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A CELL BOUNDARY ELEMENT METHOD FOR A FLUX CONTROL PROBLEM

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Abstract. We consider a distributed optimal flux control problem: finding the potential of which gradient approximates the target vector field under an elliptic constraint. Introducing the Lagrange multiplier and a change of variables the Euler-Lagrange equation turns into a coupled equation of an elliptic equation and a reaction diffusion equation. The change of variables reduces iteration steps dramatically when the Gauss-Seidel iteration is considered as a solution method. For the elliptic equation solver we consider the *Cell Boundary Element* (CBE) method, which is the finite element type flux preserving methods.

1. Introduction

In this article, we consider a distributed optimal control problem for a second order partial differential equation: seek $(u, p) \in H^1(\Omega)\times L^2(\Omega)$ which minimizes the cost functional

$$\mathcal{J}(u, p) = J_{u_d}(u) + \delta N(p) \tag{1.1}$$

subject to

$$-\nabla \cdot (a\nabla u) = p \quad \text{on } \Omega, \tag{1.2}$$

where $J_{u_d}(u)$ and $N(p)$ are some (semi)norms of u and p, respectively and the parameter, δ is a positive regularization parameter. We assume that $\Omega = \cup_{j=1}^{J}\Omega_j$ is a simply connected polygonal domain with the boundary Γ. The permeability coefficient, a is a *positive definite, symmetric tensor* and it is constant on each subdomain Ω_j.

In our problem the target function $u_d \in H^1(\Omega)$ will be used to provide the target flux field ∇u_d and the Dirichlet boundary condition $u = u_d$ on Γ. Throughout this article we assume $u = u_d = 0$ on Γ to simplify our analysis.

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Thus, we consider the gradient tracking functional

$$J_{u_d}(u) = \frac{1}{2}\int_\Omega a\nabla(u-u_d)\cdot\nabla(u-u_d)\,dx. \tag{1.3}$$

The second term of the cost functional is introduced in order to bound the control function and to prove the existence of an optimal control. In general,

$$N(p) = \frac{1}{2}\int_\Omega p^2\,dx. \tag{1.4}$$

The distributed optimal control problem can be derived from many physical applications [2, 4, 5]. For a subsurface flow problem, we are looking for the optimal solution u and the source distribution p so that the gradients of a potential flow approximate the target vector field optimally in a certain measure. In approximation of the optimality solution the flux conservation property of numerical solutions is very important. In transport problems of substances embedded in a potential flow, the distribution of substances satisfies a time dependent convection diffusion equation driven by the gradients of the flow [8]. In this case flux preserving approximation of gradients for the potential flow is crucial in obtaining physically relevant time evolution of substance distribution. For this reason, we introduce a Cell Boundary Element (CBE) numerical method for the optimal control problem. The CBE method is introduced by Jeon *et al.* [6, 7] and it can be understood as a finite element version of the finite volume method (it is different from the finite volume element method). The main advantage of the CBE method is that it is a flux preserving numerical method while it does not need a costly covolume generation.

The main contributions of the paper can be summarized as follows: firstly, a change of variables is introduced to transform the Euler-Lagrange equation to an equivalent computing friendly form. The transformed equation reduces the number of iteration dramatically when it is combined with the Gauss-Seidel iteration, especially for nonlinear problems. Secondly, the CBE method is used as a numerical method for the control problem.

The paper is organized as follows. In Section 2 we derive the optimality equations and introduce a change of variables to transform it to a computing friendly system of equations. The transformed equations are composed of a standard elliptic equation and a reaction dominated diffusion equation. In Section 3 numerical methods for elliptic equations are introduced and analyzed, the CBE method for the elliptic equation and the lowest Crouzeix-Raviart finite element method for the reaction diffusion equation. In Section 4 numerical analysis for the optimality equations are presented. In Section 5 we consider a nonlinear optimal control problem and a Gauss-Seidel iteration method is introduce. In Section 6 we present our numerical examples for linear and nonlinear optimal problems.

2. The optimality system

Firstly, we introduce the standard Sobolev space, $H^s(\Omega)$ ($s \geq 0$) with the norm $\|\cdot\|_{s,\Omega}$. The L_2 space is the space of square integrable functions. The space $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ with null trace. By the nature of the problem, we consider a specialized norm:

$$H_a^1(div;\Omega) = \{u \in H^1(\Omega) : \nabla \cdot (a\nabla u) \in L_2(\Omega)\}.$$

Introducing a Lagrange multiplier, the constrained minimization problem (1.1)-(1.2) becomes a min-max problem of the functional:

$$\begin{aligned} L(u,p,\lambda) = \frac{1}{2}\int_\Omega a\nabla(u-u_d)\cdot\nabla(u-u_d)\,dx + \frac{\delta}{2}\int_\Omega p^2\,dx \\ + \int_\Omega (-\nabla\cdot(a\nabla u) - p)\lambda\,dx. \end{aligned} \tag{2.1}$$

Then, the Euler-Lagrange equation (the first optimality condition),

$$\frac{\partial L}{\partial u} = 0, \quad \frac{\partial L}{\partial p} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0,$$

induces a coupled system of differential equations: for $(u,\lambda) \in H_a^1(div;\Omega) \cap H_0^1(\Omega)) \times L_2(\Omega)$,

$$(a\nabla(u-u_d),\nabla v)_\Omega + (-\nabla\cdot(a\nabla v),\lambda)_\Omega = 0, \quad v \in H_a^1(div;\Omega)\cap H_0^1(\Omega), \tag{2.2}$$

$$\delta(p,h)_\Omega - (h,\lambda)_\Omega = 0, \qquad h \in L_2(\Omega), \tag{2.3}$$

$$(-\nabla\cdot(a\nabla u) - p,\mu)_\Omega = 0, \qquad \mu \in L_2(\Omega), \tag{2.4}$$

where $(u,v) = \int_\Omega uv\,d\Omega$.

To derive a strong form of differential equations, we impose a stronger regularity and an artificial boundary condition for the Lagrange multiplier, that is, $\lambda \in H_a^1(div;\Omega)\cap H_0^1(\Omega)$. From (2.3) we set

$$p = \frac{\lambda}{\delta}. \tag{2.5}$$

The integration by parts with the boundary conditions $v = \lambda = 0$ on Γ yields

$$-\nabla\cdot(a\nabla u) - \nabla\cdot(a\nabla\lambda) = -\nabla\cdot(a\nabla u_d), \tag{2.6}$$

$$-\nabla\cdot(a\nabla u) - \frac{\lambda}{\delta} = 0, \tag{2.7}$$

on Ω with the boundary condition $u = \lambda = 0$ on Γ.

Note that the Gataux differentiability for the Lagrangian can easily be proven (see [2, 4] and references therein).

Introducing a new variable w so that $w = u + \lambda$, we can have the following computation friendly form of the equations (2.6) and (2.7): find $(w,\lambda,u) \in H_a^1(div;\Omega) \times (H_a^1(div;\Omega) \times H_0^1(\Omega)) \times H_a^1(div;\Omega)$ that satisfies

$$-\nabla\cdot(a\nabla w) = f_d, \tag{2.8}$$

$$\delta\nabla\cdot(a\nabla w) + \{-\delta\nabla\cdot(a\nabla\lambda) + \lambda\} = 0, \tag{2.9}$$

$$-w + \lambda + u = 0 \tag{2.10}$$

on Ω and the boundary conditions $w = \lambda = 0$ on Γ, where $f_d = -\nabla \cdot (a\nabla u_d)$. Introduction of a new variable w is very important when one tries to solve a nonlinear optimal control problem (see Section 5). Applying the Gauss-Seidel iteration in the equations (2.6) and (2.7) may be costly, while the Gauss-Seidel iteration for the equations (2.8)-(2.10) provides a stable, fast convergent numerical scheme.

We suggest the Gauss-Seidel iteration for the equation (2.8)-(2.10) as follows:

$$-\nabla \cdot (a\nabla w^{n+1}) = f_d, \tag{2.11}$$

$$-\delta\nabla \cdot (a\nabla \lambda^{n+1}) + \lambda^{n+1} = \delta\{-\nabla \cdot (a\nabla w^{n+1})\}, \tag{2.12}$$

$$u^{n+1} = w^{n+1} - \lambda^{n+1}, \tag{2.13}$$

with the boundary conditions, $w^{n+1} = 0$ and $\lambda^{n+1} = 0$ on Γ.

Indeed, the above Gauss-Seidel iteration ends at one iteration for a linear problem.

3. Numerical methods for elliptic equations

In this paper we are looking for the almost flux conserving numerical solution for the optimal problem under the assumption that the regularization parameter δ is small. For that we consider the *Cell Boundary Element* (CBE) method for (2.11) and the nonconforming finite element method (NcFEM) for (2.12). Since the NcFEM is not a flux preserving method, our method will not be exactly flux preserving. However, in view of the equation (2.12), the exact solution λ will satisfy the estimate, $\sqrt{\delta}|\lambda|_{1,\Omega} + \|\lambda\|_{0,\Omega} \le C\delta$. Therefore, using the NcFEM for the equation (2.12) will have a very little influence on flux conservation property of the other solutions u and w when the parameter δ is a small positive number.

In this section we review the CBE method and the NcFEM, briefly and detailed description on those method will be found [6, 7] and [1], respectively.

Let $\mathcal{T}_h$ be the regular triangulation of $\Omega = \cup_{j=1}^{J}\Omega_j$ and Γ_h the discretization of Γ, induced by $\mathcal{T}_h$. Then $\mathcal{E}_h$ and $\mathcal{E}_h^0$ denote the set of edges and interior edges generated by $\mathcal{T}_h$, respectively, and $\mathcal{M}_h$ denotes the set of midpoints of edges. The *skeleton* of a triangulation $\mathcal{T}_h$ is $\mathcal{K}_h = \cup_{e\in\mathcal{E}_h} e$.

Introduce the nonconforming P_1 approximation space for v:

(3.1)

$$S_h = \{v \in \oplus_{T\in\mathcal{T}_h} S_T : v \text{ is continuous on } \mathcal{M}_h \text{ and } v(p) = 0 \text{ for } p \in \Gamma_h \cap \mathcal{M}_h\}$$

with $S_T = \text{span}\{1, x, y\}$. Then, there is a natural interpolation $I_h : C(\Omega) \to S_h$ such that

$$I_h(u)(x) = \sum_{m\in\mathcal{M}_h} u(m)\phi_m(x). \tag{3.2}$$

We also introduce discrete Sobolev seminorms and norms; for $k \in \mathbb{Z}$,

$$|u|_{k,h}^2 = \sum_{T\in\mathcal{T}_h} |u|_{k,T}^2, \quad \|u\|_{k,h}^2 = \|u\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h} |u|_{k,T}^2. \tag{3.3}$$

The analysis for the reaction diffusion equation is based on the following norm:

$$\|u\|_{\delta,h}^2 = \delta|u|_{1,h}^2 + \|u\|_{0,\Omega}^2. \tag{3.4}$$

3.1. The cell boundary element method for the elliptic equation

Consider a Dirichlet problem: find $u \in H_0^1(\Omega)$ such that

$$-\nabla\cdot(a\nabla u) \quad = \quad f \quad \text{in } \Omega, \tag{3.5}$$

where f is a piecewise constant function, $f \in L_2(\Omega)$. When f is not a piecewise constant function we approximate f by the volume preserving approximation, that is, $f_h|_T = \frac{1}{|T|}\int_T f dx$. If $f \in L_2(\Omega)$, this approximation introduces error of the order $O(h)$ in the energy norm. Therefore, the overall approximation property of u does not change with the P_1 CBE method.

To derive the CBE method we consider the localized problem of (3.5):

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f \quad \text{in } T, \\ [[\nabla u]] &= 0 \quad \text{on } e \in \mathcal{E}_h^0. \end{aligned} \tag{3.6}$$

The jump of normal fluxes is defined as follows:

$$[[\nabla u]] \equiv \begin{cases} (a\nabla u)\cdot\nu + (a'\nabla u)\cdot\nu', & e = \partial T \cap \partial T', \\ (a\nabla u)\cdot\nu, & e = \partial T \cap \Gamma_h, \end{cases} \tag{3.7}$$

where ν and ν' are the unit outward normal vectors on ∂T and $\partial T'$, respectively. The solution u of (3.6) allows locally the following superposition:

$$u = v + G(f_h) \quad \text{on } T. \tag{3.8}$$

Here v satisfies $-\nabla\cdot(a\nabla v) = 0$ on T and $v = u$ on ∂T, and $G(f)$ is a Green bubble function such that $-\nabla(a\nabla(G(f)) = f$ on T and $G(f) = 0$ on ∂T. Then we have

$$[[\nabla u]] = [[\nabla v]] + [[\nabla G(f)]] \quad \text{on } e \in \mathcal{E}_h.$$

Using flux continuity on cell interfaces, we have

$$\int_e [[\nabla v]] ds = -\int_e [[\nabla G(f)]] ds, \quad e \in \mathcal{E}_h^0. \tag{3.9}$$

Introducing a piecewise constant test function on $\mathcal{K}_h$ such that $\overline{w_h} = \frac{1}{|e|}\int_e w_h ds$ with $w_h \in S_h$, we can rewrite the above equation:

$$\int_{\mathcal{K}_h} [[\nabla v]]\overline{w_h} ds = -\int_{\mathcal{K}_h} [[\nabla G(f)]]\overline{w_h} ds, \quad w_h \in S_h. \tag{3.10}$$

Let us introduce the natural interpolation $I_{h,T} : C(\overline{T}) \to S_T$. As long as there is no risk of misunderstanding, we denote $I_{h,T}$ by I_h for notational

simplicity. Then we consider approximation of the bubble function $G(f)$. Let $a = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ on T. Then we may take

$$G_h(f) = F(f) - I_h(F(f)) \quad \text{with } F(f) = \frac{f}{2(a_{11}+a_{22})}(x^2+y^2)$$

on T. Then we have $G_h(f) \approx G(f)$ and $-\nabla \cdot (a\nabla G_h(f)) = f$ on T. Note that f is constant on each T.

The P_1 CBE method is to find $v_h \in S_h$ such that

$$\int_{\mathcal{K}_h} [[\nabla v_h]]\overline{w_h} ds = -\int_{\mathcal{K}_h} [[\nabla G_h(f)]]\overline{w_h} ds. \tag{3.11}$$

After v_h being obtained, we obtain the approximate solution and its flux as follows.

$$u_h = v_h + G_h(f), \quad a\nabla u_h = a\nabla v_h + a\nabla G_h(f).$$

Now we introduce some results on numerical analysis [6]. Theorem 3.1 asserts that the CBE has the same stiffness matrix as the Crouzeix-Raviart P_1 finite element method, while the right hand side is different by D_h. Theorem 3.2 provides a convergence result.

Theorem 3.1. *The stiffness matrix and the right hand side of the equation* (3.11) *satisfy the following relations*: *for* $v_h,\ w_h \in S_h$

$$\int_{\mathcal{K}_h} [[\nabla v_h]]\ \overline{w}_h ds = (a\nabla v_h, \nabla w_h)_h$$

and

$$-\int_{\mathcal{K}_h} [[\nabla G_h(f)]]\overline{w_h} ds = (f, w_h)_h + D_h(f, w_h),$$

where

$$D_h(f, w_h) = -\sum_{T\in\mathcal{T}_h} [(\partial_\nu^a G(f), \overline{w_h} - w_h)_{\partial T} + (a\nabla G_h(f), \nabla w_h)_T].$$

Here, $(\nabla u, \nabla v)_h = \sum_{T\in\tau_h} (\nabla u, \nabla v)_T$ *and* $\partial_\nu^a G(f) = (a\nabla G(f)) \cdot \nu$.

Proof. Since ∇v_h is constant on each T, using the integration by parts,

$$\begin{aligned} \int_{\mathcal{K}_h} [[\nabla v_h]]\ \overline{w}_h ds &= \int_{\mathcal{K}_h} [[\nabla v_h]]\ w_h ds \\ &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} (a\nabla v_h)\cdot\nu\ w_h ds \\ &= (a\nabla v_h, \nabla w_h)_h. \end{aligned}$$

Now,

$$\begin{aligned} & -\int_{\mathcal{K}_h} [[\nabla G_h(f)]]\overline{w_h} ds \\ = & -\int_{\mathcal{K}_h} [[\nabla G_h(f)]] w_h ds - \int_{\mathcal{K}_h} [[\nabla G_h(f)]](\overline{w}_h - w_h) ds \end{aligned}$$

$$= -\sum_{T\in\mathcal{T}_h}[(\partial_\nu^a G(f), w_h)_{\partial T} + (\partial_\nu^a G(f), \overline{w}_h - w_h)_{\partial T}]$$
$$= (f, w_h)_\Omega - \sum_{T\in\mathcal{T}_h}[(a\nabla G_h(f), \nabla w_h)_T + (\partial_\nu^a G(f), \overline{w}_h - w_h)_{\partial T}].$$

Then, the theorem is proved. □

Theorem 3.2. *For $f \in L_2(\Omega)$, let u be the exact solution of* (3.5) *and u_h be the P_1 CBE approximate solution. Then*

$$\|u - u_h\|_{1,h} \le Ch\|f\|_{0,\Omega}.$$

Proof. The proof is based on the proof of the lowest order Crouzeix-Raviart finite element method. The lowest order Crouzeix-Raviart finite element solution, u_h^{CR} satisfies

$$(a\nabla u_h^{CR}, \nabla w_h)_h = (f, w_h)_\Omega, \quad w_h \in S_h.$$

Simple calculation yields that

$$(a\nabla(u_h - u_h^{CR}), \nabla w_h)_h = D_h(f, w_h).$$

Since $G_h(f) = F(f) - I_h(F(f))$,

$$|(\partial_\nu^a G(f), \nabla w_h)_T| \le Ch|F(f)|_{2,T}|w_h|_{1,T} \le C\|f\|_{0,T}|w_h|_{1,T}.$$

Using Corollary (10.17) with the scaling argument in [1],

$$|(a\nabla G_h(f), \overline{w}_h - w_h)_{\partial T}| \le Ch|G_h(f)|_{2,T}|w_h|_{1,T} \le Ch\|f\|_{0,T}|w_h|_{1,T}.$$

Then

$$|D_h| \quad \le \quad Ch\|f\|_{0,\Omega}\|w_h\|_{1,h}.$$

Therefore,

$$|(\nabla(u_h^{CR} - u_h, \nabla w_h)_T| \le Ch\|f\|_{0,T}|w_h|_{1,T}.$$

It is well-known that $\|u - u_h^{CR}\|_{1,h} \le Ch\|f\|_{0,\Omega}$. The triangle inequality with the Poincaré-Friedrichs inequality [1, 3] yields the desired convergence estimate. □

3.2. The nonconforming finite element method for the reaction diffusion equation

The finite element method of the reaction diffusion equation is described briefly. The nonconforming P_1 finite element method for the reaction diffusion equation has advantage in that it has a mass-lumping property, which prevents non-physical oscillations in numerical solutions on the region of boundary layer [9].

Consider a reaction diffusion equation with the Dirichlet boundary condition:

$$\begin{aligned} -\delta\nabla\cdot(a\nabla\lambda) + \lambda &= f \quad \text{in } \Omega, \\ \lambda &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.12}$$

The lowest order Crouzeix-Raviart finite element method is to find $\lambda_h \in S_h$ such that

$$\delta(a\nabla\lambda_h, \nabla\gamma_h)_h + (\lambda_h, \gamma_h)_\Omega = (f, \gamma_h)_\Omega, \quad \gamma_h \in S_h. \tag{3.13}$$

Then the following convergence analysis is straightforward and it is found in standard textbooks [1], [9].

Theorem 3.3. *The exact and its approximate solutions, λ and λ_h satisfy the convergence estimate*:

$$\|\lambda - \lambda_h\|_{\delta,h} \le C(h^2 + \sqrt{\delta}h)(\|f\|_{0,\Omega} + \|\lambda\|_{0,\Omega}). \tag{3.14}$$

Proof. By the integration by parts,

$$\delta(a\nabla\lambda_h, \nabla\gamma)_h + (\lambda_h, \gamma)_\Omega = \delta(a\nabla\lambda, \nabla\gamma)_h + (\lambda, \gamma)_\Omega + E_h, \quad \gamma \in S_h,$$

where

$$E_h = -\delta \sum_{T\in\mathcal{T}_h} \langle \partial_\nu^a \lambda, \gamma\rangle_{\partial T} = -\delta \sum_{T\in\mathcal{T}_h} \langle \partial_\nu^a \lambda, \gamma - \overline{\gamma}\rangle_{\partial T}$$

since $\partial_\nu^a\lambda$ is continuous and $\overline{\gamma}$ is single valued on $\mathcal{K}_h$. Using Corollary (10.17) with the scaling argument in [1], we have

$$\begin{aligned} |E_h| &\le \delta \sum_{T\in\mathcal{T}_h} |\langle \partial_\nu^a \lambda, \gamma - \overline{\gamma}\rangle_{\partial T}| \\ &\le Ch\delta \sum_{T\in\mathcal{T}_h} |\lambda|_{2,T}|\gamma|_{1,T} \\ &\le Ch\delta|\lambda|_{2,h}|\gamma|_{1,h}. \end{aligned}$$

Then, for $\mu \in S_h$,

$$\begin{aligned} &\delta(a\nabla(\lambda_h - \mu), \nabla\gamma)_h + ((\lambda_h - \mu), \gamma)_\Omega \\ &= \delta(a\nabla(\lambda - \mu), \nabla\gamma)_h + ((\lambda - \mu), \gamma)_\Omega + E_h, \quad \gamma \in S_h. \end{aligned}$$

Taking $\gamma = \lambda_h - \mu$, we have

$$\begin{aligned} \|\lambda_h - \mu\|_{\delta,h} &\le \sqrt{\delta}\|\lambda - \mu\|_{1,h} + \|\lambda - \mu\|_{0,\Omega} + C\sqrt{\delta}h|\lambda|_{2,h} \\ &\le C(\sqrt{\delta}h + h^2)\|\lambda\|_{2,h} \end{aligned}$$

for an optimal $\mu \in S_h$. Since $\|\lambda\|_{2,\Omega} \le C(\|f\|_{0,\Omega} + \|\lambda\|_{0,\Omega})$ [3], the triangle inequality yields

$$\|\lambda - \lambda_h\|_{\delta,h} \le C(h^2 + \sqrt{\delta}h)(\|f\|_{0,\Omega} + \|u\|_{0,\Omega}). \qquad \square$$

4. Numerical analysis of the optimal control equations

As mentioned before, the equation (2.11) is solved by the CBE method in Subsection 3.1 and the equation (2.12) is solved by the lowest order Crouzeix-Raviart finite element method in Subsection 3.2.

Finally, we prove the convergence of (u_h, p_h) to the exact solution (u, p). It is easy to see that the above iteration ends with one iteration.

Theorem 4.1. *Let (u_h, p_h) be the approximate solution by the above Gauss-Seidel iteration, while (u, p) is the exact solution of* (2.8)-(2.10). *Then we have the following convergence estimates*:

$$\begin{aligned} \|u - u_h\|_{1,h} &\leq C(h\|f_d\|_{0,\Omega} + (\delta h + \sqrt{\delta}h^2)\|p\|_{0,\Omega}), \\ \|p - p_h\|_{0,\Omega} &\leq C(h^2 + \sqrt{\delta}h)(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}), \end{aligned}$$

where $f_d = -\nabla \cdot (a\nabla u_d)$.

Proof. From (2.11)-(2.13), w and $p = \frac{\lambda}{\delta}$ satisfy

$$\begin{aligned} -\nabla(a\nabla w) &= f_d, \\ -\delta\nabla(a\nabla p) + p &= f_d \end{aligned}$$

on Ω with boundary conditions, $w = 0$ and $p = 0$ on Γ. Analysis in Section 3 assert that w_h and p_h have the following convergence properties:

$$\begin{aligned} \|w - w_h\|_{1,h} &\leq Ch\|f_d\|_{0,\Omega}, \\ \|p - p_h\|_{\delta,h} &\leq C(h^2 + \sqrt{\delta}h)(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}). \end{aligned}$$

Therefore,

$$\begin{aligned} \|p - p_h\|_{0,h} &\leq C(h^2 + \sqrt{\delta}h)(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}), \\ |p - p_h|_{1,h} &\leq C(h + \frac{h^2}{\sqrt{\delta}})(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}). \end{aligned}$$

Since $\lambda = p\delta$,

$$|\lambda - \lambda_h|_{1,h} \leq C(\delta h + \sqrt{\delta}h^2)(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}).$$

Therefore,

$$\begin{aligned} \|u - u_h\|_{1,h} &\leq \|w - w_h\|_{1,h} + \|\lambda - \lambda_h\|_{1,h} \\ &\leq Ch\|f_d\|_{0,\Omega} + C(\delta h + \sqrt{\delta}h^2)(\|f_d\|_{0,\Omega} + \|p\|_{0,\Omega}) \\ &\leq C(h\|f_d\|_{0,\Omega} + (\delta h + \sqrt{\delta}h^2)\|p\|_{0,\Omega}). \end{aligned}$$

□

5. Nonlinear problem

Let us consider a nonlinear problem: minimize the cost functional

$$\mathcal{J}(u, p) = J_{u_d}(u) + \delta N(p) \tag{5.1}$$

subject to

$$-\nabla \cdot (a\nabla u) + \mathcal{F}(u) = p \quad \text{on } \Omega, \tag{5.2}$$

where $\mathcal{F}$ is a nonlinear operator. By following the same process in Section 2, we obtain the strong form of optimality system: find $(u, \lambda) \in [H_a^1(div; \Omega) \cap H_0^1(\Omega)]^2$ such that

$$-\nabla \cdot (a\nabla u) - \nabla \cdot (a\nabla \lambda) + \mathcal{F}'(u)^*\lambda = f_d, \tag{5.3}$$

$$-\nabla \cdot (a\nabla u) + \mathcal{F}(u) - \frac{\lambda}{\delta} = 0. \tag{5.4}$$

Here, $\mathcal{F}'(u)^*$ is the adjoint of $\mathcal{F}'(u)$ and $\mathcal{F}'$ is the Frechet derivative of $\mathcal{F}$. Introducing a new variable $w = u + \lambda$, we obtain the computing friendly form of equations:

$$\begin{aligned} -\nabla \cdot (a\nabla w) + \mathcal{F}'(u)^*\lambda &= f_d, \\ \delta\nabla \cdot (a\nabla w) + \{-\delta\nabla \cdot (a\nabla\lambda) + \lambda\} - \delta\mathcal{F}(u) &= 0, \\ -w + \lambda + u &= 0. \end{aligned}$$

Then the Gauss-Seidel iteration can provide an efficient numerical algorithm for the problem as follows:

$$-\nabla \cdot (a\nabla w^{n+1}) = -\mathcal{F}'(u^n)^*\lambda^n + f_d, \tag{5.5}$$

$$-\delta\nabla \cdot (a\nabla\lambda^{n+1}) + \lambda^{n+1} = \delta\{-\nabla \cdot (a\nabla w^{n+1}) + \mathcal{F}(u^n)\}, \tag{5.6}$$

$$u^{n+1} = w^{n+1} - \lambda^{n+1}. \tag{5.7}$$

Remark 5.1. Let us apply the Gauss-Seidel iteration directly to the system (5.3) and (5.4) as follows:

$$\begin{aligned} -\nabla \cdot (a\nabla u^{n+1}) - \nabla \cdot (a\nabla\lambda^n) + \mathcal{F}'(u^n)^*\lambda^n &= f_d, \\ -\nabla \cdot (a\nabla u^{n+1}) + \mathcal{F}(u^{n+1}) - \frac{\lambda^{n+1}}{\delta} &= 0. \end{aligned}$$

For simplification of our discussion, we assume $\mathcal{F} = 0$. Simple calculation yields that the sequence, $\{\lambda^n\}_{n=0}^{\infty}$ satisfies

$$\lambda^{n+1} = \delta\nabla(a\nabla\lambda^n) + \delta f_d.$$

Since the eigenvalues of the operator $-\delta\nabla(a\nabla\cdot)$ is unbounded, convergence may not happen unless the initial condition λ^0 is properly chosen.

Table 1: Numerical results with Example 6.1: $\delta = 10^{-4}$

n	$\|u - u_h\|_0$	α	$\|u - u_h\|_1$	α	$\|p - p_h\|_0$	α	$err(flux_D)$
4	9.7728e-02		3.3098e-01		7.0505e-01		7.4093e-03
8	2.5181e-02	1.96	1.5689e-01	1.08	1.8643e-01	1.92	7.6618e-03
16	6.3434e-03	1.99	7.7309e-02	1.02	4.7258e-02	1.98	7.7162e-03
32	1.5890e-03	2.00	3.8532e-02	1.00	1.1852e-02	2.00	7.7660e-03

Table 2: Numerical results with Example 6.1: $\delta = 10^{-6}$

n	$\|u - u_h\|_0$	α	$\|u - u_h\|_1$	α	$\|p - p_h\|_0$	α	$err(flux_D)$
4	9.7973e-02		3.3211e-01		7.0603e-01		7.4971e-05
8	2.5244e-02	1.96	1.5742e-01	1.08	1.8670e-01	1.92	7.7931e-05
16	6.3589e-03	1.99	7.7513e-02	1.02	4.7339e-02	1.98	7.8676e-05
32	1.5927e-03	2.00	3.8603e-02	1.01	1.1876e-02	2.00	7.8843e-05

6. Numerical examples

The Gauss-Seidel iteration for the linear problem (1.1) and (1.2) involves solving only two elliptic equations once. For the nonlinear problem (5.1) and

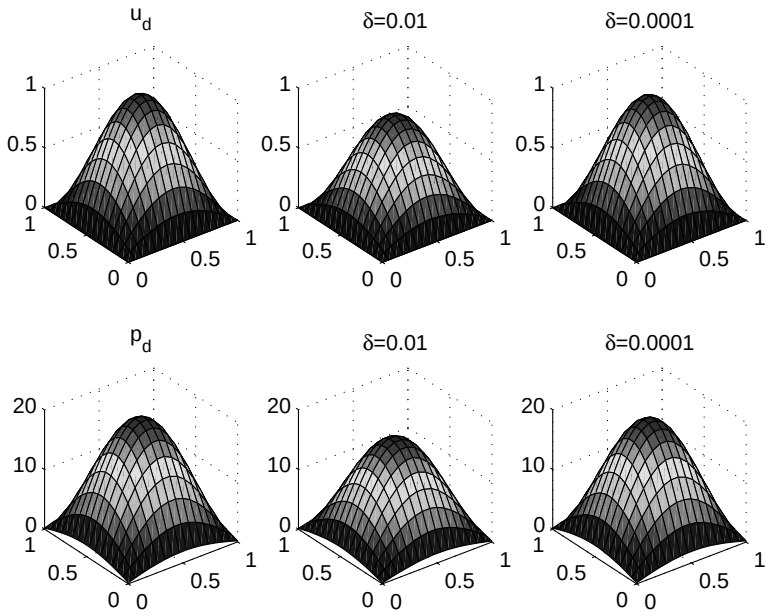

Figure 1: The controlled solutions u_h with the target solution u_d (top) and p_h with the target solution p_d (bottom).

(5.2) the Gauss-Seidel iteration requires for elliptic equations to be solved repeatedly and the algorithm (5.5)-(5.7) seems to provide a stable, fast converging numerical scheme when δ is small, at least for numerical examples considered in this section. Here, we provide both the linear and nonlinear examples.

Example 6.1. Let us consider a linear distributed optimal control problem:

$$\min_{u\in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\nabla(u-u_d)|^2 dx + \frac{\delta}{2}\int_\Omega p^2 dx \tag{6.1}$$

subject to

$$-\Delta u = p \quad \text{on } \Omega. \tag{6.2}$$

For the target function $u_d(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2)$, we can obtain the exact solution by solving (2.8)-(2.10). The exact solution is

$$\begin{aligned} u(x_1,x_2) &= \frac{1}{1+2\delta\pi^2}\sin(\pi x_1)\sin(\pi x_2), \\ p(x_1,x_2) &= \frac{2\pi^2}{1+2\delta\pi^2}\sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

The domain is the unit square and we consider a uniform triangulation of $\Omega = (0,1) \times (0,1)$, which consists of the right isosceles.

To investigate the flux approximation property of our algorithm, we consider a subdomain $D = (0,1) \times (0,1/2)$ and the approximate flux as follows:

$$flux_D = \int_{\partial D} (\nabla u_d \cdot \nu) ds \approx \int_{\partial D} (\nabla u_h \cdot \nu) ds.$$

Note that this approximation is worth for a small $0 < \delta << 1$ and it can be a measure of how our approximate solution matches the target function in sense of total flux on a certain region when δ is small.

As shown in Tables 1-2, the convergence of u_h in the energy norm is of order $O(h)$ as predicted in Theorem 4.1 and the convergence of p_h is of order $O(h^2)$ for any $\delta > 0$, which is better than that predicted in theory, $O(h^2 + \sqrt{\delta}h)$. Even though our analysis does not mention on the L_2 convergence we expect the $O(h^2)$ convergence and it is shown in Tables. Tables 3-4 represents numerical results for nonlinear problem (6.3). We observe desired numerical results. The error in $flux_D$ seems to be dependent only on δ. Indeed, $C\delta$ is the bound of the error in $flux_D$ for some constant $C > 0$ since the error comes only from the lowest order Crouzeix-Raviart finite element solver for the Lagrange multiplier λ. Comparing the linear and nonlinear problems we have the same convergence rates for u and p. Flux conservation property becomes a little worse because of the existence of a nonlinear term, which can not be approximated in a flux preserving manner.

Figure 1 represents numerical results obtained by changing control parameters for Example 6.1. It shows that the approximate solutions u_h and p_h converge to the target solutions as the control parameter δ becomes smaller.

Example 6.2. Let us consider a nonlinear distributed optimal control problem:

$$\min_{u \in H_0^1(\Omega), f} \frac{1}{2} \int_\Omega |\nabla(u - u_d)|^2 dx + \frac{\delta}{2} \int_\Omega p^2 dx \tag{6.3}$$

subject to

$$-\Delta u + 10u^2 = p \quad \text{on } \Omega. \tag{6.4}$$

The target function is $u_d(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$. Since the exact solution u is not known, we take u_h and p_h with $h = 1/96$ as the exact solutions.

Table 3: Numerical results for Example 6.2: $\delta = 10^{-4}$

n	$\|u - u_h\|_0$	α	$\|u - u_h\|_1$	α	$\|p - p_h\|_0$	α	$err(flux_D)$
4	9.7005e-02		4.4260e-01		2.4881e+00		1.3995e-02
8	2.4979e-02	1.96	2.1796e-01	1.02	6.7758e-01	1.88	1.4760e-02
16	6.2912e-03	1.99	1.0860e-01	1.01	1.7441e-01	1.96	1.4746e-02
32	1.3748e-03	2.19	5.4260e-02	1.00	4.5164e-02	1.95	1.4701e-02

Table 4: Numerical results for Example 6.2: $\delta = 10^{-6}$

n	$\|u-u_h\|_0$	α	$\|u-u_h\|_1$	α	$\|p-p_h\|_0$	α	$err(flux_D)$
4	9.7965e-02		4.4575e-01		2.4431e+00		1.4129e-04
8	2.5241e-02	1.96	2.1942e-01	1.02	6.7969e-01	1.85	1.4932e-04
16	6.3579e-03	1.99	1.0921e-01	1.01	1.7601e-01	1.95	1.4915e-04
32	1.3898e-03	2.19	5.4504e-02	1.00	4.4924e-02	1.97	1.4802e-04

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