

LINEAR TRANSFORMATIONS THAT PRESERVE TERM RANK BETWEEN DIFFERENT MATRIX SPACES

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ABSTRACT. The term rank of a matrix A is the least number of lines (rows or columns) needed to include all the nonzero entries in A . In this paper, we obtain a characterization of linear transformations that preserve term ranks of matrices over antinegative semirings. That is, we show that a linear transformation T from a matrix space into another matrix space over antinegative semirings preserves term rank if and only if T preserves any two term ranks k and l .

1. Introduction

There are many papers on linear operators on a matrix space that preserve matrix functions over various algebraic structures. But there are few papers of linear transformations from one matrix space into another matrix space that preserve matrix functions over an algebraic structure. In this paper we consider linear transformations from $m \times n$ matrices into $p \times q$ matrices that preserve term rank.

A *semiring* [2] is a set \mathbb{S} equipped with two binary operations $+$ and \cdot such that $(\mathbb{S}, +)$ is a commutative monoid with identity element 0 and (\mathbb{S}, \cdot) is a monoid with identity element 1 . In addition, the operations $+$ and \cdot are connected by distributivity of \cdot over $+$, and 0 annihilates \mathbb{S} .

A semiring \mathbb{S} is called *antinegative* if 0 is the only element to have an additive inverse. The following are some examples of antinegative semirings which occur in combinatorics. Let $\mathbb{B} = \{0, 1\}$. Then $(\mathbb{B}, +, \cdot)$ is an antinegative semiring (the *binary Boolean semiring*) if arithmetic in \mathbb{B} follows the usual rules except that $1 + 1 = 1$. If \mathbb{P} is any subring of \mathbb{R} with identity, the reals (under real addition and multiplication), and \mathbb{P}^+ denotes the nonnegative part of \mathbb{P} , then \mathbb{P}^+ is an antinegative semiring. In particular \mathbb{Z}^+ , the nonnegative integers, is

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an antinegative semiring. A nonzero $s \in \mathbb{S}$ is a *zero divisor* if $s's = 0$ for some nonzero $s' \in \mathbb{S}$.

Hereafter, \mathbb{S} will denote an arbitrary commutative and antinegative semiring. Let $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{M}_{p,q}(\mathbb{S})$ be the set of all $m \times n$ and $p \times q$ matrices respectively with entries in a semiring \mathbb{S} . Algebraic operations on $\mathbb{M}_{m,n}(\mathbb{S})$ and $\mathbb{M}_{p,q}(\mathbb{S})$ are defined as if the underlying scalars were in a field.

The *term rank* $\tau(A)$ of a matrix A is the minimal number k such that all the nonzero entries of A are contained in h rows and $k - h$ columns. Term rank plays a central role in combinatorial matrix theory and has many applications in network and graph theory (see [4]).

From now on we will assume that $2 \leq m \leq n$. It follows that $1 \leq \tau(A) \leq m$ for all nonzero $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

Let $\Xi_k^{(r,s)}$ denote the set of all matrices in $\mathbb{M}_{r,s}(\mathbb{S})$ whose term rank is k .

Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. If f is a function defined on $\mathbb{M}_{m,n}(\mathbb{S})$ and on $\mathbb{M}_{p,q}(\mathbb{S})$, then T *preserves* the function f if $f(T(A)) = f(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$. If \mathbb{X} is a subset of $\mathbb{M}_{m,n}(\mathbb{S})$ and \mathbb{Y} is a subset of $\mathbb{M}_{p,q}(\mathbb{S})$, then T *preserves* the pair (\mathbb{X}, \mathbb{Y}) if $A \in \mathbb{X}$ implies $T(A) \in \mathbb{Y}$. Further, T *strongly preserves* the pair (\mathbb{X}, \mathbb{Y}) if $A \in \mathbb{X}$ if and only if $T(A) \in \mathbb{Y}$. Further, we say that T (*strongly*) *preserves term rank k* if T (*strongly*) preserves the pair $(\Xi_k^{(m,n)}, \Xi_k^{(p,q)})$.

Beasley and Pullman ([2]) have characterized linear operators on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserve term rank, and the following are main results of their work: for a linear operator $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{m,n}(\mathbb{S})$,

(1.1) T preserves term rank if and only if T preserves term ranks 1 and 2;

(1.2)

T preserves term rank if and only if T strongly preserves term rank 1 or m .

Kang, Song and Beasley ([5]) also have characterized linear operators on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserve term rank, and the following are main results of their work: for a linear operator $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{m,n}(\mathbb{S})$,

(1.3) T preserves term rank if and only if T preserves term ranks 1 and k .

Note that if $1 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ preserves term rank k , then necessarily $k \leq \min(p, q)$.

In this paper, their work is continued. A sectional summary is as follows: Some definitions and preliminaries are presented in Section 2. Section 3 generalizes (1.1)~(1.3) by showing that $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ preserves term rank (of Boolean matrices) if and only if T preserves term ranks k and l , where $1 \leq k < l \leq m \leq n$. In Section 4, we show that $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ preserves the term rank of matrices over antinegative semiring \mathbb{S} if and only if it preserves any two term ranks if and only if it strongly preserves any one term rank.

2. Preliminaries

The matrix $A^{(m,n)}$ denotes a matrix in $\mathbb{M}_{m,n}(\mathbb{B})$, $O^{(m,n)}$ is the $m \times n$ zero matrix, I_n is the $n \times n$ identity matrix, $I_k^{(m,n)} = I_k \oplus O_{m-k,n-k}$, and $J^{(m,n)}$ is the $m \times n$ matrix all of whose entries are 1. Let $E_{i,j}^{(m,n)}$ be the $m \times n$ matrix whose (i, j) th entry is 1 and whose other entries are all 0, and we call $E_{i,j}^{(m,n)}$ a *cell*. An $m \times n$ matrix $L^{(m,n)}$ is called a *full line matrix* if

$$L^{(m,n)} = \sum_{l=1}^n E_{i,l}^{(m,n)} \quad \text{or} \quad L^{(m,n)} = \sum_{k=1}^m E_{k,j}^{(m,n)}$$

for some $i \in \{1, \dots, m\}$ or for some $j \in \{1, \dots, n\}$; $R_i^{(m,n)} = \sum_{l=1}^n E_{i,l}^{(m,n)}$ is the *i th full row matrix* and $C_j^{(m,n)} = \sum_{k=1}^m E_{k,j}^{(m,n)}$ is the *j th full column matrix*. We will suppress the subscripts or superscripts on these matrices when the orders are evident from the context and we write $A, O, I, I_k, J, E_{i,j}, L, R_i$ and C_j respectively.

The following is obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.1. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{S})$, we have $\tau(A + B) \leq \tau(A) + \tau(B)$ and $\tau(A) \leq \tau(A + B)$.*

If A and B are matrices in $\mathbb{M}_{m,n}(\mathbb{S})$, we say that B *dominates* A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j . This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}(\mathbb{S})$.

The following is also obvious by the definition of term rank of matrices over antinegative semirings.

Lemma 2.2. *For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{S})$, $A \sqsubseteq B$ implies that $\tau(A) \leq \tau(B)$.*

As usual, for any matrix A and lists L_1 and L_2 of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows L_1 and columns L_2 from A and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows L_1 and columns L_2 from A .

Definition. For matrices A and B in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $A \circ B$ denotes the *Hadamard* or *Schur product*. That is, the (i, j) th entry of $A \circ B$ is $a_{i,j}b_{i,j}$.

Definition. If $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$, then T is a (P, Q) -*block-transformation* if there are permutation matrices $P \in \mathbb{M}_p(\mathbb{B})$ and $Q \in \mathbb{M}_q(\mathbb{B})$ such that

- $m \leq p$ and $n \leq q$, and $T(A) = P[A \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$ or
- $m \leq q$ and $n \leq p$, and $T(A) = P[A^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Definition. If \mathbb{S} is a commutative antinegative semiring without zero divisors, $1 \leq m, n$ and $1 \leq p, q$, and $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$, then T is a (P, Q, B) -block-transformation if there are permutation matrices $P \in \mathbb{M}_p(\mathbb{S})$ and $Q \in \mathbb{M}_q(\mathbb{S})$, and $B \in \mathbb{M}_{m,n}(\mathbb{S})$ such that

- $m \leq p$ and $n \leq q$, and $T(A) = P[(A \circ B) \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$ or
- $m \leq q$ and $n \leq p$, and $T(A) = P[(A \circ B)^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

3. A characterization of term rank preservers of Boolean matrices.

For a linear transformation $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$, we say that T

- (1) *preserves term rank k* if $\tau(T(X)) = k$ whenever $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B})$, or equivalently if T preserves the pair $(\Xi_k^{(m,n)}, \Xi_k^{(p,q)})$;
- (2) *strongly preserves term rank k* if $\tau(T(X)) = k$ if and only if $\tau(X) = k$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B})$, or equivalently if T strongly preserves the pair $(\Xi_k^{(m,n)}, \Xi_k^{(p,q)})$;
- (3) *preserves term rank* if it preserves term rank k for every $k (\leq m)$.

In this section we provide characterizations of linear transformations $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ that preserve term ranks k and l , where $1 \leq k < l \leq m \leq n$.

Theorem 3.1. *Let $1 \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$. Then T strongly preserves term rank 1 if and only if T is a (P, Q) -block-transformation (Necessarily, either $m \leq p$ and $n \leq q$, or $m \leq q$ and $n \leq p$).*

Proof. It is routine to show that if T is a (P, Q) -block transformation, then T strongly preserves term rank 1.

Assume that T strongly preserves term rank 1. Then, the image of each line in $\mathbb{M}_{m,n}(\mathbb{B})$ is a line in $\mathbb{M}_{p,q}(\mathbb{B})$. We may assume that either $T(R_1^{(m,n)}) \subseteq R_1^{(p,q)}$ or $T(R_1^{(m,n)}) \subseteq C_1^{(p,q)}$.

Case 1. $T(R_1^{(m,n)}) \subseteq R_1^{(p,q)}$. Suppose that $T(C_j^{(m,n)}) \subseteq R_i^{(p,q)}$. Then, since $E_{1,j}^{(m,n)}$ is in both $R_1^{(m,n)}$ and $C_j^{(m,n)}$ and since $T(E_{1,j}^{(m,n)}) \neq O$, we must have $i = 1$. But then, for $j \neq k$ $T(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) \subseteq R_1^{(m,n)}$ and hence, has term rank 1. But $\tau(E_{2,j}^{(m,n)} + E_{1,k}^{(m,n)}) = 2$, a contradiction. Thus the image of any column is dominated by a column. Similarly, the image of any row is dominated by a row. Further, since the sum of two rows (columns) has term rank 2, the image of distinct rows (columns) must be dominated by distinct columns. Let $\phi : \{1, \dots, m\} \rightarrow \{1, \dots, p\}$ be a mapping defined by $\phi(i) = j$ if $T(R_i^{(m,n)}) \subseteq R_j^{(p,q)}$ and define $\theta : \{1, \dots, n\} \rightarrow \{1, \dots, p\}$ by $\theta(i) = j$ if $T(C_i^{(m,n)}) \subseteq C_j^{(p,q)}$. Then, it is easily seen that ϕ and θ are one-to-one mappings, and hence, $m \leq p$ and $n \leq q$. Let $\phi' : \{1, \dots, p\} \rightarrow \{1, \dots, p\}$ and $\theta' : \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ be one-to-one mappings such that $\phi' \upharpoonright_{\{1, \dots, m\}} = \phi$ and $\theta' \upharpoonright_{\{1, \dots, n\}} = \theta$. Let $P_{\phi'}$ and $Q_{\theta'}$ denote the permutation matrices corresponding to the permutations ϕ' and θ' .

In this case we have that $m \leq p$ and $n \leq q$, and $T(A) = P_{\phi'}[A \oplus O]Q_{\theta'}$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, that is, T is a (P, Q) -block-transformation.

Case 2. $T(R_1^{(m,n)}) \subseteq C_1^{(p,q)}$. As in Case 1, a parallel argument shows that $m \leq q$ and $n \leq p$, and $T(A) = P[A^t \oplus O]Q$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, and consequently that T is a (P, Q) -block-transformation. \square

Lemma 3.2. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves term rank k and term rank 1, then T strongly preserves term rank 1.*

Proof. If $k = 2$, then clearly T strongly preserves term rank 1. Assume that $k \geq 3$. Suppose a term rank 2 matrix is mapped to a term rank 1 matrix. Without loss of generality, $\tau(T(E_{1,1} + E_{2,2})) = 1$. But then, since T preserves term rank 1, $\tau(T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2}) + T(E_{3,3}) + \cdots + T(E_{k,k})) \leq \tau(T(E_{1,1} + E_{2,2})) + \tau(T(E_{3,3})) + \cdots + \tau(T(E_{k,k})) = 1 + (k-2) < k$, a contradiction. Thus, T strongly preserves term rank 1. \square

Corollary 3.3. *Let $1 < k \leq m, n$ and $1 \leq p, q$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation. Then T preserves term rank 1 and term rank k if and only if T is a (P, Q) -block-transformation.*

Proof. By Lemma 3.2, T strongly preserves term rank 1. By Theorem 3.1, the corollary follows. \square

We now come to the main theorem of this section:

Theorem 3.4. *Let $1 \leq k < l \leq m \leq n$ and $k + 1 < m$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves term rank k and term rank l , or if T strongly preserves term rank k , then T is a (P, Q) -block-transformation.*

The proof of this theorem relies upon nine lemmas which now follow.

Lemma 3.5. *Let $2 \leq k \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves term rank k . If T does not preserve term rank 1, then there is some term rank 1 matrix whose image has term rank at least 2.*

Proof. Suppose that T does not preserve term rank 1 and $\tau(T(A)) \leq 1$ for all A with $\tau(A) = 1$. Then, there is some cell $E_{i,j}$ such that $T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since $\tau(E_{1,1} + E_{2,2} + \cdots + E_{k,k}) = k$ and T preserves term rank k , we have $\tau(T(E_{2,2} + E_{3,3} + \cdots + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k,k})) = k$. Let $X = T(E_{2,2} + \cdots + E_{k,k})$ then we can choose a set of cells $Y = \{F_1, F_2, \dots, F_k\}$ such that $X \supseteq F_i$ for all $i = 1, \dots, k$, and $\tau(F_1 + F_2 + \cdots + F_k) = k$. Since $T(E_{2,2} + \cdots + E_{k,k}) = X$, there is some cell in $\{E_{2,2}, \dots, E_{k,k}\}$ whose image under T dominates two cells in Y , a contradiction. This contradiction establishes the lemma. \square

Lemma 3.6. *Let $1 \leq k \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves term rank k . If $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and $\tau(A) \leq k$, then $\tau(T(A)) \leq k$.*

Proof. If $\tau(A) = k$, then $\tau(T(A)) = k$ since T preserves term rank k . Suppose that $\tau(A) = h < k$, and $\tau(T(A)) > k$. Then there exists a matrix B such that $\tau(A + B) = k$ and hence $\tau(T(A + B)) = k$, but by Lemma 2.1, $\tau(T(A + B)) = \tau(T(A) + T(B)) \geq \tau(T(A)) > k$, a contradiction. Thus $\tau(T(A)) \leq k$. \square

Recall that the matrix J is the matrix whose entries are all ones.

Lemma 3.7. *Let $2 \leq k \leq m \leq n$ and $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves term rank k . If T does not preserve term rank 1, then $\tau(T(J)) \leq k + 2$.*

Proof. By Lemma 3.5, if T does not preserve term rank 1, then there is some rank 1 matrix whose image has term rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2}$.

Suppose that $\tau(T(J)) \geq k + 3$. Then, $\tau(T(J)[3, \dots, p | 3, \dots, q]) \geq k - 1$. Without loss of generality, we may assume that $T(J)[3, \dots, p | 3, \dots, q] \supseteq E_{3,3} + E_{4,4} + \dots + E_{k+1,k+1}$. Thus, there are $k - 1$ cells, F_3, F_4, \dots, F_{k+1} such that $T(F_3 + F_4 + \dots + F_{k+1}) \supseteq E_{3,3} + E_{4,4} + \dots + E_{k+1,k+1}$. Then, $T(E_{1,1} + E_{1,2} + F_3 + F_4 + \dots + F_{k+1}) \supseteq I_{k+1}$. But, $\tau(E_{1,1} + E_{1,2} + F_3 + F_4 + \dots + F_{k+1}) \leq k$ while $\tau(T(E_{1,1} + E_{1,2} + F_3 + F_4 + \dots + F_{k+1})) \geq k + 1$, a contradiction. Thus, $\tau(T(J)) \leq k + 2$. \square

Lemma 3.8. *Let $1 \leq k$, $k + 3 \leq l \leq m \leq n$. Let $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ be a linear transformation that preserves term rank k and term rank l , then T preserves term rank 1.*

Proof. Suppose that T does not preserve term rank 1. By Lemma 3.5, there is some term rank 1 matrix whose image has term rank at least 2. Let A be such a term rank 1 matrix. Then, A is dominated by a row or column and the image of the sum of two cells in that line has term rank at least two. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2}$. Now, by Lemma 3.7, if $B = T(C)$ is in the image of T , $\tau(B) \leq k + 2 < l$. But if we take $B = T(I_l)$, then $T(I_l)$ must have term rank l , a contradiction.

That is, $\tau(T(A)) \leq 1$. Since A was an arbitrary term rank 1 matrix, T preserves term rank 1. \square

Lemma 3.9. *Let $1 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves term rank k and term rank $k + 2$, then T strongly preserves term rank $k + 1$.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Case 1. Suppose that $\tau(A) = k + 1$ and $\tau(T(A)) \geq k + 2$. Let A_1, A_2, \dots, A_{k+1} be matrices of term rank 1 such that $A = A_1 + A_2 + \dots + A_{k+1}$. Without loss of generality we may assume that $T(A) \supseteq E_{1,1} + E_{2,2} + \dots + E_{k+2,k+2}$ and, since the image of some A_i must have term rank at least 2, we may assume that $\tau(T(A_1 + A_2 + \dots + A_i)) \geq i + 1$ for every $i = 1, 2, \dots, k + 1$. But then $\tau(A_1 + A_2 + \dots + A_k) = k$ while $\tau(T(A_1 + A_2 + \dots + A_k)) \geq k + 1$, a contradiction. Thus if $\tau(A) = k + 1$, $\tau(T(A)) \leq k + 1$.

Case 2. Suppose that $\tau(A) = k + 1$ and $\tau(T(A)) = s \leq k$. Without loss of generality, we may assume that $A = E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1}$ and $T(A) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Then there are s members of

$$\{T(E_{1,1}), T(E_{2,2}), \dots, T(E_{k+1,k+1})\}$$

whose sum dominates $E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Say, without loss of generality, that $T(E_{1,1} + E_{2,2} + \cdots + E_{s,s}) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s}$. Now, $\tau(A + E_{k+2,k+2}) = k + 2$ so that $\tau(T(A + E_{k+2,k+2})) = k + 2$. But since $\tau(T(A + E_{k+2,k+2})) = \tau((T(A) + T(E_{k+2,k+2}))) \leq \tau(T(A)) + \tau(T(E_{k+2,k+2}))$, it follows that $\tau(T(E_{k+2,k+2})) \geq k + 2 - s$ and there are s members of $\{T(E_{1,1}), T(E_{2,2}), \dots, T(E_{k+1,k+1})\}$ whose sum together with $T(E_{k+2,k+2})$ has term rank $k + 2$, say $\tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2$. Since $s \leq k$, $\tau(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2}) \leq k + 1$ and $\tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2$. By Case 1, we again arrive at a contradiction.

Therefore T strongly preserves term rank $k + 1$. \square

Lemma 3.10. *Let $1 \leq k < r, s$. If $\tau(E_{1,1} + \cdots + E_{k,k} + A) \geq k + 1$ and $A[k + 1, \dots, r | k + 1, \dots, s] = O$, then there is some $i, 1 \leq i \leq k$, such that $\tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1} + \cdots + E_{k,k} + A) \geq k + 1$.*

Proof. Suppose that $B = E_{1,1} + \cdots + E_{k,k} + A$ and $\tau(B) \geq k + 1$. Then there are $k + 1$ cells F_1, F_2, \dots, F_{k+1} such that $B \supseteq F_1 + F_2 + \cdots + F_{k+1}$ and $\tau(F_1 + F_2 + \cdots + F_{k+1}) = k + 1$. If $F_1 + F_2 + \cdots + F_{k+1} \supseteq I_k \oplus O$, then one cell F_j must be a cell $E_{a,b}$ where $a, b \geq k + 1$, which contradicts the assumption $A[k + 1, \dots, r | k + 1, \dots, s] = O$. Thus $F_1 + F_2 + \cdots + F_{k+1}$ does not dominate $I_k \oplus O$. That is, there is some $i, 1 \leq i \leq k$, such that $\tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1} + \cdots + E_{k,k} + A) \geq k + 1$. \square

Lemma 3.11. *Let $2 \leq k + 1 < m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that preserves term rank k and term rank $k + 1$, then T preserves term rank 1.*

Proof. If $k = 1$, the lemma vacuously holds. Suppose that $k \geq 2$.

Suppose that T does not preserve term rank 1. Then there is some matrix of term rank 1 whose image has term rank at least 2. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2}$. By Lemma 3.7 we have that $\tau(T(J)) \leq k + 2$. Since T preserves term rank $k + 1$, $\tau(T(J)) \geq k + 1$.

Thus, $\tau(T(J)) = k + i$ for either $i = 1$ or $i = 2$. Now, we may assume that for some r, s with $r + s = k + i$, $T(J)[r + 1, \dots, p | s + 1, \dots, q] = O$. Further, we may assume, without loss of generality, that there are $k + i$ cells F_1, F_2, \dots, F_{k+i} such that $T(F_l) \supseteq E_{l,k+i-l+1}$ for $l = 1, \dots, k + i$. Suppose the image of one of the cells in F_1, F_2, \dots, F_{k+i} dominates more than one cell in $\{E_{1,k+i}, E_{2,k+i-1}, \dots, E_{k+1,i}\}$. Say, without loss of generality, that $T(F_1) \supseteq E_{1,k+i} + E_{2,k+i-1}$, then, $T(F_1 + F_3 + \cdots + F_{k+1}) \supseteq E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}$, a contradiction since $\tau(F_1 + F_3 + \cdots + F_{k+1}) \leq k$, and hence $\tau(T(F_1 + F_3 + \cdots + F_{k+1})) \leq k$, and $\tau(E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}) = k + 1$. It

follows that for each $j = 1, \dots, k+1$, if $T(F_l) \sqsupseteq E_{j,k+i-j+1}$, then $l = j$ since $T(F_j) \sqsupseteq E_{j,k+i-j+1}$ is unique. Further, by permuting we may assume that $F_1 + F_2 + \dots + F_k \sqsubseteq \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix}$.

Now, let $O \neq A \in \mathbb{M}_{m,n}(\mathbb{B})$ have term rank 1, and suppose that

$$A[1, 2, \dots, k \mid 1, 2, \dots, n] = O \text{ and } A[1, \dots, m \mid 1, \dots, k] = O.$$

So that $A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_1 \end{bmatrix}$. If $T(A)[k+1, \dots, p \mid 1, i] = O$, then, since $\tau(F_1 + \dots + F_k + A) = k+1$, $\tau(T(F_1 + \dots + F_k + A)) = k+1$. Applying Lemma 3.10, we have that there is some j such that $\tau(T(F_1 + \dots + F_{j-1} + F_{j+1} + \dots + F_k + A)) = k+1$. But $\tau(F_1 + \dots + F_{j-1} + F_{j+1} + \dots + F_k + A) = k$ while $\tau(T(F_1 + \dots + F_{j-1} + F_{j+1} + \dots + F_k + A)) = k+1$, a contradiction. So we must have that $T(E_{k+1,1})[k+1, \dots, p \mid 1, i] \neq O$. If $T(E_{k+1,1})[k+1, \dots, p \mid 1, i] \neq O$, then $\tau(T(F_1 + \dots + F_k + E_{k+1,1})) = k+1$, a contradiction since $\tau(F_1 + \dots + F_k + E_{k+1,1}) = k$. Suppose that the $(k, i+1)$ entry of $T(E_{k,k+1})$ is nonzero, then, $\tau(T(F_1 + \dots + F_{k-1} + E_{k,k+1} + E_{k+1,k+1})) = k+1$, a contradiction, since $\tau(F_1 + \dots + F_{k-1} + E_{k,k+1} + E_{k+1,k+1}) = k$.

Consider $T(F_1 + \dots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2})$. This must have term rank $k+1$ and dominates $E_{1,k+i} + E_{2,k+i-1} + \dots + E_{k-1,i+2} + E_{k+1,j}$ for some $j \in \{1, i\}$. Thus, by Lemma 3.10, there is some cell in $\{F_1, \dots, F_{k-1}\}$, say F_j such that $\tau(T(F_1 + \dots + F_{j-1} + F_{j+1} + \dots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2})) = k+1$. But $\tau(F_1 + \dots + F_{j-1} + F_{j+1} + \dots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2}) = k$, a contradiction.

It follows that T must preserve term rank 1. \square

Lemma 3.12. *Let $2 \leq k \leq m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that strongly preserves term rank k , then T preserves term rank $k-1$.*

Proof. If $k = 2$, the lemma holds. Suppose that $k \geq 3$.

Let $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and $\tau(A) = k-1$, and suppose that $\tau(T(A)) = s < k-1$. Without loss of generality, we may assume that $\tau(T(E_{1,1} + \dots + E_{k-1,k-1})) = s < k-1$. Since $\tau(T(E_{1,1} + \dots + E_{k,k})) = k$, we have that $\tau(T(E_{k,k})) \geq k-s$. Without loss of generality we may assume that $T(E_{1,1} + \dots + E_{k,k}) \sqsupseteq E_{1,1} + \dots + E_{k,k}$ and that $T(E_{k,k}) \sqsupseteq E_{t+1,t+1} + \dots + E_{k,k}$ for some $t \leq s$. Then, there are t cells $\{E_{i_1,i_1}, \dots, E_{i_t,i_t}\}$ in $\{E_{1,1}, \dots, E_{k,k}\}$ such that $T(E_{i_1,i_1} + \dots + E_{i_t,i_t}) \sqsupseteq E_{1,1} + \dots + E_{t,t}$. Then $T(E_{i_1,i_1} + \dots + E_{i_t,i_t} + E_{k,k}) \sqsupseteq E_{1,1} + \dots + E_{k,k}$. Thus $\tau(T(E_{i_1,i_1} + \dots + E_{i_t,i_t} + E_{k,k})) = k$. But $\tau(E_{1,1} + \dots + E_{t,t} + E_{k,k}) = t+1 \leq s+1 < (k-1)+1 = k$, which contradicts the assumption of T . Hence $\tau(T(A)) \geq k-1$. Further, $\tau(T(A)) \leq k-1$, since T strongly preserves term rank k . Thus, T preserves term rank $k-1$. \square

Lemma 3.13. *Let $2 \leq k < m \leq n$. If $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation that strongly preserves term rank k , then T preserves term rank 1.*

Proof. By Lemma 3.12, T preserves term rank $k - 1$. By Lemma 3.11, T preserves term rank 1. \square

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.4. By hypothesis, Lemma 3.8, Lemma 3.11, or Lemma 3.13, T preserves term rank 1. By Lemma 3.2, T strongly preserves term rank 1. By Theorem 3.1, the theorem follows. \square

4. Term rank preservers of matrices over antinegative semirings.

Throughout this section, \mathbb{S} will denote any commutative and antinegative semiring without zero divisors.

In this section we provide characterizations of linear transformations $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ that preserve term rank. Let $A \in \mathbb{M}_{m,n}(\mathbb{S})$ and define $\bar{A} \in \mathbb{M}_{m,n}(\mathbb{B})$ to be the matrix $[\bar{a}_{i,j}]$ where $\bar{a}_{i,j} = 1$ if and only if $a_{i,j} \neq 0$. \bar{A} is called the *support* or *pattern* of A . Clearly $\tau(\bar{A}) = \tau(A)$. Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. Define $\bar{T} : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ by $\bar{T}(E_{i,j}) = \overline{T(E_{i,j})}$, and extend linearly. Then $\bar{T} : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{p,q}(\mathbb{B})$ is a linear transformation over binary Boolean semiring.

Lemma 4.1. *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. Then T preserves term rank k if and only if \bar{T} preserves term rank k , for any $1 \leq k \leq m$.*

Proof. The proof is left to the reader. \square

Theorem 4.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{p,q}(\mathbb{S})$ be a linear transformation. Then the following are equivalent:*

1. T preserves term rank;
2. T preserves term rank k and term rank l , with $1 \leq k < l \leq m \leq n$ and $k + 1 < m$;
3. T strongly preserves term rank h , with $1 \leq h < m \leq n$;
4. T is a (P, Q, B) -block transformation.

Proof. It is obvious that 1 implies 2 and 3, and 4 implies 1, 2 and 3. In order to show that 2 (or 3) implies 4, assume that T preserves term rank k and term rank l , with $1 \leq k < l \leq m \leq n$. By Lemma 4.1, \bar{T} preserves term rank k and term rank l , with $1 \leq k < l \leq m \leq n$. Thus, by Theorem 3.4, \bar{T} is a (P, Q) -block transformation. It follows that for every cell $E_{i,j}$, there is some nonzero $b_{i,j} \in \mathbb{S}$ such that for $B = [b_{i,j}]$ either

$$(1) \quad T(E_{i,j}) = b_{i,j}(P[E_{i,j} \oplus O]Q), \text{ and}$$

$$T(X) = T\left(\sum_{i=1}^m \sum_{j=1}^n x_{i,j} E_{i,j}\right) = \sum_{i=1}^m \sum_{j=1}^n x_{i,j} T(E_{i,j})$$

$$= \sum_{i=1}^m \sum_{j=1}^n x_{i,j} b_{i,j} (P[E_{i,j} \oplus O]Q) = P[(X \circ B) \oplus O]Q$$

for every $X \in \mathbb{M}_{m,n}(\mathbb{S})$; or

(2) $m = n$ and $T(E_{i,j}) = b_{i,j}(P[E_{i,j} \oplus O]^tQ)$, and

$$\begin{aligned} T(X) &= T\left(\sum_{i=1}^m \sum_{j=1}^n x_{i,j} E_{i,j}\right) = \sum_{i=1}^m \sum_{j=1}^n x_{i,j} T(E_{i,j}) \\ &= \sum_{i=1}^m \sum_{j=1}^n x_{i,j} b_{i,j} (P[E_{i,j} \oplus O]^tQ) = P[(X \circ B) \oplus O]^tQ \end{aligned}$$

for every $X \in \mathbb{M}_{m,n}(\mathbb{S})$.

Thus, T is a (P, Q, B) -block transformation.

In order to show that 3 implies 4, if we apply Lemma 4.1 and Theorem 3.4, the proof is parallel to the above. \square

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