0$, and hence N is ϕ-torsion free. □

Theorem 2.5. *Let R be a commutative ring with prime nil ideal $Nil(R)$ and $\{M_i \mid i \in \Gamma\}$ be a family of ϕ-torsion modules. Then $\bigoplus\limits_{i\in\Gamma} M_i$ is ϕ-torsion.*

Proof. We have that $\mathrm{Hom}_R(\bigoplus_{i\in\Gamma} M_i, N) \cong \prod_{i\in\Gamma} \mathrm{Hom}_R(M_i, N)$. □

Theorem 2.6. *Let $f : R \to T$ be an monomorphism from rings R to T. If M is a ϕ-torsion R-module, then $M \bigotimes_R T$ is a ϕ-torsion T-module.*

Proof. If I is a nonnil ideal of R, then $f(I)$ is a nonnil ideal of T. □

Corollary 2.7. *If M is a ϕ-torsion R-module, then $M[x] = M \bigotimes_R R[x]$, as an $R[x]$-module, is also a ϕ-torsion module.*

Corollary 2.8. *Let M be a ϕ-torsion R-module, and S be a regular multiplicative set in the ring R. Then $S^{-1}M$ is a ϕ-torsion $S^{-1}R$-module.*

Proof. If I is a nonnil ideal of R, then $S^{-1}I$ is a nonnil ideal of $S^{-1}R$. □

Theorem 2.9. *Let $f : R \to T$ be an epimorphism from rings R to T. If M is a ϕ-torsion T-module, then M, as a R-module, is also a ϕ-torsion module.*

Proof. If J is a nonnil ideal of T, then $f^{-1}(J)$ is a nonnil ideal of R. □

Corollary 2.10. *Let M be an R-module. If M/IM is a ϕ-torsion R/I-module, then M is a ϕ-torsion R-module.*

3. On ϕ-flat modules

An R-module M is said to be flat if for every monomorphism $f : A \to B$, $f \otimes \mathbf{1} : A \bigotimes_R M \to B \bigotimes_R M$ is also monomorphic; equivalently, if $0 \to A \to B \to C \to 0$ is exact, then $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is exact. We give the definition of ϕ-flat modules as follows.

Definition 3.1. An R-module M is said to be ϕ-flat, if for every monomorphism $f : A \to B$ with ϕ-torsion coker(f), $f \otimes \mathbf{1} : A \bigotimes_R M \to B \bigotimes_R M$ is also monomorphic; equivalently, if $0 \to A \to B \to C \to 0$ is an exact R-sequence where C is ϕ-torsion, then $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is exact.

Recall from [20] that M is flat if and only if $0 \to I \bigotimes_R M \to R \bigotimes_R M$ is exact for any ideal I (or any finitely generated ideal I), if and only if the natural homomorphism $\sigma : I \bigotimes_R M \to IM$ given by $\sigma(a \otimes x) = ax$, $a \in I$, $x \in M$, is isomorphic for any ideal I (or any finitely generated ideal I), if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for any ideal I (or any finitely generated ideal I), if and only if for any submodule N of a free R-module F, $0 \to N \bigotimes_R M \to F \bigotimes_R M$ is exact, if and only if the character module $M^+ = \mathrm{Hom}_Z(M, Q/Z)$ is injective. We have the following results for ϕ-flat modules.

Theorem 3.2. *The following conditions are equivalent for a R-module M.*

(a) *M is ϕ-flat.*

(b) *$\mathrm{Tor}_1^R(P, M) = 0$ for all ϕ-torsion R-modules P.*

(c) *$\mathrm{Tor}_1^R(R/I, M) = 0$ for all nonnil ideals I of R.*

(d) *$0 \to I \bigotimes_R M \to R \bigotimes_R M$ is an exact sequence for all nonnil ideals I of R.*

(e) *$I \bigotimes_R M \cong IM$ for all nonnil ideals I of R.*

(f) *$- \bigotimes_R M$ is exact for every exact R-sequence $0 \to N \to F \to C \to 0$, where N, F, C are finitely generated, C is a ϕ-torsion R-module, and F is free.*

(g) *$- \bigotimes_R M$ is exact for every exact R-sequence $0 \to N \to F \to C \to 0$, where C is a ϕ-torsion R-module, and F is free.*

(h) *$\mathrm{Tor}_1^R(R/I, M) = 0$ for all finitely generated nonnil ideals I of R.*

(i) $0 \to I \bigotimes_R M \to R \bigotimes_R M$ *is an exact sequence for all finitely generated nonnil ideals I of R.*

(j) $I \bigotimes_R M \cong IM$ *for all finitely generated nonnil ideals I of R.*

(k) $\mathrm{Ext}^1_R(I, M^+) = 0$ *for any nonnil ideal I of R, where M^+ denote by the character module* $\mathrm{Hom}_Z(M, Q/Z)$.

(l) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \bigcap FI = IK$ for all nonnil ideals I of R.*

(m) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \bigcap FI = IK$ for all finite generated nonnil ideal I of R.*

Proof. (a)⇔(b). We only need the long exact sequence

$$0 = \mathrm{Tor}^R_1(C, M) \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0.$$

(b)⇒(c). If I is a nonnil ideal of R, then R/I is ϕ-torsion R-module.

(c)⇔(d)⇔(e), (d)⇔(k), (d)⇔(l)⇔(m). It is similar to the flat modules.

(d)⇔(h)⇔(i)⇔(j). Every nonnil ideal I of R is the direct limit of all finitely generated nonnil subideals I_i of I, i.e., $I = \varinjlim I_i$.

(d)⇒(f). Let $X = \{e_i\}_{i=1}^n$ be a basis of F. The case for $n = 1$ is true by hypothesis and the following result. If $0 \to I \to R \to R/I \to 0$ is exact, and R/I is a ϕ-torsion R-module, then $I = Ann_R(\overline{1}) \nsubseteq Nil(R)$. Therefore, I is a nonnil ideal of R.

Suppose $n > 1$. Set $F_1 = Re_2 \bigoplus \cdots \bigoplus Re_n$ and $A = N \bigcap Re_1$. Let $I = \{r \in R \mid re_1 \subseteq A\}$. Then $A = Ie_1 \cong I$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & N & \xrightarrow{\pi} & N/A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & Re_1 & \longrightarrow & F & \xrightarrow{p} & F_1 & \longrightarrow & 0
\end{array}$$

where π is the natural homomorphism, p is the projection, and f is the homomorphism induced by the left square. If $u \in N$ with $f(\overline{u}) = p(u) = 0$, then $u \in Re_1$. Thus $u \in A$, whence f is monomorphic.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & N & \xrightarrow{\pi} & N/A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & Re_1 & \longrightarrow & F & \xrightarrow{p} & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all columns and rows are exact. C is a ϕ-torsion R-module imply that C', C'' are ϕ-torsion R-modules.

Set $N' = \ker(A \bigotimes_R M \to N \bigotimes_R M)$. Tensoring by M we have the following commutative diagram with the top row exact:

$$\begin{array}{ccccccccc}
N' & \longrightarrow & A \bigotimes_R M & \longrightarrow & N \bigotimes_R M & \longrightarrow & N/A \bigotimes_R M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Re_1 \bigotimes_R M & \longrightarrow & F \bigotimes_R M & \longrightarrow & F_1 \bigotimes_R M & \longrightarrow & 0
\end{array}$$

The bottom row is also exact because that $F \bigotimes_R M \cong (Re_1 \bigoplus F_1) \bigotimes M \cong (Re_1 \bigotimes M) \bigoplus (F_1 \bigotimes M)$. Notice that $A \bigotimes_R M = Ie_1 \bigotimes_R M \to Re_1 \bigotimes_R M$ is monomorphic by hypothesis and $N/A \bigotimes_R M \to F_1 \bigotimes_R M$ is monomorphic by induction. Hence we obtain that $N \bigotimes_R M \to F \bigotimes_R M$ is monomorphic by *Five Lemma*.

(f)$\Rightarrow$(g). Let $u_i \in N$ and $x_i \in M$ such that $\sum_{i=1}^m u_i \otimes x_i = 0$ in $F \bigotimes_R M$. We show $\sum_{i=1}^m u_i \otimes x_i = 0$ in $N \bigotimes_R M$. Set $N_0 = Ru_1 + \cdots + Ru_m$. Then there are a finitely generated free submodule F_0 and a free submodule F_1 of F such that $F = F_0 \bigoplus F_1$ and $N_0 \subseteq F_0$. In the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_0 & \longrightarrow & F_0 & \stackrel{\pi}{\longrightarrow} & F_0/N_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & N & \longrightarrow & F & \stackrel{p}{\longrightarrow} & C & \longrightarrow & 0
\end{array}$$

f is monomorphic by *Five Lemma*, C is a ϕ-torsion R-module imply that F_0/N_0 is a ϕ-torsion R-module. Thus $N_0 \bigotimes_R M \to F_0 \bigotimes_R M$ is monomorphic by hypothesis. Consider the following commutative diagram:

$$\begin{array}{ccc}
N_0 \bigotimes_R M & \longrightarrow & N \bigotimes_R M \\
\downarrow & & \downarrow \\
F_0 \bigotimes_R M & \longrightarrow & F \bigotimes_R M
\end{array}$$

Since $F_0 \bigotimes_R M \to F \bigotimes_R M$ is monomorphic, and $\sum_{i=1}^m u_i \otimes x_i = 0$ in $F_0 \bigotimes_R M$. Hence $\sum_{i=1}^m u_i \otimes x_i = 0$ in $N_0 \bigotimes_R M$ by hypothesis. Thus we see $\sum_{i=1}^m u_i \otimes x_i = 0$ in $N \bigotimes_R M$ from this diagram.

(g)$\Rightarrow$(a). Let A be a submodule of a module B. Pick a free module F and an epimorphism $g : F \to B$. Set $N = g^{-1}(A)$ and $K = \ker(g)$. Then we have the following commutative diagram (a pullback diagram) with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Tensoring by M we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K\bigotimes_R M & \longrightarrow & N\bigotimes_R M & \xrightarrow{g} & A\bigotimes_R M & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow f & & \\
K\bigotimes_R M & \longrightarrow & F\bigotimes_R M & \xrightarrow{g} & B\bigotimes_R M & \longrightarrow & 0
\end{array}$$

Since

$$N\bigotimes_R M \to F\bigotimes_R M$$

is monomorphic by hypothesis, $A\bigotimes_R M \to B\bigotimes_R M$ is monomorphic by *Five Lemma*. □

Example 3.3. Every flat R-module is ϕ-flat. If R is a domain, then every ϕ-flat R-module is flat.

We know that flatness of R-modules is a local property. The following two results imply ϕ-flatness is also a local property.

Theorem 3.4. *Let M be a ϕ-flat R-module, and S be a multiplicative set in the ring R. Then M_S is a ϕ-flat R-module.*

Proof. If M is a ϕ-flat R-module, then $0 \to IM \to M \to M/IM \to 0$ is an exact sequence for any nonnil ideal I. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IM\bigotimes_R R_S & \longrightarrow & M\bigotimes_R R_S & \longrightarrow & M/IM\bigotimes_R R_S & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & I\bigotimes_R M_S & \longrightarrow & R\bigotimes_R M_S & \longrightarrow & R/I\bigotimes_R M_S & \longrightarrow & 0
\end{array}$$

Thus M_S is a ϕ-flat R-module. □

Theorem 3.5. *Let M be a R-module. The following conditions are equivalent*:

(a) *M is a ϕ-flat R-module.*
(b) *M_P is a ϕ-flat R_P-module for each prime ideal P of R.*
(c) *M_m is a ϕ-flat R_m-module for each prime ideal m of R.*

Proof. (a)⇒(b). Let J be an ideal of R_P with $J \nsubseteq Nil(R_P) = (Nil(R))_P$. Set $I = \{r \in R \mid \frac{r}{1} \in R_P\}$, we have $I_P = J \nsubseteq (Nil(R))_P$, thus I ia a nonnil ideal of R. The exact sequence $0 \to I \bigotimes_R M \to R \bigotimes_R M$ implies that $0 \to I_P \bigotimes_{R_P} M_P \to R_P \bigotimes_{R_P} M_P$ is exact. Therefore, M_P is a ϕ-flat R_P-module for each prime ideal P of R.

(b)⇒(c). It is trivial.

(c)⇒(a). If $Nil(R)$ is a maximal ideal of R, then $R_{Nil(R)} = R$, and $M_{Nil(R)} \cong R_{Nil(R)} \bigotimes_R M \cong R \bigotimes_R M \cong M$. Suppose that $Nil(R)$ is not a maximal ideal of R. If I is a nonnil ideal of R, then I_m is a nonnil ideal of R_m for any maximal ideal m. The exact sequence $0 \to I_m \bigotimes_{R_m} M_m \to R_m \bigotimes_{R_m} M_m$ implies that $0 \to I \bigotimes_R M \to R \bigotimes_R M$ is exact, thus M is a ϕ-flat R-module. □

Theorem 3.6. *Let $f : R \to T$ be an epimorphism from rings R to T. If M is a ϕ-flat R-module, then $M \bigotimes_R T$ is a ϕ-flat T-module.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact T-sequence, where C is a ϕ-torsion module. By Theorem 2.9, $0 \to A \to B \to C \to 0$ is also an exact R-sequence, and C is a ϕ-torsion module. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \bigotimes_R M & \longrightarrow & B \bigotimes_R M & \longrightarrow & C \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & A \bigotimes_T T \bigotimes_R M & \longrightarrow & B \bigotimes_T T \bigotimes_R M & \longrightarrow & C \bigotimes_T T \bigotimes_R M & \longrightarrow & 0
\end{array}$$

The above row exact implies the below row exact, thus $M \bigotimes_R T$ is a ϕ-flat T-module. □

Corollary 3.7. *Let M be a ϕ-flat R-module and I be an ideal of R. Then M/IM is a ϕ-flat R/I-module.*

Theorem 3.8. *Let R be a ϕ-ring, M be a R-module and I be an ideal of R. If $I \subseteq Nil(R)$ and $I \bigotimes_R M \cong IM$. Then M is a ϕ-flat R-module if and only if M/IM is a ϕ-flat R/I-module.*

Proof. We suppose M/IM is a ϕ-flat R/I-module. For any nonnil ideal J of R, consider the following commutative diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & J/I \bigotimes_{R/I} R/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_{R/I} R/I \bigotimes_R M \\
 & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & J/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_R M
\end{array}$$

The above row exact implies the below row exact, thus consider the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_R M & \longrightarrow & R/J \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & JM/IM & \longrightarrow & M/IM & \longrightarrow & M/JM & \longrightarrow & 0
\end{array}$$

Thus, $J/I \bigotimes_R M \cong Jm/IM$. Consider the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \bigotimes_R M & \longrightarrow & J \bigotimes_R M & \longrightarrow & J/I \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & IM & \longrightarrow & JM & \longrightarrow & JM/IM & \longrightarrow & 0
\end{array}$$

We obtain $J \bigotimes_R M \cong JM$, thus M is a ϕ-flat R-module. □

4. On ϕ-von Neumann regular rings

Recall that a ring R is said to be von Neumann regular if every R-module is flat. A ring R is von Neumann regular, if and only if there is an element $x \in R$ such that $a = xa^2$ for each $a \in R$, if and only if every principal ideal I of R is generated by an idempotent, if and only if every finitely generated ideal I of R is generated by an idempotent. A ring R is π-regular if for each $r \in R$ there is a positive integer n and an element $x \in R$ such that $r^{2n}x = r^n$.

We define a ϕ-ring R to be a *ϕ-von Neumann regular ring* if every R-module is ϕ-flat.

Theorem 4.1. *Let R be a ϕ-ring. The following conditions are equivalent*:

(a) *R is a ϕ-von Neumann regular ring.*

(b) *R is zero-dimensional.*

(c) *There is a nonnil element $x \in R$ such that $a = xa^2$ for any nonnilpotent element $a \in R$.*

(d) *Every nonnil principal ideal I of R is generated by an idempotent element $e \in R$.*

(e) *Every finite generated nonnil ideal I of R is generated by an idempotent element $e \in R$.*

(f) *$R/Nil(R)$ is a von Neumann regular ring.*

(g) *R is π-regular.*

Proof. (a)⇒(b). It is trivial.

(a)⇒(c). For each nonnilpotent $a \in R$, $0 \to Ra \to R \to R/Ra \to 0$ is exact. Since R/Ra is ϕ-flat, $Ra = Ra \bigcap Ra = Ra^2$ by Theorem 3.2. Then there is a nonnilpotent element $x \in R$ such that $a = xa^2$.

(c)⇒(d). Let $I = Ra$, $a \overline{\in} Nil(R)$. Then $a = xa^2$ for some $x \overline{\in} Nil(R)$. Thus $e = xa \in R$ is idempotent. By $e = xa \in Ra$ and $a = ea \in Re$, we have $I = Ra = Re$.

(d)⇒(e). Let $I = Ra_1 + \cdots + Ra_n$ be a nonnil ideal of R. By the condition that R is a ϕ-ring, we may assume that each a_i is idempotent. For any $x \in I$, $x = r_1a_1 + \cdots + r_na_n = r_1a_1^2 + \cdots + r_na_n^2 \in I^2$. Thus $I^2 = I$. therefore, I is generated by an idempotent element.

(e)⇒(a). Let B be an R-module and let $0 \to A \to F \to B \to 0$ be exact, where F is free. Let I be a finitely generated nonnil ideal of R. Then $I = Re$ for some idempotent e by hypothesis. For $x \in A \bigcap IF$, we have $x = ey =$

$e^2y = ex \in IA$ (where $y \in F$). Hence $A \bigcap IF = IA$. Therefore, B is ϕ-flat by Theorem 3.2

(d)⇒(f). Let $I = Ra/Nil(R) = (\overline{a})R/Nil(R)$ be a principal ideal of $R/Nil(R)$. We have Ra is a principal ideal of R, thus $Ra = Re$ for some idempotent element $e \in R$. Therefore, I is generated by an idempotent element $\overline{e} \in R/Nil(R)$.

(f)⇒(e). Let I be a finitely generated nonnil ideal of R, then $I/Nil(R)$ is also a finitely generated nonzero ideal of $R/Nil(R)$. Therefore, $I/Nil(R) = (\overline{e})$ for some idempotent $e \in R$, and I is generated by e.

(f)⇔(g). See Theorem 3.1 in [17]. □

Example 4.2. Let k be a field, and B be a k-linear space. $R = k(+)B$ is the idealization of k in B. Set $N = 0(+)B$, then $R/N \cong k$ is a field, and $Nil(R) = N \neq 0$. Therefore, R is a ϕ-von Neumann regular ring, but not a von Neumann regular ring.

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