J. Korean Math. Soc. **50** (2013), No. 1, pp. 137–159
http://dx.doi.org/10.4134/JKMS.2013.50.1.137

ESSENTIAL NORMS OF LINEAR COMBINATIONS OF COMPOSITION OPERATORS ON h^∞

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Abstract. It is studied the linear combinations of composition operators on the Banach space of bounded harmonic functions on the open unit disk. We determine the essential norm of them.

1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ the unit circle. We denote by $h^\infty = h^\infty(\mathbb{D})$ and $H^\infty = H^\infty(\mathbb{D})$ the sets of bounded harmonic and analytic functions on $\mathbb{D}$, respectively. Then h^∞ and H^∞ are the Banach spaces with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

We denote by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. For $\varphi \in \mathcal{S}(\mathbb{D})$ and a harmonic function f, the composite function $f \circ \varphi$ is also harmonic on $\mathbb{D}$. So each self-map φ induces the composition operator C_φ defined on h^∞ by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in h^\infty.$$

Composition operators have been investigated on various analytic function spaces (see [2, 13]). Recently, the norm, the essential norm and the topological structure of composition operators on H^∞ have been studied (see [5, 6, 8, 9, 10, 12]). But the exact value of the essential norm of the difference of composition operators $\|C_\varphi - C_\psi\|_e$ on H^∞ is not yet known. In [1], Choa, Ohno and the first author studied composition operators on h^∞ and determined the exact value of $\|C_\varphi - C_\psi\|_e$ on h^∞.

In [5], Gorkin and Mortini studied the norm and the essential norm of linear combinations of endomorphisms on uniform algebras. They gave a sufficient condition for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ to satisfy $\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\|_e = \sum_{j=1}^N |\lambda_j|$. In [11], Ohno and the first author studied the norm and the essential norm of linear

Received January 17, 2012.
2010 *Mathematics Subject Classification.* 47B33, 46J15.
Key words and phrases. essential norm, linear combination of composition operators, Banach space of bounded harmonic functions.
The first author is partially supported by Grant-in-Aid for Scientific Research (No. 21540166), Japan Society for the Promotion of Science.

combinations of composition operators on H^∞. They gave a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on H^∞ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$ and also gave a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on H^∞ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e = \sum_{j=1}^N |\lambda_j|$ under the assumption that $\operatorname{Re}\lambda_j > 0$ for every $1 \le j \le N$.

In Section 2, we study the norm of linear combinations of composition operators on h^∞, and we shall give a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on h^∞ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$ on h^∞. We also characterize the compactness of linear combinations of composition operators on h^∞. In Section 3, we shall determine the essential norm of linear combinations of composition operators on h^∞, and give a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on h^∞ to satisfy $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e = \sum_{j=1}^N |\lambda_j|$. In [11], the essential norm $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on H^∞ was studied, but the exact value of $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on H^∞ is not known. One reason is that it is not known the existence of enough many concrete compact operators on H^∞. But in the case of h^∞, there are a lot of concrete compact operators on h^∞, so we may give the exact value of $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$ on h^∞.

Generally it holds

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e \le \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| \le \sum_{j=1}^N |\lambda_j|.$$

In Section 4, we shall give some examples concerning with the above inequalities.

2. Norms of linear combinations

For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |z-w|/|1-\overline{z}w|.$$

The spaces $L^1(\partial\mathbb{D})$ and $L^\infty(\partial\mathbb{D})$ stand for the standard Lebesgue spaces with the norms $\|f\|_1$ and $\|f\|_\infty$, respectively. For $f \in L^1(\partial\mathbb{D})$, let

$$\hat{f}(z) = \int_{\partial\mathbb{D}} f(e^{i\theta}) P_z(e^{i\theta})\, d\sigma(e^{i\theta}), \quad z \in \mathbb{D},$$

where

$$P_z(e^{i\theta}) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}, \quad z \in \mathbb{D},$$

is the Poisson kernel for $z \in \mathbb{D}$ and σ is the normalized Lebesgue measure on $\partial\mathbb{D}$. For each $f \in h^\infty$, there exists the radial limit function f^* on $\partial\mathbb{D}$ defined by $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on $\partial\mathbb{D}$. It is well known that $\{f^* : f \in h^\infty\} = L^\infty(\partial\mathbb{D})$ and $f = \widehat{f^*}$, so identifying f with f^* we may consider $h^\infty = L^\infty(\partial\mathbb{D})$. Note that

$$(C_\varphi f)^*(e^{i\theta}) = \begin{cases} f(\varphi^*(e^{i\theta})) & \text{for } |\varphi^*(e^{i\theta})| < 1 \\ f^*(\varphi^*(e^{i\theta})) & \text{for } |\varphi^*(e^{i\theta})| = 1 \end{cases}$$

a.e. on $\partial\mathbb{D}$.

We denote by $B(h^\infty)$ the closed unit ball of h^∞. For a measurable subset E of $\partial\mathbb{D}$, let χ_E denote the characteristic function for E.

Lemma 2.1. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then*

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\| = \sup_{z\in\mathbb{D}} \Big\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z)} \Big\|_1.$$

Proof. For $g \in B(h^\infty)$, we have

$$\begin{aligned} \Big\| \Big(\sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big) g \Big\|_\infty &= \sup_{z\in\mathbb{D}} \Big| \sum_{j=1}^{N} \lambda_j g(\varphi_j(z)) \Big| \\ &= \sup_{z\in\mathbb{D}} \Big| \int_{\partial\mathbb{D}} \Big(\sum_{j=1}^{N} \lambda_j P_{\varphi_j(z)} \Big) g^* \, d\sigma \Big|. \end{aligned}$$

Thus we get the assertion. $\square$

By the proof of Lemma 2.1, one easily sees the following.

Lemma 2.2. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then there exists a sequence $\{z_n\}_n$ in $\mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ satisfying*

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\| = \lim_{n\to\infty} \Big\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z_n)} \Big\|_1.$$

The following is an elementary property of Poisson kernels.

Lemma 2.3. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/|\lambda_1| \neq \lambda_2/|\lambda_2|$. Let $\{z_n\}_n, \{w_n\}_n$ be sequences in $\mathbb{D}$. If $\|\lambda_1 P_{z_n} + \lambda_2 P_{w_n}\|_1 \to |\lambda_1| + |\lambda_2|$, then $\|P_{z_n} - P_{w_n}\|_1 \to 2$ as $n \to \infty$.*

By [4, p. 42], we have the following.

Lemma 2.4. *For $z, w \in \mathbb{D}$,*

$$\|P_z - P_w\|_1 = 2 - \frac{4\cos^{-1}\rho(z,w)}{\pi}.$$

Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$. We use the same notations as in [11]. We denote by $\mathcal{Z}(\{\varphi_j\})$ the set of sequences $\{z_n\}_n$ in $\mathbb{D}$ satisfying the following four conditions:

(a) $\{z_n\}_n$ is a convergent sequence.
(b) $\{\varphi_j(z_n)\}_n$ is a convergent sequence for every $1 \leq j \leq N$.
(c) $\lim_{n\to\infty} |\varphi_j(z_n)| = 1$ for some $1 \leq j \leq N$.
(d) $\{\rho(\varphi_i(z_n), \varphi_j(z_n))\}_n$ is a convergent sequence for every $1 \leq i, j \leq N$.

In this paper, the set $\mathcal{Z}(\{\varphi_j\})$ acts an important role. Note that for a sequence $\{z_n\}_n$ in $\mathbb{D}$, if $|\varphi_j(z_n)| \to 1$ as $n \to \infty$ for some $1 \le j \le N$, then it is easy to see that there exists a subsequence $\{z_{n_i}\}_i$ of $\{z_n\}_n$ satisfying $\{z_{n_i}\}_i \in \mathcal{Z}(\{\varphi_j\})$.

Let $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$. We write $\alpha_j = \lim_{n\to\infty} \varphi_j(z_n)$ for every $1 \le j \le N$. Let

$$I(\{z_n\}) = \{j : |\alpha_j| = 1, 1 \le j \le N\}. \tag{2.1}$$

We define the equivalence relation $i \sim j$ in $I(\{z_n\})$ by

$$\lim_{n\to\infty} \rho(\varphi_i(z_n), \varphi_j(z_n)) = 0. \tag{2.2}$$

By condition (c) and (2.1), $I(\{z_n\}) \neq \emptyset$. For each $t \in I(\{z_n\})$, let

$$I(\{z_n\}, t) = \{j \in I(\{z_n\}) : j \sim t, 1 \le j \le N\}. \tag{2.3}$$

For $s, t \in I(\{z_n\})$, either $I(\{z_n\}, s) = I(\{z_n\}, t)$ or $I(\{z_n\}, s) \cap I(\{z_n\}, t) = \emptyset$ holds. Hence there is a subset $\{t_1, t_2, \dots, t_\ell\} \subset I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I(\{z_n\}, t_p)$ and $I(\{z_n\}, t_p) \cap I(\{z_n\}, t_q) = \emptyset$ for $p \neq q$.

Generally, we have $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| \le \sum_{j=1}^N |\lambda_j|$. If $\lambda_1/|\lambda_1| = \lambda_j/|\lambda_j|$ for every $1 \le j \le N$, one easily sees that $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \| = \sum_{j=1}^N |\lambda_j|$. The other case, we have the following.

Theorem 2.5. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \le j \le N$. Suppose that $\lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|$ for some $1 \le i, j \le N$. Then the following conditions are equivalent*:

(i) $\displaystyle \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| = \sum_{j=1}^n |\lambda_j|$ *on* h^∞.

(ii) $\displaystyle \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| = \sum_{j=1}^n |\lambda_j|$ *on* H^∞.

(iii) *There exists a sequence $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$ satisfying $\rho(\varphi_i(z_n), \varphi_j(z_n)) \to 1$ as $n \to \infty$ for every $1 \le i, j \le N$ with $\lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|$.*

Proof. (ii) $\Leftrightarrow$ (iii) was proven in [11, Theorem 3.1]. (ii) $\Rightarrow$ (i) is trivial.

We shall prove (i) $\Rightarrow$ (iii). Suppose that (i) holds. By Lemma 2.2, there is a sequence $\{z_n\}_n$ in $\mathbb{D}$ with $|z_n| \to 1$ such that

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^N \lambda_j P_{\varphi_j(z_n)} \Big\|_1 = \sum_{j=1}^N |\lambda_j|.$$

Suppose that $\lambda_{j_1}/|\lambda_{j_1}| \neq \lambda_{j_2}/|\lambda_{j_2}|$. Then

$$\sum_{j=1}^N |\lambda_j| = \lim_{n\to\infty} \Big\| \sum_{j=1}^N \lambda_j P_{\varphi_j(z_n)} \Big\|_1$$

$$\le \liminf_{n\to\infty} \left\| \lambda_{j_1} P_{\varphi_{j_1}(z_n)} + \lambda_{j_2} P_{\varphi_{j_2}(z_n)} \right\|_1 + \sum_{j:j\neq j_1,j_2} |\lambda_j|.$$

Hence

$$\left\| \lambda_{j_1} P_{\varphi_{j_1}(z_n)} + \lambda_{j_2} P_{\varphi_{j_2}(z_n)} \right\|_1 \to |\lambda_{j_1}| + |\lambda_{j_2}|$$

as $n \to \infty$. By Lemma 2.3, $\|P_{\varphi_{j_1}(z_n)} - P_{\varphi_{j_2}(z_n)}\|_1 \to 2$. By Lemma 2.4, $\rho(\varphi_{j_1}(z_n), \varphi_{j_2}(z_n)) \to 1$. Hence $\max\{|\varphi_{j_1}(z_n)|, |\varphi_{j_2}(z_n)|\} \to 1$ as $n \to \infty$. Thus we get (iii). □

To study the compactness of $\sum_{j=1}^N \lambda_j C_{\varphi_j}$, we use the following lemma which follows from that $B(h^\infty)$ is a normal family.

Lemma 2.6. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \le j \le N$. Then $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on h^∞ if and only if $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} f_n\|_\infty \to 0$ as $n \to \infty$ for every sequence $\{f_n\}_n$ in $B(h^\infty)$ such that $\{f_n\}_n$ converges to 0 uniformly on any compact subset of $\mathbb{D}$.*

From this, if $\|\varphi\|_\infty < 1$, then C_φ is compact on h^∞. The following is a characterization of the compactness of $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ on h^∞.

Theorem 2.7. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \le j \le N$. Then the following conditions are equivalent*:

(i) *$\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on h^∞.*
(ii) *$\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on H^∞.*
(iii) *$\sum \{\lambda_i : i \in I(\{z_n\}, t)\} = 0$ for every $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$ and $t \in I(\{z_n\})$.*

Proof. (ii) ⇔ (iii) was proven in [11, Theorem 2.2]. (i) ⇒ (ii) is trivial.

The proof (iii) ⇒ (i) is the same as the one in [11, Theorem 2.2] essentially. Suppose that $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is not compact on h^∞. By Lemma 2.6, there is a sequence $\{f_n\}_n$ in $B(h^\infty)$ such that $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$ and $\| \sum_{j=1}^N \lambda_j f_n \circ \varphi_j\|_\infty \not\to 0$ as $n \to \infty$. Considering a subsequence of $\{f_n\}_n$, we may assume that there exists $\delta > 0$ such that

$$\Big\| \sum_{j=1}^N \lambda_j f_n \circ \varphi_j \Big\|_\infty > \delta \quad \text{for every } n \ge 1.$$

Take a sequence $\{z_n\}_n$ in $\mathbb{D}$ satisfying $|z_n| \to 1$ and

$$\Big| \sum_{j=1}^N \lambda_j f_n(\varphi_j(z_n)) \Big| > \delta \quad \text{for every } n \ge 1.$$

We may assume that $\varphi_j(z_n) \to \alpha_j \in \overline{\mathbb{D}}$ for every $1 \le j \le N$. Since $f_n \to 0$ uniformly on any compact subset of $\mathbb{D}$, $|\alpha_j| = 1$ for some j. Considering a subsequence of $\{f_n\}_n$, we may assume that $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$. By (2.1), $I(\{z_n\}) = \{j : |\alpha_j| = 1, 1 \le j \le N\}$. Then we have

$$\liminf_{k\to\infty} \Big| \sum_{j\in I(\{z_n\})} \lambda_j f_k(\varphi_j(z_k)) \Big| \ge \delta. \tag{2.4}$$

Let $\{t_1, t_2, \dots, t_\ell\} \subset I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{\ell} I(\{z_n\}, t_p)$ and $I(\{z_n\}, t_p) \cap I(\{z_n\}, t_q) = \emptyset$ for $p \ne q$. Let $j \in I(\{z_n\}, t_p)$. By (2.2) and (2.3), $\rho(\varphi_j(z_k), \varphi_{t_p}(z_k)) \to 0$ as $k \to \infty$. By Lemma 2.4, $\|P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)}\|_1 \to 0$. Since $\{f_k(\varphi_j(z_k))\}_k$ is a bounded sequence, considering a subsequence of $\{f_k\}_k$, we may assume that $f_k(\varphi_j(z_k)) \to \gamma_j \in \overline{\mathbb{D}}$ as $k \to \infty$ for every $1 \le j \le N$. We have

$$\begin{aligned} |f_k(\varphi_j(z_k)) - f_k(\varphi_{t_p}(z_k))| &= \Big| \int_{\partial\mathbb{D}} f_k^*(P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)})\, d\sigma \Big| \\ &\le \|P_{\varphi_j(z_k)} - P_{\varphi_{t_p}(z_k)}\|_1 \\ &\to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

Thus we get $\gamma_j = \gamma_{t_p}$ for every $j \in I(\{z_n\}, t_p)$. Therefore

$$\begin{aligned} \lim_{k\to\infty} \sum_{j\in I(\{z_n\})} \lambda_j f_k(\varphi_j(z_k)) &= \lim_{k\to\infty} \sum_{p=1}^{\ell} \sum_{j\in I(\{z_n\},t_p)} \lambda_j f_k(\varphi_j(z_k)) \\ &= \sum_{p=1}^{\ell} \gamma_{t_p} \sum_{j\in I(\{z_n\},t_p)} \lambda_j \\ &= 0 \qquad \text{by condition (iii).} \end{aligned}$$

This contradicts with (2.4). Thus we get (iii) $\Rightarrow$ (i). □

3. Essential norms of linear combinations

Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \ne \varphi_j$ for $i \ne j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \ne 0$ for every $1 \le j \le N$. Let $\mathcal{K}$ be the set of compact operators on h^∞. The essential norm is defined by

$$\Big\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \Big\|_e = \inf_{K\in\mathcal{K}} \Big\| K + \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \Big\|.$$

For each $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$, we define

$$\Gamma(\{z_n\}) = \liminf_{k\to\infty} \Big\| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big\|_1. \tag{3.1}$$

This term is used to determine the values of essential norms of linear combinations of composition operators. The following is an elementary property of Poisson kernels.

Lemma 3.1. *Let $\{z_{j,k}\}_k$ be sequences in $\mathbb{D}$ such that $z_{j,k} \to \alpha_j$ as $k \to \infty$ and $|\alpha_j| = 1$ for every $1 \le j \le N$. Let U be an open subset of $\partial\mathbb{D}$ satisfying $\{\alpha_j\}_{j=1}^N \subset U$. Then for $\lambda_j \in \mathbb{C}, 1 \le j \le N$, we have*

$$\lim_{k\to\infty} \int_{\partial\mathbb{D}} \chi_U \Big| \sum_{j=1}^N \lambda_j P_{z_{j,k}} \Big| \, d\sigma = \lim_{k\to\infty} \Big\| \sum_{j=1}^N \lambda_j P_{z_{j,k}} \Big\|_1.$$

First, we give a lower estimate of $\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \|_e$.

Theorem 3.2. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \ne \varphi_j$ for $i \ne j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \ne 0$ for every $1 \le j \le N$. Then*

$$\sup_{\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})} \Gamma(\{z_n\}) \le \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e.$$

Proof. Let $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$. We may assume that $\Gamma(\{z_n\}) > 0$. Considering a subsequence of $\{z_n\}_n$, we may assume that

$$\Gamma(\{z_n\}) = \lim_{k\to\infty} \Big\| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big\|_1 \tag{3.2}$$

and

$$\Big\| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big\|_1 \ne 0 \quad \text{for every } k \ge 1.$$

By condition (b) in Section 2, $\varphi_j(z_k) \to \alpha_j \in \overline{\mathbb{D}}$ as $k \to \infty$ for every j. Recall that $|\alpha_j| = 1$ for $j \in I(\{z_n\})$ and $|\alpha_j| < 1$ for $j \notin I(\{z_n\})$.

By induction, we shall take a subsequence $\{z_{n_k}\}_k$ of $\{z_n\}_n$ and a sequence of open subsets $\{U_k\}_k$ of $\partial\mathbb{D}$ satisfying the following two conditions;

$$\{\alpha_j : j \in I(\{z_n\})\} \subset U_{k+1} \subset U_k, \quad \{\alpha_j : j \in I(\{z_n\})\} = \bigcap_{k=1}^\infty U_k \tag{3.3}$$

and

$$\int_{\partial\mathbb{D}} \chi_{(U_k \setminus U_{k+1})} \Big| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \Big| \, d\sigma > \Gamma(\{z_n\}) \Big(1 - \frac{1}{k}\Big). \tag{3.4}$$

Put $n_1 = 1$. Then there is an open subset U_1 of $\partial\mathbb{D}$ with $\{\alpha_j : j \in I(\{z_n\})\} \subset U_1$ such that

$$\int_{\partial\mathbb{D}} \chi_{U_1} \Big| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_1})} \Big| \, d\sigma > 0.$$

We may take an open subset U_2 of $\partial\mathbb{D}$ with $\{\alpha_j : j \in I(\{z_n\})\} \subset U_2 \subset U_1$ such that

$$\int_{\partial\mathbb{D}} \chi_{(U_1 \setminus U_2)} \Big| \sum_{j \in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_1})} \Big| \, d\sigma > 0.$$

Let m be a positive integer. We assume that $\{z_{n_1}, z_{n_2}, \dots, z_{n_m}\}$ and $\{U_1, U_2, \dots, U_{m+1}\}$ are taken satisfying conditions (3.3) and (3.4). We have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\partial\mathbb{D}} \chi_{U_{m+1}} \Big| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big| \, d\sigma \\
&= \lim_{k\to\infty} \Big\| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big\|_1 \qquad \text{by Lemma 3.1} \\
&= \Gamma(\{z_n\}) \qquad \text{by (3.2).}
\end{aligned}$$

Hence there exists a positive integer n_{m+1} such that

$$\int_{\partial\mathbb{D}} \chi_{U_{m+1}} \Big| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_{m+1}})} \Big| \, d\sigma > \Gamma(\{z_n\}) \Big(1 - \frac{1}{m+1} \Big).$$

It is not difficult to take an open subset U_{m+2} of $\partial\mathbb{D}$ with

$$\{\alpha_j : j \in I(\{z_n\})\} \subset U_{m+2} \subset U_{m+1}$$

such that

$$\int_{\partial\mathbb{D}} \chi_{(U_{m+1}\setminus U_{m+2})} \Big| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_{m+1}})} \Big| \, d\sigma > \Gamma(\{z_n\}) \Big(1 - \frac{1}{m+1} \Big).$$

Of course we may take $\{U_k\}_k$ satisfying the second condition in (3.3). This completes the induction.

For each positive integer k, there exists a function $g_k \in L^\infty(\partial\mathbb{D})$ such that

$$g_k \Big(\sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \Big) = \Big| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_{n_k})} \Big| \quad \text{a.e. on } \partial\mathbb{D}.$$

Note that $|g_k| = 1$ a.e. on $\partial\mathbb{D}$. Let $h_k = \chi_{(U_k\setminus U_{k+1})} g_k$. Then $\|h_k\|_\infty = 1$, and by (3.3) $h_k \to 0$ weakly in $L^\infty(\partial\mathbb{D})$, so $\hat{h}_k \to 0$ weakly in h^∞. Let K be a compact operator on h^∞. Then we have $\|K\hat{h}_k\|_\infty \to 0$ and

$$\Big| \sum_{j\notin I(\{z_n\})} \lambda_j \hat{h}_k(\varphi_j(z_{n_k})) \Big| \to 0 \quad \text{as } k \to \infty.$$

Therefore

$$\begin{aligned}
&\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} + K \Big\| \\
&\geq \limsup_{k\to\infty} \Big\| \Big(\sum_{j=1}^N \lambda_j C_{\varphi_j} \Big) \hat{h}_k + K\hat{h}_k \Big\|_\infty \\
&= \limsup_{k\to\infty} \Big\| \sum_{j=1}^N \lambda_j \hat{h}_k \circ \varphi_j \Big\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{k\to\infty}\Big(\Big|\sum_{j\in I(\{z_n\})}\lambda_j\hat{h}_k(\varphi_j(z_{n_k}))\Big| - \Big|\sum_{j\notin I(\{z_n\})}\lambda_j\hat{h}_k(\varphi_j(z_{n_k}))\Big|\Big)\\
&= \limsup_{k\to\infty}\Big|\sum_{j\in I(\{z_n\})}\lambda_j\hat{h}_k(\varphi_j(z_{n_k}))\Big|\\
&= \limsup_{k\to\infty}\Big|\int_{\partial\mathbb{D}} h_k\Big(\sum_{j\in I(\{z_n\})}\lambda_j P_{\varphi_j(z_{n_k})}\Big)\,d\sigma\Big|\\
&= \limsup_{k\to\infty}\int_{\partial\mathbb{D}}\chi_{(U_k\setminus U_{k+1})}\Big|\sum_{j\in I(\{z_n\})}\lambda_j P_{\varphi_j(z_{n_k})}\Big|\,d\sigma\\
&\geq \Gamma(\{z_n\}) \qquad \text{by (3.4).}
\end{aligned}$$

Thus we get the assertion. □

Next we shall study an upper estimate of $\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\|_e$ and this is the main subject of this paper. For $g \in L^\infty(\partial\mathbb{D})$, we define the bounded linear operator M_g on h^∞ by

$$(M_g f)(z) = \int_{\partial\mathbb{D}} g f^* P_z\,d\sigma, \quad f\in h^\infty, \quad z\in\mathbb{D}.$$

Let U, V be measurable subsets of $\partial\mathbb{D}$. Then $M_{\chi_U}M_{\chi_V} = M_{\chi_{(U\cap V)}}$ and $I = M_{\chi_U} + M_{\chi_{U^c}}$, where I is the identity operator on h^∞.

Lemma 3.3. *Let $\varphi \in \mathcal{S}(\mathbb{D})$ with $\|\varphi\|_\infty = 1$. For $0 < \delta < 1$, let U be a measurable subset of $\partial\mathbb{D}$ with $U \subset \{e^{i\theta}\in\partial\mathbb{D} : |\varphi^*(e^{i\theta})| \leq \delta\}$. Then $M_{\chi_U}C_\varphi$ is compact on h^∞.*

Proof. Let $\{f_n\}_n$ be a sequence in $B(h^\infty)$ such that $\{f_n\}_n$ converges uniformly on any compact subset of $\mathbb{D}$. Then $\sup_{e^{it}\in U}|f_n(\varphi^*(e^{it}))| \to 0$ as $n\to\infty$. Hence

$$\|M_{\chi_U}C_\varphi f_n\|_\infty = \|\chi_U(f_n\circ\varphi)^*\|_\infty = \sup_{z\in\mathbb{D}}\Big|\int_U f_n(\varphi^*)P_z\,d\sigma\Big| \to 0$$

as $n\to\infty$. By Lemma 2.6, $M_{\chi_U}C_\varphi$ is compact on h^∞. □

One easily checks the following.

Lemma 3.4. *Let $U_1, U_2, \ldots, U_m$ be measurable subsets of $\partial\mathbb{D}$ with $U_i\cap U_j = \emptyset$ for $i\neq j$. For every bounded linear operators $T_1, T_2, \ldots, T_m$ on h^∞, we have*

$$\Big\|\sum_{j=1}^m M_{\chi_{U_j}}T_j\Big\| = \max_{1\leq j\leq m}\|M_{\chi_{U_j}}T_j\|.$$

Lemma 3.5. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ satisfying $\varphi_i\neq\varphi_j$ for $i\neq j$ and $\lambda_j\in\mathbb{C}$ with $\lambda_j\neq 0$ for every $1\leq j\leq N$. Then*

$$\Big\|M_{\chi_U}\sum_{j=1}^N\lambda_j C_{\varphi_j}\Big\|_e = \inf_{K\in\mathcal{K}}\Big\|M_{\chi_U}\Big(K + \sum_{j=1}^N\lambda_j C_{\varphi_j}\Big)\Big\|.$$

Proof. It is trivial that

$$\Big\| M_{\chi_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e \leq \inf_{K \in \mathcal{K}} \Big\| M_{\chi_U} \Big(K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big) \Big\|.$$

Let $K \in \mathcal{K}$. Then

$$\begin{aligned} &\Big\| K + M_{\chi_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\| \\ &= \Big\| (M_{\chi_U} + M_{\chi_{U^c}})K + M_{\chi_U} \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\| \\ &= \Big\| M_{\chi_U} \Big(K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big) + M_{\chi_{U^c}} K \Big\| \\ &\geq \Big\| M_{\chi_U} \Big(K + \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big) \Big\| \qquad \text{by Lemma 3.4.} \end{aligned}$$

Therefore we get the assertion. □

Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$. For $0 < \delta < 1$, write

$$W_{\delta,j} = \{ e^{i\theta} \in \partial\mathbb{D} : |\varphi_j^*(e^{i\theta})| > \delta \}.$$

We define the family Λ by

$$\Lambda = \{ p = (p_1, p_2, \ldots, p_N) : p_j = 0 \text{ or } 1, 1 \leq j \leq N \}.$$

We use the following notations;

$$W_{\delta,j}^0 = W_{\delta,j} \quad \text{and} \quad W_{\delta,j}^1 = W_{\delta,j}^c = \partial D \setminus W_{\delta,j}. \tag{3.5}$$

For each $p = (p_1, p_2, \ldots, p_N) \in \Lambda$, write

$$W_{\delta,p} = \bigcap_{j=1}^{N} W_{\delta,j}^{p_j} \quad \text{and} \quad \tilde{p} = \{ j : p_j = 0, 1 \leq j \leq N \}. \tag{3.6}$$

Note that $W_{\delta,p}$ may be an empty set for some $p \in \Lambda$, and $W_{\delta,p} \cap W_{\delta,q} = \emptyset$ holds for $p, q \in \Lambda$ with $p \neq q$. We have

$$1 = \prod_{j=1}^{N} (\chi_{W_{\delta,j}} + \chi_{W_{\delta,j}^c}) = \sum_{p \in \Lambda} \chi_{W_{\delta,p}} \quad \text{on } \partial\mathbb{D}.$$

Hence $I = \sum_{p \in \Lambda} M_{\chi_{W_{\delta,p}}}$ on h^∞.

Lemma 3.6. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. For a measurable subset U of ∂D, $0 < \delta < 1$ and a nonempty subset $L \subset \{1, 2, \dots, N\}$, we have*

$$\Big\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|_e = \max_{p \in \Lambda} \Big\| M_{\chi_{(U \cap W_{\delta,p})}} \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e.$$

Proof. Let $p \in \Lambda$. If $j \notin \tilde{p}$, then by (3.5) and (3.6) $|\chi_{W_{\delta,p}} \varphi_j^*| \leq \delta$ on $\partial\mathbb{D}$. Hence by Lemma 3.3, $M_{\chi_{W_{\delta,p}}} C_{\varphi_j} \in \mathcal{K}$. Let $K_p \in \mathcal{K}$ for every $p \in \Lambda$. We have

$$\begin{aligned}
&\Big\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|_e \\
&= \Big\| M_{\chi_U} \Big(\sum_{p \in \Lambda} M_{\chi_{W_{\delta,p}}} \Big) \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|_e \\
&= \Big\| M_{\chi_U} \sum_{p \in \Lambda} \Big(M_{\chi_{W_{\delta,p}}} \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big) \Big\|_e \\
&\leq \Big\| \sum_{p \in \Lambda} M_{\chi_{(U \cap W_{\delta,p})}} \Big(K_p + \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big) \Big\| \\
&= \max_{p \in \Lambda} \Big\| M_{\chi_{(U \cap W_{\delta,p})}} \Big(K_p + \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big) \Big\| \qquad \text{by Lemma 3.4.}
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|_e \\
&\leq \max_{p \in \Lambda} \inf_{K_p \in \mathcal{K}} \Big\| M_{\chi_{(U \cap W_{\delta,p})}} \Big(K_p + \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big) \Big\| \\
&= \max_{p \in \Lambda} \Big\| M_{\chi_{(U \cap W_{\delta,p})}} \sum_{j \in L \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e \qquad \text{by Lemma 3.5.}
\end{aligned}$$

There also exists a sequence $\{K_n\}_n$ in $\mathcal{K}$ satisfying

$$\Big\| M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|_e = \lim_{n \to \infty} \Big\| K_n + M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\|.$$

We have

$$\begin{aligned}
&\Big\| K_n + M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big\| \\
&= \Big\| \Big(\sum_{p \in \Lambda} M_{\chi_{W_{\delta,p}}} \Big) \Big(K_n + M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big) \Big\| \\
&= \max_{p \in \Lambda} \Big\| M_{\chi_{W_{\delta,p}}} \Big(K_n + M_{\chi_U} \sum_{j \in L} \lambda_j C_{\varphi_j} \Big) \Big\| \qquad \text{by Lemma 3.4}
\end{aligned}$$

$$
\begin{aligned}
&= \max_{p\in\Lambda} \Big\| M_{\chi_{W_{\delta,p}}} \Big(K_n + M_{\chi_U} \sum_{j\in L\cap\tilde{p}} \lambda_j C_{\varphi_j} + M_{\chi_U} \sum_{j\in L\setminus\tilde{p}} \lambda_j C_{\varphi_j} \Big) \Big\| \\
&\geq \max_{p\in\Lambda} \Big\| M_{\chi_{(U\cap W_{\delta,p})}} \sum_{j\in L\cap\tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e .
\end{aligned}
$$

Therefore we get

$$
\Big\| M_{\chi_U} \sum_{j\in L} \lambda_j C_{\varphi_j} \Big\|_e \geq \max_{p\in\Lambda} \Big\| M_{\chi_{(U\cap W_{\delta,p})}} \sum_{j\in L\cap\tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e . \qquad \square
$$

Lemma 3.7. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Let U be a measurable subset of $\partial\mathbb{D}$ with $\sigma(U) > 0$ and a nonempty subset $L \subset \{1, 2, \dots, N\}$. Let $0 < \delta_i < 1$ for $i = 1, 2$. Suppose that $|\varphi_j^*| > \delta_1$ a.e. on U for every $j \in L$ and $|\varphi_j^*| \leq \delta_2$ a.e. on U for every $j \notin L$. Then there is a sequence $\{z_n\}_n$ in $\mathbb{D}$ satisfying the following conditions*:

(i) $\Big\| M_{\chi_U} \sum_{j\in L} \lambda_j C_{\varphi_j} \Big\| \leq \liminf_{n\to\infty} \Big\| \sum_{j\in L} \lambda_j P_{\varphi_j(z_n)} \Big\|_1 .$

(ii) $\Big| \Big(\prod_{j\in L} \varphi_j \Big)(z_n) \Big| > \delta_1^N$ *for every* $n \geq 1$.

(iii) $|\varphi_j(z_n)| < (1 + \delta_2)/2$ *for every* $n \geq 1$ *and* $j \notin L$.

Proof. Note that

$$
\Big| \prod_{j\in L} \varphi_j^* \Big| > \delta_1^N \quad \text{a.e. on } U \tag{3.7}
$$

and

$$
A := \Big\| M_{\chi_U} \sum_{j\in L} \lambda_j C_{\varphi_j} \Big\| > 0.
$$

For each positive integer n, there exists $f_n \in B(h^\infty)$ satisfying

$$
A - \frac{1}{n} < \Big\| \Big(M_{\chi_U} \sum_{j\in L} \lambda_j C_{\varphi_j} \Big) f_n \Big\|_\infty \leq A,
$$

that is,

$$
A - \frac{1}{n} < \Big\| \chi_U \sum_{j\in L} \lambda_j (f_n \circ \varphi_j)^* \Big\|_\infty \leq A.
$$

By (3.7), there is $z_n \in \mathbb{D}$ such that

$$
\Big| \Big(\prod_{j\in L} \varphi_j \Big)(z_n) \Big| > \delta_1^N
$$

and

$$
A - \frac{1}{n} < \Big| \int_{\partial\mathbb{D}} \chi_U \Big(\sum_{j\in L} \lambda_j (f_n \circ \varphi_j)^* \Big) P_{z_n} \, d\sigma \Big| < A. \tag{3.8}
$$

We have

$$\begin{aligned}A-\frac{1}{n} &< \Big|\int_U \sum_{j\in L}\lambda_j (f_n\circ\varphi_j)^* P_{z_n}\,d\sigma\Big|\\ &= \Big|\int_{\partial\mathbb{D}} \sum_{j\in L}\lambda_j (f_n\circ\varphi_j)^* P_{z_n}\,d\sigma - \int_{U^c} \sum_{j\in L}\lambda_j (f_n\circ\varphi_j)^* P_{z_n}\,d\sigma\Big|\\ &\le \Big|\sum_{j\in L}\lambda_j f_n(\varphi_j(z_n))\Big| + \Big|\int_{U^c} \sum_{j\in L}\lambda_j (f_n\circ\varphi_j)^* P_{z_n}\,d\sigma\Big|\\ &:= I_1(n)+I_2(n) \qquad \text{say.}\end{aligned}$$

We have

$$I_1(n)\le \Big\|\sum_{j\in L}\lambda_j P_{\varphi_j(z_n)}\Big\|_1.$$

Also by (3.8),

$$A-\frac{1}{n} < \Big|\int_U \sum_{j\in L}\lambda_j (f_n\circ\varphi_j)^* P_{z_n}\,d\sigma\Big| \le A\int_U P_{z_n}\,d\sigma,$$

so we get $A\int_{U^c} P_{z_n}\,d\sigma < 1/n$. Since $A\neq 0$, $\int_{U^c} P_{z_n}\,d\sigma\to 0$. Hence we have $I_2(n)\to 0$ as $n\to\infty$. Therefore

$$A\le \liminf_{n\to\infty}\Big\|\sum_{j\in L}\lambda_j P_{\varphi_j(z_n)}\Big\|_1.$$

Thus we get (i).

Let $j\notin L$. By the assumption, $|\varphi_j^*|\le\delta_2$ a.e. on U. Then

$$|\varphi_j(z_n)|\le \Big|\int_U \varphi_j^* P_{z_n}\,d\sigma\Big| + \Big|\int_{U^c} \varphi_j^* P_{z_n}\,d\sigma\Big| \le \delta_2 + \Big|\int_{U^c} P_{z_n}\,d\sigma\Big|.$$

Thus we get

$$\limsup_{n\to\infty}|\varphi_j(z_n)|\le\delta_2.$$

Considering a subsequence of $\{z_n\}_n$, we have (iii). □

Theorem 3.8. *Let $\varphi_1,\varphi_2,\dots,\varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty=1$ satisfying $\varphi_i\neq\varphi_j$ for $i\neq j$ and $\lambda_j\in\mathbb{C}$ with $\lambda_j\neq 0$ for every $1\le j\le N$. Then there exists $\{z_n\}_n\in\mathcal{Z}(\{\varphi_j\})$ satisfying*

$$\Big\|\sum_{j=1}^N\lambda_j C_{\varphi_j}\Big\|_e\le\Gamma(\{z_n\}).$$

Proof. We may assume that

$$\Big\|\sum_{j=1}^N\lambda_j C_{\varphi_j}\Big\|_e>0. \tag{3.9}$$

Take a sequence $\{\delta_n\}_n$ satisfying $0 < \delta_n < \delta_{n+1} < 1$ and $\delta_n \to 1$. By Lemma 3.6,

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \max_{p \in \Lambda} \Big\| M_{\chi_{W_{\delta_1,p}}} \sum_{j \in \tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e .$$

Hence there exists $p^1 \in \Lambda$ satisfying

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \Big\| M_{\chi_{W_{\delta_1,p^1}}} \sum_{j \in \widetilde{p^1}} \lambda_j C_{\varphi_j} \Big\|_e .$$

By Lemma 3.6 again,

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \max_{p \in \Lambda} \Big\| M_{\chi_{(W_{\delta_1,p^1} \cap W_{\delta_2,p})}} \sum_{j \in \widetilde{p^1} \cap \tilde{p}} \lambda_j C_{\varphi_j} \Big\|_e .$$

Hence there exists $p^2 \in \Lambda$ satisfying

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \Big\| M_{\chi_{(W_{\delta_1,p^1} \cap W_{\delta_2,p^2})}} \sum_{j \in \widetilde{p^1} \cap \widetilde{p^2}} \lambda_j C_{\varphi_j} \Big\|_e .$$

Repeating the same argument, there exists a sequence $\{p^\ell\}_\ell$ in Λ satisfying

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \Big\| M_{\chi_{(\bigcap_{\ell=1}^k W_{\delta_\ell,p^\ell})}} \sum_{j \in \bigcap_{\ell=1}^k \widetilde{p^\ell}} \lambda_j C_{\varphi_j} \Big\|_e .$$

Since $\widetilde{p^\ell} \subset \{1, 2, \ldots, N\}$, there exists a positive integer k_0 satisfying

$$L_0 := \bigcap_{\ell=1}^{k_0} \widetilde{p^\ell} = \bigcap_{\ell=1}^{k} \widetilde{p^\ell} \quad \text{for every } k \geq k_0. \tag{3.10}$$

By (3.9), we have $L_0 \neq \emptyset$,

$$\sigma\Big(\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell} \Big) > 0$$

and

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \Big\| M_{\chi_{(\bigcap_{\ell=1}^k W_{\delta_\ell,p^\ell})}} \sum_{j \in L_0} \lambda_j C_{\varphi_j} \Big\|_e \tag{3.11}$$

for every $k \geq k_0$.

Let $k \geq k_0$ and $j \in L_0$. By (3.10), $j \in \widetilde{p^k}$, so by (3.5), $p^k_j = 0$ and by (3.6),

$$\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell} \subset W_{\delta_k,p^k} \subset W^{p^k_j}_{\delta_k,j} = W_{\delta_k,j}.$$

Since $W_{\delta_k,j} = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi_j^*(e^{i\theta})| > \delta_k\}$, we get

$$|\varphi_j^*| > \delta_k \text{ a.e. on } \bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell} \text{ for } j \in L_0.$$

Let $j \notin L_0$. By (3.10), there is an integer i with $1 \le i \le k_0$ such that $j \notin \widetilde{p^i}$, so $p^i_j = 1$. Hence by (3.5) and (3.6),

$$\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell} \subset W_{\delta_i,p^i} \subset W_{\delta_i,j}^{p^i_j} = W^c_{\delta_i,j}.$$

Since $|\varphi_j^*| \le \delta_i$ a.e. on $W^c_{\delta_i,j}$ and $\delta_i \le \delta_{k_0}$, we have

$$|\varphi_j^*| \le \delta_{k_0} \text{ a.e. on } \bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell} \text{ for } j \notin L_0.$$

Applying Lemma 3.7, for each $k \ge k_0$ there is a sequence $\{z_{k,n}\}_n$ in $\mathbb{D}$ satisfying

$$\Big\| M_{\chi_{(\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell})}} \sum_{j\in L_0} \lambda_j C_{\varphi_j} \Big\| \le \liminf_{n\to\infty} \Big\| \sum_{j\in L_0} \lambda_j P_{\varphi_j(z_{k,n})} \Big\|_1, \tag{3.12}$$

$$\Big| \Big(\prod_{j\in L_0} \varphi_j \Big)(z_{k,n}) \Big| > \delta_k^N \quad \text{for every } n \ge 1 \tag{3.13}$$

and

$$|\varphi_j(z_{k,n})| < (1+\delta_{k_0})/2 \quad \text{for every } n \ge 1 \text{ and } j \notin L_0. \tag{3.14}$$

We have

$$\begin{aligned} \Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e &= \Big\| M_{\chi_{(\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell})}} \sum_{j\in L_0} \lambda_j C_{\varphi_j} \Big\|_e && \text{by (3.11)} \\ &\le \Big\| M_{\chi_{(\bigcap_{\ell=1}^{k} W_{\delta_\ell,p^\ell})}} \sum_{j\in L_0} \lambda_j C_{\varphi_j} \Big\| \\ &\le \liminf_{n\to\infty} \Big\| \sum_{j\in L_0} \lambda_j P_{\varphi_j(z_{k,n})} \Big\|_1 && \text{by (3.12).} \end{aligned}$$

For each $k \ge k_0$, we may take a positive integer n_k satisfying

$$\liminf_{n\to\infty} \Big\| \sum_{j\in L_0} \lambda_j P_{\varphi_j(z_{k,n})} \Big\|_1 - \frac{1}{k} \le \Big\| \sum_{j\in L_0} \lambda_j P_{\varphi_j(z_{k,n_k})} \Big\|_1.$$

Then

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e \le \liminf_{k\to\infty} \Big\| \sum_{j\in L_0} \lambda_j P_{\varphi_j(z_{k,n_k})} \Big\|_1. \tag{3.15}$$

By (3.13),

$$\Big|\Big(\prod_{j\in L_0}\varphi_j\Big)(z_{k,n_k})\Big| > \delta_k^N \quad \text{for every } k \ge k_0.$$

Since $\delta_k \to 1$,

$$\lim_{k\to\infty}\Big|\Big(\prod_{j\in L_0}\varphi_j\Big)(z_{k,n_k})\Big| = 1.$$

By (3.14),

$$\limsup_{k\to\infty}|\varphi_j(z_{k,n_k})| \le (1+\delta_{k_0})/2 \quad \text{for every } j \notin L_0.$$

Considering a subsequence of $\{z_{k,n_k}\}_k$, we may assume that $\{z_{k,n_k}\}_k \in \mathcal{Z}(\{\varphi_j\})$ and $I(\{z_{k,n_k}\}) = L_0$. Hence

$$\begin{aligned}\Big\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\Big\|_e &\le \liminf_{m\to\infty}\Big\|\sum_{j\in I(\{z_{k,n_k}\})}\lambda_j P_{\varphi_j(z_{m,n_m})}\Big\|_1 \qquad \text{by (3.15)}\\ &= \Gamma(\{z_n\}) \qquad \text{by (3.1)}.\end{aligned}$$

This completes the proof. □

Combining Theorems 3.2 with 3.8, we have the main theorem.

Theorem 3.9. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \ne \varphi_j$ for $i \ne j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \ne 0$ for every $1 \le j \le N$. Then*

$$\Big\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\Big\|_e = \max_{\{z_n\}_n\in\mathcal{Z}(\{\varphi_j\})}\Gamma(\{z_n\}).$$

Corollary 3.10. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \ne \varphi_j$ for $i \ne j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \ne 0$ for every $1 \le j \le N$. Then $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is a compact operator on h^∞ if and only if $\Gamma(\{z_n\}) = 0$ for every $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$.*

In the last part of this section, we give a characterization for $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ to satisfy

$$\Big\|\sum_{j=1}^N \lambda_j C_{\varphi_j}\Big\|_e = \sum_{j=1}^N |\lambda_j|.$$

Theorem 3.11. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then*

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \sum_{j=1}^{N} |\lambda_j|$$

if and only if there is $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$ satisfying the following conditions:

(i) $\Big| \Big(\prod_{j=1}^{N} \varphi_j \Big)(z_n) \Big| \to 1$ *as $n \to \infty$.*

(ii) $\rho(\varphi_i(z_n), \varphi_j(z_n)) \to 1$ *as $n \to \infty$ for every $1 \leq i, j \leq N$ with $\lambda_i/|\lambda_i| \neq \lambda_j/|\lambda_j|$.*

To prove the above theorem, we need a lemma.

Lemma 3.12. *Let $\{F_{j,n}\}_n, 1 \leq j \leq m$, be sequences of positive functions in $L^1(\partial\mathbb{D})$. Suppose that $\|F_{j,n}\|_1 \to c_j \neq \infty$ for every $1 \leq j \leq N$ and $\|F_{i,n} - F_{j,n}\|_1 \to c_i + c_j$ for every $1 \leq i, j \leq N$ with $i \neq j$ as $n \to \infty$. Then for $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{C}$, we have*

$$\Big\| \sum_{j=1}^{m} \lambda_j F_{j,n} \Big\|_1 \to \sum_{j=1}^{m} |\lambda_j| c_j \quad \text{as } n \to \infty.$$

Proof. Let

$$E_{i,j,n} = \big\{ e^{i\theta} \in \partial\mathbb{D} : (F_{i,n} - F_{j,n})(e^{i\theta}) > 0 \big\}, \quad i \neq j.$$

Then $E_{i,j,n}^c = E_{j,i,n}$. Since

$$\|F_{i,n} - F_{j,n}\|_1 = \int_{E_{i,j,n}} F_{i,n} - F_{j,n} \, d\sigma + \int_{E_{i,j,n}^c} F_{j,n} - F_{i,n} \, d\sigma$$

and $\|F_{i,n} - F_{j,n}\|_1 \to c_i + c_j$, we have

$$\int_{E_{i,j,n}} F_{i,n} \, d\sigma \to c_i \quad \text{and} \quad \int_{E_{i,j,n}^c} F_{j,n} \, d\sigma \to c_j.$$

Hence

$$\int_{E_{j,i,n}} F_{j,n} \, d\sigma \to c_j \quad \text{for } i \neq j.$$

For each $1 \leq j \leq N$, we write

$$\tilde{E}_{j,n} = \bigcap_{i : i \neq j} E_{j,i,n}.$$

Then

$$\int_{\tilde{E}_{j,n}} F_{j,n} \, d\sigma \to c_j \quad \text{and} \quad \int_{\tilde{E}_{j,n}^c} F_{j,n} \, d\sigma \to 0. \tag{3.16}$$

For $j_1 \neq j_2$, we have

$$\begin{aligned}\tilde{E}_{j_1,n} \cap \tilde{E}_{j_2,n} &= \Big(\bigcap_{t:t\neq j_1} E_{j_1,t,n} \Big) \cap \Big(\bigcap_{s:s\neq j_2} E_{j_2,s,n} \Big) \\ &\subset E_{j_1,j_2,n} \cap E_{j_2,j_1,n} \\ &= E_{j_1,j_2,n} \cap E^c_{j_1,j_2,n} = \emptyset.\end{aligned}$$

Hence

$$\begin{aligned}\Big\| \sum_{j=1}^m \lambda_j F_{j,n} \Big\|_1 &\geq \sum_{j=1}^m \int_{\tilde{E}_{j,n}} \Big| \sum_{i=1}^m \lambda_i F_{i,n} \Big| \, d\sigma \\ &\geq \sum_{j=1}^m \Big(\int_{\tilde{E}_{j,n}} |\lambda_j| F_{j,n} \, d\sigma - \sum_{i\neq j} |\lambda_i| \int_{\tilde{E}_{j,n}} F_{i,n} \, d\sigma \Big) \\ &\to \sum_{j=1}^m |\lambda_j| c_j \qquad \text{as } n \to \infty \text{ by (3.16).}\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.11. Suppose that

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e = \sum_{j=1}^N |\lambda_j|.$$

By Theorem 3.9, there exists a sequence $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$ satisfying

$$\Gamma(\{z_n\}) = \sum_{j=1}^N |\lambda_j|.$$

By (3.1),

$$\liminf_{k\to\infty} \Big\| \sum_{j\in I(\{z_n\})} \lambda_j P_{\varphi_j(z_k)} \Big\|_1 = \sum_{j=1}^N |\lambda_j|.$$

This shows that $I(\{z_n\}) = \{1, 2, \ldots, N\}$, so (i) holds, and we have

$$\lim_{k\to\infty} \Big\| \sum_{j=1}^N \lambda_j P_{\varphi_j(z_k)} \Big\|_1 = \sum_{j=1}^N |\lambda_j|.$$

By the proof of Theorem 2.5, we get condition (ii).

Suppose that there is $\{z_n\}_n \in \mathcal{Z}(\{\varphi_j\})$ satisfying (i) and (ii). Then $I(\{z_n\}) = \{1, 2, \ldots, N\}$. For each $1 \leq j \leq N$, let

$$J_j = \{i : \lambda_i/|\lambda_i| = \lambda_j/|\lambda_j|.$$

Then there exist $j_1, j_2, \ldots, j_\ell$ such that $J_{j_t} \cap J_{j_s} = \emptyset$ for $t \neq s$ and $\bigcup_{t=1}^{\ell} J_{j_t} = \{1, 2, \ldots, N\}$. We have

$$\Big\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z_n)} \Big\|_1 = \Big\| \sum_{t=1}^{\ell} \sum_{i \in J_{j_t}} \lambda_i P_{\varphi_i(z_n)} \Big\|_1$$
$$= \Big\| \sum_{t=1}^{\ell} \frac{\lambda_{j_t}}{|\lambda_{j_t}|} \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)} \Big\|_1 .$$

Let $1 \leq t, s \leq \ell$ with $t \neq s$. By condition (ii), for every $i_1 \in J_{j_t}$ and $i_2 \in J_{j_s}$ we have $\rho(\varphi_{i_1}(z_n), \varphi_{i_2}(z_n)) \to 1$. By Lemma 2.4,

$$\|P_{\varphi_{i_1}(z_n)} - P_{\varphi_{i_2}(z_n)}\|_1 \to 2 \quad \text{as } n \to \infty.$$

We write

$$F_{t,n} = \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)} .$$

Then $\|F_{t,n}\|_1 = \sum_{i \in J_{j_t}} |\lambda_i|$, and

$$\|F_{t,n} - F_{s,n}\|_1 \to \Big(\sum_{i \in J_{j_t}} |\lambda_i| \Big) + \Big(\sum_{i \in J_{j_s}} |\lambda_i| \Big) \quad \text{as } n \to \infty.$$

Therefore by Lemma 3.12,

$$\Big\| \sum_{j=1}^{N} \lambda_j P_{\varphi_j(z_n)} \Big\|_1 = \Big\| \sum_{t=1}^{\ell} \frac{\lambda_{j_t}}{|\lambda_{j_t}|} \sum_{i \in J_{j_t}} |\lambda_i| P_{\varphi_i(z_n)} \Big\|_1$$
$$\to \sum_{t=1}^{\ell} \sum_{i \in J_{j_t}} |\lambda_i| \qquad \text{as } n \to \infty$$
$$= \sum_{j=1}^{N} |\lambda_j| .$$

Thus we get $\Gamma(\{z_n\}) = \sum_{j=1}^{N} |\lambda_j|$. By Theorem 3.9,

$$\Gamma(\{z_n\}) \leq \Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e \leq \sum_{j=1}^{N} |\lambda_j| ,$$

so we get

$$\Big\| \sum_{j=1}^{N} \lambda_j C_{\varphi_j} \Big\|_e = \sum_{j=1}^{N} |\lambda_j| ,$$

□

4. Examples

Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_j\|_\infty = 1$ satisfying $\varphi_i \neq \varphi_j$ for $i \neq j$ and $\lambda_j \in \mathbb{C}$ with $\lambda_j \neq 0$ for every $1 \leq j \leq N$. Then we have

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e \leq \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| \leq \sum_{j=1}^N |\lambda_j|. \tag{4.1}$$

In this section, we shall give examples $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ satisfying the following conditions, respectively:

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e < \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| < \sum_{j=1}^N |\lambda_j|, \tag{4.2}$$

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e < \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| = \sum_{j=1}^N |\lambda_j|, \tag{4.3}$$

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e = \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| < \sum_{j=1}^N |\lambda_j|, \tag{4.4}$$

$$\Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\|_e = \Big\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \Big\| = \sum_{j=1}^N |\lambda_j|. \tag{4.5}$$

Example 4.1. Let $\varphi_1(z) = sz+1-s$ for $0 < s < 1$ and $\varphi_2(z) = \varphi_1(z)+t(z-1)^b$. For $b > 2$ and t is real and $|t|$ is so small, we have $\varphi_1 \in \mathcal{S}(\mathbb{D})$. By [12, Example 1], $C_{\varphi_1} - C_{\varphi_2}$ is compact on H^∞. By Theorem 2.7, $C_{\varphi_1} - C_{\varphi_2}$ is compact on h^∞. Hence $\|C_{\varphi_1} - C_{\varphi_2}\|_e = 0$. By [12, Theorem 3],

$$\lim_{z\to 1} \rho(\varphi_1(z), \varphi_2(z)) = 0.$$

By Theorem 2.5, $\|C_{\varphi_1} - C_{\varphi_2}\| < 2$. It is easy to see that $0 < \|C_{\varphi_1} - C_{\varphi_2}\|$. Thus $C_{\varphi_1} - C_{\varphi_2}$ is an example satisfying (4.2).

Example 4.2. Let $\varphi_1(z) = (z+2)/3$ and $\varphi_2(z) = (z-2)/3$. Then $\|\varphi_1\|_\infty = \|\varphi_2\|_\infty = 1 > \|\varphi_1\varphi_2\|_\infty$. By Theorem 2.5, it is easy to see that $\|C_{\varphi_1} - C_{\varphi_2}\| = 2$. Let $\{z_n\}_n \in \mathcal{Z}(\{\varphi_1, \varphi_2\})$. If $1 \in I(\{z_n\})$, then $\varphi_1(z_n) \to 1$, so $z_n \to 1$. Hence $\varphi_2(z_n) \to 0$ as $n \to \infty$. Thus $I(\{z_n\}) = \{1\}$. Similarly if $2 \in I(\{z_n\})$, then $I(\{z_n\}) = \{2\}$. Hence by (3.1), $\Gamma(\{z_n\}) = 1$. Therefore

$$\max_{\{z_n\}_n \in \mathcal{Z}(\{\varphi_1, \varphi_2\})} \Gamma(\{z_n\}) = 1.$$

By Theorem 3.9, $\|C_{\varphi_1} - C_{\varphi_2}\|_e = 1$. So $C_{\varphi_1} - C_{\varphi_2}$ satisfies (4.3).

Example 4.3. This example is similar to the one given in [9]. Let $\varphi_1(z) = z$ and $\varphi_2(z) = z^2$. Let $\{z_n\}_n$ be a sequence of real numbers in $\mathbb{D}$ with $z_n \to -1$. Then $\{z_n\}_n \in \mathcal{Z}(\{\varphi_1, \varphi_2\})$ and $I(\{z_n\}) = \{1, 2\}$. Since $\varphi_1(z_n) \to -1$ and $\varphi_2(z_n) \to 1$ as $n \to \infty$, we have

$$\Gamma(\{z_n\}) = \liminf_{n\to\infty} \|P_{\varphi_1(z_n)} - P_{\varphi_2(z_n)}\|_1 = 2.$$

By Theorem 3.9, $\|C_{\varphi_1} - C_{\varphi_2}\|_e = 2$. By (4.1), $C_{\varphi_1} - C_{\varphi_2}$ satisfies (4.5).

Example 4.4. Take $\varphi_1 \in \mathcal{S}(\mathbb{D})$ satisfying $\|\varphi_1\|_\infty = 1$ and

$$\int_{\partial\mathbb{D}} \log(1 - |\varphi_1^*|)\, d\sigma > -\infty.$$

Then there exists an outer function $\omega(z) \in H^\infty$ satisfying

$$|\omega^*| = 1 - |\varphi_1^*| \quad \text{a.e. on } \partial\mathbb{D} \tag{4.6}$$

(see [3, 7]). We have

$$|\omega| + |\varphi_1| \le 1 \quad \text{on } \mathbb{D}. \tag{4.7}$$

By (4.6), there is a sequence $\{z_n\}_n$ in $\mathbb{D}$ satisfying

$$\frac{1 - |\varphi_1(z_n)|}{|\omega(z_n)|} \to 1 \quad \text{as } n \to \infty$$

and

$$|\varphi_1(z_n)| \to 1 \quad \text{as } n \to \infty. \tag{4.8}$$

Here we may assume that

$$\frac{1 - |\varphi_1(z_n)|}{\omega(z_n)} \to 1 \quad \text{as } n \to \infty. \tag{4.9}$$

For $0 < t < 1$, let

$$\varphi_2(z) = \varphi_1(z) + t\omega(z)\varphi_1(z).$$

By (4.6), $\varphi_2 \in \mathcal{S}(\mathbb{D})$. Since $\omega(z_n) \to 0$, $|\varphi_2(z_n)| \to 1$. We have

$$\begin{aligned}
\rho(\varphi_1(z), \varphi_2(z)) &= \left| \frac{t\omega(z)\varphi_1(z)}{1 - |\varphi_1(z)|^2 - t\omega(z)|\varphi_1(z)|^2} \right| \\
&\le \frac{t|\varphi_1(z)|}{\left|\frac{1-|\varphi_1(z)|^2}{\omega(z)}\right| - t|\varphi_1(z)|^2} \\
&\le \frac{t|\varphi_1(z)|}{1 + |\varphi_1(z)| - t|\varphi_1(z)|^2} \qquad \text{by (4.7)} \\
&\le \frac{t}{2-t}.
\end{aligned}$$

Hence

$$\rho(\varphi_1(z), \varphi_2(z)) \le \frac{t}{2-t}, \quad z \in \mathbb{D}.$$

On the other hand,

$$\limsup_{n\to\infty}\rho(\varphi_1(z_n),\varphi_2(z_n)) = \limsup_{n\to\infty}\left|\frac{t\varphi_1(z_n)}{\frac{1-|\varphi_1(z_n)|^2}{\omega(z_n)} - t|\varphi_1(z_n)|^2}\right| = \frac{t}{2-t} \qquad \text{by (4.8) and (4.9).}$$

Therefore

$$\sup_{z\in\mathbb{D}}\rho(\varphi_1(z),\varphi_2(z)) = \limsup_{n\to\infty}\rho(\varphi_1(z_n),\varphi_2(z_n)) = \frac{t}{2-t}.$$

By Lemmas 2.1 and 2.4,

$$\begin{aligned}\|C_{\varphi_1}-C_{\varphi_2}\| &= \sup_{z\in\mathbb{D}}\|P_{\varphi_1(z)}-P_{\varphi_2(z)}\|_1\\ &= \limsup_{n\to\infty}\|P_{\varphi_1(z_n)}-P_{\varphi_2(z_n)}\|_1\\ &= 2-\frac{4\cos^{-1}\frac{t}{2-t}}{\pi} < 2.\end{aligned}$$

By (3.1) and Theorem 3.2, we have

$$\|C_{\varphi_1}-C_{\varphi_2}\|_e \geq \limsup_{n\to\infty}\|P_{\varphi_1(z_n)}-P_{\varphi_2(z_n)}\|_1 = 2-\frac{4\cos^{-1}\frac{t}{2-t}}{\pi}.$$

Hence $\|C_{\varphi_1}-C_{\varphi_2}\| = \|C_{\varphi_1}-C_{\varphi_2}\|_e$. Therefore $C_{\varphi_1}-C_{\varphi_2}$ satisfies (4.4).

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