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EXISTENCE AND CONTROLLABILITY RESULTS FOR NONDENSELY DEFINED STOCHASTIC EVOLUTION DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we investigate the existence and controllability results for a class of abstract stochastic evolution differential inclusions with nonlocal conditions where the linear part is nondensely defined and satisfies the Hille-Yosida condition. The results are obtained by using integrated semigroup theory and a fixed point theorem for condensing map due to Martelli.

1. Introduction

In this paper, we are interested in the existence and controllability problem of the following system

$$\begin{cases} dx(t) \in [Ax(t) + F(t, x(t))]dt + g(t, x(t))dW(t), & t \in [0, T], \\ x(0) + \theta(x) = x_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a nondensely defined closed linear operator on a separable Hilbert space H. Let K be another separable Hilbert space. Suppose $W(t)$ is a given K-valued Brownian motion with a finite trace nuclear covariance operator $Q \geq 0$. $\mathcal{P}(H)$ denotes the space of nonempty subsets of the space H. Assume that $F : [0, T] \times H \to \mathcal{P}(H)$, $g : [0, T] \times H \to L_2^0(K, H)$ are two measurable mappings, where $L_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H and it will be described in detail Section 2 and $\theta : H \to L^2(\Omega, H)$ is a given function.

Stochastic differential equations of inclusions play a very important role in mechanical, electrical, control engineering as well as physical, economic, and

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social science. Therefore, the theory of stochastic differential equations of inclusions has been developed at early stage. There has been extensive existence and controllability results for stochastic differential equations or inclusions, where the operator A is densely defined and satisfies the Hille-Yosida condition or equivalently, A generates a C_0 semigroup. Many important results can be found in [3, 6, 7, 8, 11, 20, 25, 26] and references cited therein. However, as indicated in [12], we sometimes need to deal with nondensely defined operators. For example, age-dependent population models can be written as abstract semilinear functional differential equations with a nondensely defined Hille-Yosida operator. See [12] for more examples and remarks concerning non-densely defined operators. When the operator A is nondensely defined and the problem (1.1) is deterministic, existence of integral solutions and controllability results have been obtained in many works by using integrated semigroup theory, and the readers can refer to [1, 2, 10, 15, 16, 17, 22, 23] and references cited therein.

To the best of our knowledge, there is no work reported on the existence of integral solutions and controllability results for the stochastic functional differential equations or inclusions with nondensely defined operators. Since stochastic effects exist widely in realistic situations, it is necessary to discuss stochastic differential equations or inclusions. The aim of this paper is to close the gap. The main tools in the approach followed in this work are the theory of integrated semigroups, a fixed point theorem of multivalued map due to Martelli and the Itô formula for stochastic integral in Hilbert spaces.

This paper will be organized as follows. In Section 2, we will recall some basic definitions and preliminary facts from multivalued analysis, integrated semigroups and stochastic integral in Hilbert spaces which will be used later. Section 3 is devoted to the existence of integral solutions to problem (1.1). Section 4 is reserved for controllability results.

2. Preliminaries

Let $\{\Omega, \mathfrak{F}, P\}$ be a complete probability space equipped with some σ-algebras $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let H and K be two real separable Hilbert spaces. Without the risk of confusion, we just use $\langle\cdot,\cdot\rangle$ for the inner product and $|\cdot|$ for the norm.

Let $\beta_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of real-valued one dimensional standard Brownian motions mutually independent over $\{\Omega, \mathfrak{F}, P\}$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are nonnegative real numbers and e_n $(n = 1, 2, \ldots)$ is a complete orthonormal basis in K. Let $Q \in L(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then the above K-valued stochastic process $W(t)$ is called a Q Winner process. We denote by $L(K, H)$ the set of all linear bounded operators from K into H. For $\Psi \in$

$L(K,H)$, we define

$$\|\Psi\|_{L_2^0}^2 = \mathrm{Tr}(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Psi e_n\|^2.$$

If $\|\Psi\|_{L_2^0} < \infty$, then Ψ is called a Q-Hilbert-Schmidt operators. Let $L_2^0(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to H$. Let $\Psi : [0,T] \to L_2^0(K,H)$ be predicable process such that

$$\int_0^t E\|\Psi\|_{L_2^0}^2 ds < \infty.$$

Then we can define the H-valued stochastic integral $\int_0^t \Psi(s)dW(s)$, which is a continuous square integrable martingale. For more details of this construction, see Da. Prato [13].

Definition 2.1 (see [4])**.** Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:
(i) $S(0) = 0$;
(ii) $t \to S(t)$ is strongly continuous;
(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \geq 0$.

Definition 2.2 (see [18])**.** An operator A is called a generator of an integrated semigroup, if there exists $\omega \in R$ such that $(\omega, +\infty) \subset \rho(A)$ and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.

Definition 2.3. We say that linear operator A satisfies the Hille-Yosida condition if there exist $M \geq 0$ and $\omega \in R$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n \|R(\lambda, A)^n\|, n \in N, \lambda > \omega\} \leq M.$$

Theorem 2.4 (see [18])**.** *The following assertions are equivalent*:
(i) *A is the generator of a locally Lipschitz continuous integrated semigroup*;
(ii) *A satisfies the Hille-Yosida condition.*

Here and hereafter, we assume that A satisfies the Hille-Yosida condition. Let us introduce the part A_0 of A in $\overline{D(A)}$: $A_0 = A$ on $D(A_0) = \{x \in D(A); Ax \in \overline{D(A)}\}$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. We note that $(S'(t))_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|S'(t)\| \leq Me^{\omega t}$, $t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition (see [18, 23]).

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition it is easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \leq M|x|$.

For more properties on integral semigroup theory, the interested reader may refer to [5, 22].

Let (X, d) be a metric space, $\mathcal{P}(X)$ denotes the family for all nonempty subsets of X. We use the notations:

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}, \quad P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},$$

$$P_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}, \quad P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}.$$

A multivalued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$, F is bounded on bounded sets if $F(B) = U_{x\in B}F(x)$ is bounded in X for all $B \in P_b(X)$, i.e., $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} \leq \infty$. F is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subset \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map F is completely continuous with nonempty compact valued, then F is u.s.c. if and only if F has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

Definition 2.5 (see [9])**.** An upper semicontinuous map $G : H \to H$ is said to be condensing if for any bounded subset $V \subset H$ with $\alpha(V) \neq 0$, we have $\alpha(G(V)) < \alpha(V)$, where α denotes the Kuratowski measure of noncompactness.

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

Theorem 2.6 (see [19])**.** *Let J be a compact interval and E be a Banach space. Let $F : J \times C(J, E) \to P_{b,cl,c}(E)$, $(t, u) \mapsto F(t, u)$ be measurable with respect to t for each $u \in E$, upper semicontinuous with respect to u for each $t \in J$. Moreover, for each fixed $u \in C(J, E)$ the set*

$$N_{F,u} = \{f \in L^2(J, E) : f(t) \in F(t, u) \text{ for a.e. } t \in [0, T]\}$$

is nonempty. Also let Γ be a linear continuous mapping from $L^2(J, E)$ to $C(J, E)$, then the operator

$$\Gamma \circ N_F : C(J, E) \to P_{b,cl,c}(C(J, E)), \quad u \to (\Gamma \circ N_F)(u) = \Gamma(N_{F,u})$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Theorem 2.7 (see [21])**.** *Let E be a Banach space and $\Phi : E \to P_{b,cl,c}(E)$ a condensing map. If the set*

$$U = \{x \in E : \delta x \in \Phi x \text{ for some } \delta > 1\}$$

is bounded, then Φ has a fixed point.

More details on multivalued maps can be found in the book of Deimling [14].

3. Existence results

Let $C = C([0,T], L^2(\Omega, H))$ denote the class of H-valued stochastic process $\{\xi(t) : t \in [0,T]\}$ which are $\mathfrak{F}_t$-adapt and $\|\xi\| = \|\xi\|_C = \sup_{t\in[0,T]}(E|\xi(t)|^2)^{\frac{1}{2}}$. It is easy to verify that C furnished with the norm topology as defined above is a Banach space.

Definition 3.1. We say that $x \in C$ is an integral solution of problem (1.1) if

(i) $\int_0^t x(s)ds \in D(A),\ \ t \in [0,T]$,

(ii) there exists a function $f \in L^2([0,T], H)$ such that $f(t) \in F(t, x(t))$ a.e. $t \in [0,T]$ and

(3.1)

$$x(t) = x_0 - \theta(x) + A\int_0^t x(s)ds + \int_0^t f(s)ds + \int_0^t g(s, x(s))dW(s), \quad t \in [0,T].$$

From this definition, we deduce that for an integral solution x, we have $x(t) \in \overline{D(A)}$ for all $t \in [0,T]$, because $x(t) = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} x(s)ds$ and $\int_t^{t+h} x(s)ds \in D(A)$. In particular, $x_0 - \theta(x) \in L^2(\Omega, \overline{D(A)})$. So, if we assume that $x_0 \in L^2(\Omega, \overline{D(A)})$ we conclude that $\theta(x) \in L^2(\Omega, \overline{D(A)})$.

Lemma 3.2. *If x is an integral solution of problem* (1.1)*, then for $t \in [0,T]$, $x(t)$ is given by*

$$x(t) = S'(t)(x_0 - \theta(x)) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds$$
$$+ \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s).$$

Proof. Let $x(t)$ be an integral solution of problem (1.1) and set $x_\lambda(t) = B_\lambda x(t)$. Then, by applying B_λ to both sides of (3.1), we get

$$x_\lambda(t) = B_\lambda(x_0 - \theta(x)) + \int_0^t Ax_\lambda(s)ds + \int_0^t B_\lambda f(s)ds + \int_0^t B_\lambda g(s, x(s))dW(s).$$

Set $u(s, x_\lambda(s)) = S'(t-s)x_\lambda(s)$, then by Itô formula (see [13]), we get that

$$\begin{aligned}
&dS'(t-s)x_\lambda(s)\\
&= S'(t-s)B_\lambda g(s, x(s))dW(s)\\
&\quad + [-AS'(t-s)x_\lambda(s) + S'(t-s)(Ax_\lambda(s) + B_\lambda f(s))]ds\\
&= S'(t-s)B_\lambda g(s, x(s))dW(s) + S'(t-s)B_\lambda f(s)ds.
\end{aligned}$$

Integrating the equality above from 0 to t and noting that $S'(0) = I$ and $x(0) = x_0 - \theta(x)$, we have

$$x_\lambda(t) = S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds$$
$$+ \int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s).$$

Let $\lambda \to \infty$, since for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$, we have

$$\begin{aligned} x(t) &= S'(t)(x_0 - \theta(x)) + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda f(s)ds \\ &\quad + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s). \end{aligned}$$

This completes the proof. □

We are now in a position to state and prove our main result for the existence of solutions of problem (1.1).

Let us list the following hypotheses:

(H1) A satisfies the Hille-Yosida condition.

(H2) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

(H3) $F : [0,T] \times H \to P_{b,cl,c}(H)$ for each $u \in H$, $F(\cdot, u)$ is measurable and for each $t \in [0,T]$, $F(t,\cdot)$ is upper semicontinuous. For each fixed $u \in H$, the set $N_{F,u} = \{f \in L^2([0,T],H) : f(t) \in F(t,u) \text{ for a.e. } t \in [0,T]\}$ is not empty.

(H4) $g : [0,T] \times H \to L_2^0(K,H)$ is continuous with respect to u and there exist constants c_1, $c_2 \geq 0$ such that $|g(t,u)|^2 \leq c_1|u|^2 + c_2$ for $u \in H$.

(H5) $x_0 \in L^2(\Omega, \overline{D(A)})$, $\theta : H \to L^2(\Omega, \overline{D(A)})$ and there exists $L > 0$ such that $E|\theta(x)|^2 \leq L$ for all $x \in H$.

(H6) $|F(t,u)|^2 = \sup\{|v|^2 : v \in F(t,u)\} \leq \eta(t)\Psi((|u|^2))$ for almost all $t \in [0,T]$ and $u \in H$, where $\eta \in L^1([0,T],\mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \to (0,\infty)$ is continuous concave and increasing with

$$\int_0^T \overline{m}(s)ds < \int_{c_0}^\infty \frac{d\tau}{1+\tau+\Psi(\tau)},$$

where

$$\begin{aligned} c_0 &= 6M^2 e^{2|\omega|T}(E|x_0|^2 + L), \\ \overline{m}(t) &= \max\{c_3 e^{-2\omega t}\eta(t), c_1 c_4 e^{-2\omega t},\ c_2 c_4 e^{-2\omega t}\}, \\ c_3 &= 3M^4 T e^{2|\omega|T}, \\ c_4 &= 3M^4 e^{2|\omega|T}. \end{aligned}$$

Theorem 3.3. *Assume that hypotheses* (H1)-(H6) *hold. Then the problem* (1.1) *has at least one integral solution on* $[0,T]$.

Proof. Denote $C_0 = C([0,T],\ L^2(\Omega, \overline{D(A)}))$, which is a closed subset of C. Obviously, C_0 with the same norm in C is also a Banach space. Consider the multivalued map $\Phi : C_0 \to \mathcal{P}(C_0)$ defined by

$$\begin{aligned} \Phi x = \Big\{ h \in C_0 : B_\lambda h(t) &= S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds \\ &\quad + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s), \quad t \in [0,T] \Big\}, \end{aligned}$$

where $f \in N_{F,x} = \{f \in L^2([0,T] \times \Omega, H) : f(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\}$.

Since $\lim_{\lambda\to\infty} B_\lambda h = h$ for $h \in C_0$, it is clear that the fixed points of Φ are integral solutions to problem (1.1). We shall prove that Φ satisfies Theorem 2.7 in the following steps.

Step 1. $\Phi(x)$ is convex for each $x \in C_0$.

Indeed, if h_1 and h_2 belong to Φx, then there exist f_1, $f_2 \in N_{F,x}$ such that for each $t \in [0,T]$, we have

$$B_\lambda h_i(t) = S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f_i(s)ds + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s), \quad i = 1, 2.$$

Let $0 \le k \le 1$, then for each $t \in [0,T]$, we have

$$\begin{aligned} &B_\lambda(kh_1 + (1-k)h_2)(t) \\ &= S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda(kf_1(s) + (1-k)f_2(s))ds \\ &\quad + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s). \end{aligned}$$

Since $N_{F,x}$ is convex, we have $kh_1 + (1-k)h_2 \in \Phi x$. The proof of Step 1 is completed.

Step 2. Φ maps bounded sets into bounded sets in C_0.

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in \Phi x$, $x \in B_q = \{x \in C_0, \|x\|^2 \le q\}$ one has $\|h\|^2 \le l$.

Let $h \in \Phi x$, then there exists $f \in N_{F,x}$ such that for $t \in [0,T]$, we have

$$B_\lambda h(t) = S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s).$$

Thus

$$\begin{aligned} &E|B_\lambda h(t)|^2 \\ &= E\left|S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds \right.\\ &\qquad \left. + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s)\right|^2 \\ &\le 3E|S'(t)B_\lambda(x_0 - \theta(x))|^2 + 3E|\int_0^t S'(t-s)B_\lambda f(s)ds|^2 \\ &\quad + 3E|\int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s)|^2 \end{aligned}$$

$$\leq 3M^2e^{2\omega t}E|B_\lambda(x_0-\theta(x))|^2+3M^4e^{2\omega t}TE\int_0^t e^{-2\omega s}|f(s)|^2ds$$
$$+3M^4e^{2\omega t}E\int_0^t e^{-2\omega s}|g(s,x(s))|^2ds.$$

Hence,

$$\lim_{\lambda\to\infty}E|B_\lambda h(t)|^2$$
$$\leq 3M^2e^{2\omega t}\lim_{\lambda\to\infty}E|B_\lambda(x_0-\theta(x))|^2+3M^4e^{2\omega t}TE\int_0^t e^{-2\omega s}|f(s)|^2ds$$
$$+3M^4e^{2\omega t}E\int_0^t e^{-2\omega s}|g(s,x(s))|^2ds.$$

From the Fatou lemma, Fubini theorem and the fact that $\lim_{\lambda\to\infty}B_\lambda x=x$ for all $x\in\overline{D(A)}$, we get that

$$E|h(t)|^2\leq 3M^2e^{2\omega t}E|(x_0-\theta(x))|^2+3M^4e^{2\omega t}T\int_0^t e^{-2\omega s}E|f(s)|^2ds$$
$$+3M^4e^{2\omega t}\int_0^t e^{-2\omega s}E|g(s,x(s))|^2ds.$$

From (H4)-(H6), we have for each $t\in[0,T]$,

$$E|h(t)|^2\leq 6M^2e^{2\omega t}(E|x_0|^2+E|\theta(x)|^2)+3M^4e^{2\omega t}T\int_0^t e^{-2\omega s}E|f(s)|^2ds$$
$$+3M^4e^{2\omega t}\int_0^t e^{-2\omega s}E|g(s,x(s))|^2ds$$
$$\leq 6M^2e^{2|\omega|T}(E|x_0|^2+L)+3M^4Te^{2|\omega|T}\Psi(q)\int_0^t e^{-2\omega s}\eta(s)ds$$
$$+M^4e^{2|\omega|T}\int_0^t e^{-2\omega s}(c_1q+c_2)ds.$$

Then, for each $h\in\Phi(B_q)$ we have

$$\|h\|^2=\sup_{t\in[0,T]}E|h(t)|^2$$
$$\leq 6M^2e^{2|\omega|T}(E|x_0|^2+L)+3M^4Te^{2|\omega|T}\Psi(q)\int_0^T e^{-2\omega s}\eta(s)ds$$
$$+3M^4e^{2|\omega|T}\int_0^T e^{-2\omega s}(c_1q+c_2)ds$$
$$:=l.$$

Step 3. Φ maps bounded sets into equicontinuous sets of C_0.

Let $t_1,\, t_2 \in [0,T]$, $t_1 < t_2$ and $B_q = \{x \in C_0, \|x\|^2 \le q\}$ be a bounded set of C_0. For each $x \in B_q$ and $h \in \Phi x$, there exists $f \in N_{F,x}$ such that

$$\begin{aligned} B_\lambda h(t) = S'(t)B_\lambda(x_0-\theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds \\ + \int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s), \quad t\in[0,T]. \end{aligned}$$

Therefore we have

$$\begin{aligned} & E|B_\lambda(h(t_2)-h(t_1))|^2 \\ &= E\bigg|S'(t_2)B_\lambda(x_0-\theta(x)) + \int_0^{t_2} S'(t_2-s)B_\lambda f(s)ds \\ &\quad + \int_0^{t_2} S'(t_2-s)B_\lambda g(s,x(s))dW(s) - S'(t_1)B_\lambda(x_0-\theta(x)) \\ &\quad - \int_0^{t_1} S'(t_1-s)B_\lambda f(s)ds - \int_0^{t_1} S'(t_1-s)B_\lambda g(s,x(s))dW(s)\bigg|^2 \\ &= E\bigg|(S'(t_2)-S'(t_1))B_\lambda(x_0-\theta(x)) + \int_0^{t_2}(S'(t_2-s)-S'(t_1-s))B_\lambda f(s)ds \\ &\quad + \int_0^{t_2}(S'(t_2-s)-S'(t_1-s))B_\lambda g(s,x(s))dW(s) \\ &\quad + \int_{t_1}^{t_2} S'(t_1-s)B_\lambda f(s)ds + \int_{t_1}^{t_2} S'(t_1-s)B_\lambda g(s,x(s))dW(s)\bigg|^2 \\ &\le 5|S'(t_2)-S'(t_1)|^2E|B_\lambda(x_0-\theta(x))|^2 \\ &\quad + 5E\left|\int_0^{t_2}(S'(t_2-s)-S'(t_1-s))B_\lambda f(s)ds\right|^2 \\ &\quad + 5\left|\int_0^{t_2}(S'(t_2-s)-S'(t_1-s))B_\lambda g(s,x(s))dW(s)\right|^2 \\ &\quad + 5E\left|\int_{t_1}^{t_2} S'(t_1-s)B_\lambda f(s)ds\right|^2 + 5E\left|\int_{t_1}^{t_2} S'(t_1-s)B_\lambda g(s,x(s))dW(s)\right|^2 \\ &\le 5|S'(t_2)-S'(t_1)|^2E|B_\lambda(x_0-\theta(x))|^2 \\ &\quad + 5TM^2\int_0^{t_2}|S'(t_2-s)-S'(t_1-s)|^2E|f(s)|^2ds \\ &\quad + 5M^2\int_0^{t_2}|S'(t_2-s)-S'(t_1-s)|^2E|g(s,x(s))|^2ds \\ &\quad + 5TM^2\int_{t_1}^{t_2}|S'(t_1-s)|^2E|f(s)|^2ds \end{aligned}$$

$$+ 5M^2 \int_{t_1}^{t_2} |S'(t_1 - s)|^2 E|g(s, x(s))|^2 ds.$$

Let $\lambda \to \infty$, we get

$$\begin{aligned}
&E|h(t_2) - h(t_1)|^2 \\
\leq\ & 5|S'(t_2) - S'(t_1)|^2 E|(x_0 - \theta(x))|^2 \\
&+ 5TM^2 \int_0^{t_2} |S'(t_2 - s) - S'(t_1 - s)|^2 E|f(s)|^2 ds \\
&+ 5M^2 \int_0^{t_2} |S'(t_2 - s) - S'(t_1 - s)|^2 E|g(s, x(s))|^2 ds \\
&+ 5TM^2 \int_{t_1}^{t_2} S'(t_1 - s)^2 E|f(s)|^2 ds + 5M^2 \int_{t_1}^{t_2} |S'(t_1 - s)|^2 E|g(s, x(s))|^2 ds.
\end{aligned}$$

(H2) implies that $S'(t)$ for $t > 0$ is continuous in the uniform operator topology. Combining this with (H4)-(H6), we have the right side of the above inequality tend to zero as $t_2 \to t_1$. This completes the proof.

Step 4. $(\Phi B_q)(t)$ is relatively compact in C_0 for each t, where $(\Phi B_q)(t) = \{h(t) : h \in \Phi B_q\}$, $t \in [0, T]$ and $B_q = \{x \in C_0, \|x\|^2 \leq q\}$.

It is obvious that $(\Phi B_q)(t)$ is relatively compact in C_0 for $t = 0$.

Let $0 < t \leq T$ be fixed and $0 < \epsilon < t$. For $x \in B_q$ and $h \in \Phi x$, there exists $f \in N_{F,x}$ such that

$$\begin{aligned}
B_\lambda h(t) = S'(t)B_\lambda(x_0 - \theta(x)) &+ \int_0^{t-\epsilon} S'(t - s)B_\lambda f(s)ds \\
&+ \int_{t-\epsilon}^{t} S'(t - s)B_\lambda f(s)ds + \int_0^{t-\epsilon} S'(t - s)B_\lambda g(s, x(s))dW(s) \\
&+ \int_{t-\epsilon}^{t} S'(t - s)B_\lambda g(s, x(s))dW(s).
\end{aligned}$$

We define

$$\begin{aligned}
B_\lambda h_\epsilon(t) = S'(t)B_\lambda(x_0 - \theta(x)) &+ \int_0^{t-\epsilon} S'(t - s)B_\lambda f(s)ds \\
&+ \int_0^{t-\epsilon} S'(t - s)B_\lambda g(s, x(s))dW(s).
\end{aligned}$$

Let $\lambda \to \infty$, we get

$$\begin{aligned}
h_\epsilon(t) = S'(t)(x_0 - \theta(x)) + S'(\epsilon) \lim_{\lambda \to \infty} \Big[&\int_0^{t-\epsilon} S'(t - \epsilon - s)B_\lambda f(s)ds \\
&+ \int_0^{t-\epsilon} S'(t - \epsilon - s)B_\lambda g(s, x(s))dW(s) \Big].
\end{aligned}$$

Since $S'(t)$ is a compact operator, the set $V_\epsilon(t) = \{h_\epsilon(t), h \in \Phi B_q\}$ is relative compact in C_0 for each ϵ, $0 < \epsilon \leq t$. Then we have

$$\begin{aligned}
&E|B_\lambda(h(t) - h_\epsilon(t))|^2 \\
&= E\left|\int_{t-\epsilon}^{t} S'(t-s)B_\lambda f(s)ds - \int_{t-\epsilon}^{t} S'(t-s)B_\lambda g(s, x(s))dW(s)\right|^2 \\
&\leq 2M\epsilon^2 \int_{t-\epsilon}^{t} |S'(t-s)|^2 E|f(s)|^2 ds + 2M\int_{t-\epsilon}^{t} |S'(t-s)|^2 E|g(s, x(s))|^2 ds.
\end{aligned}$$

Hence, $E|h(t) - h_\epsilon(t)|^2 = \lim_{\lambda\to\infty} E|B_\lambda(h(t) - h_\epsilon(t))|^2 \to 0$ as $\epsilon \to 0$.

Therefore, there are relative compact sets arbitrarily close to the set $\{h(t), h \in \Phi(B_q)\}$. Thus the set $\{h(t), h \in \Phi(B_q)\}$ is relative compact in C_0. As a consequence of Steps 2, 3, 4 and the Arzela-Ascoli theorem it is concluded that $\Phi : C_0 \to \mathcal{P}(C_0)$ is a completely continuous map.

Step 5. Φ has a closed graph.

Let $x_n \to x_*$, $h_n \in \Phi x_n$ and $h_n \to h_*$ as $n \to \infty$, we shall prove that $h_* \in \Phi x_*$. $h_n \in \Phi x_n$ means that there exists $f_n \in N_{F,x_n}$ such that

$$\begin{aligned}
B_\lambda h_n(t) = S'(t)B_\lambda(x_0 - \theta(x_n)) &+ \int_0^t S'(t-s)B_\lambda f_n(s)ds \\
&+ \int_0^t S(t-s)B_\lambda g(s, x_n(s))dW(s), \quad t \in [0, T].
\end{aligned}$$

We must prove that there exists $f_* \in N_{F,x_*}$ such that

$$\begin{aligned}
B_\lambda h_*(t) = S'(t)B_\lambda(x_0 - \theta(x_*)) &+ \int_0^t S'(t-s)B_\lambda f_*(s)ds \\
&+ \int_0^t S'(t-s)B_\lambda g(s, x_*(s))dW(s), \quad t \in [0, T].
\end{aligned}$$

Since $u \to g(t, u)$ and θ are continuous we have that

$$\begin{aligned}
&\Bigg\| B_\lambda h_n - S'(t)B_\lambda(x_0 - \theta(x_n)) - \int_0^t S'(t-s)B_\lambda g(s, x_n(s))dW(s) \\
&- \left[B_\lambda h_*(t) - S'(t)B_\lambda(x_0 - \theta(x_*)) - \int_0^t S'(t-s)B_\lambda g(s, x_*(s))dW(s)\right]\Bigg\|
\end{aligned}$$

tends to 0 as $n \to 0$. Consider the linear continuous operator

$$\Gamma : L^2([0, T], H) \to C([0, T], H), \quad f \to (\Gamma f)(t) = \int_0^t S'(t-s)B_\lambda f(s)ds.$$

From Theorem 2.6, it follows that $\Gamma \circ N_F$ is a closed graph operator. Moreover, we have that

$$B_\lambda h_n - S'(t)B_\lambda(x_0 - \theta(x_n)) - \int_0^t S'(t-s)B_\lambda g(s, x_n(s))dW(s) \in \Gamma(N_{F,x_n}).$$

Since $x_n \to x_*$, it follows from Theorem 2.6 that there exists $f_* \in N_{F,x_*}$ such that

$$B_\lambda h_*(t) - S'(t)B_\lambda(x_0 - \theta(x_*)) - \int_0^t S'(t-s)B_\lambda g(s, x_*(s))dW(s)$$
$$= \int_0^t S'(t-s)B_\lambda f_*(s)ds.$$

Therefore Φ is a completely continuous multivalued map, u.s.c. with convex closed values. In order to prove that Φ has a fixed point, we need one more step.

Step 6. The set $U = \{x \in C_0 : \delta x \in \Phi x \text{ for some } \delta > 1\}$ is bounded.

Let $x \in U$, then $\delta x \in \Phi x$ for some $\delta > 1$. Thus there exists $f \in N_{F,x}$ such that for $t \in [0, T]$,

$$\begin{aligned} B_\lambda x(t) = \delta^{-1}S'(t)B_\lambda(x_0 - \theta(x)) + \delta^{-1}\int_0^t S'(t-s)B_\lambda f(s)ds \\ + \int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s). \end{aligned}$$

From (H4)-(H6) we have that for each $t \in [0, T]$,

$$\begin{aligned} E|x(t)|^2 = {} & \lim_{\lambda\to\infty} E\bigg|\delta^{-1}S'(t)B_\lambda(x_0 - \theta(x)) + \delta^{-1}\int_0^t S'(t-s)B_\lambda f(s)ds \\ & + \delta^{-1}\int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s)\bigg|^2 \\ \le {} & \lim_{\lambda\to\infty} 3|S'(t)|^2 E|B_\lambda(x_0 - \theta(x))|^2 + 3M^4e^{2\omega t}T\int_0^t e^{-2\omega s}E|f(s)|^2ds \\ & + 3M^4e^{2\omega t}\int_0^t e^{-2\omega s}E|g(s, x(s))|^2ds \\ \le {} & 6M^2e^{2|\omega|T}(E|x_0|^2 + L) + 3M^4Te^{2|\omega|T}\int_0^t e^{-2\omega s}\eta(s)\Psi(E|x(s)|^2)ds \\ & + 3M^4e^{2|\omega|T}\int_0^t e^{-2\omega s}(c_1E|x(s)|^2 + c_2)ds. \end{aligned}$$

We shall consider the function μ defined by $\mu(t) = \sup\{E|x(s)|^2, 0 \le s \le t\}$. By the previous inequality we have for $t \in [0, T]$,

$$\begin{aligned} \mu(t) \le 6M^2e^{2|\omega|T}(E|x_0|^2 + L) + 3M^4Te^{2|\omega|T}\int_0^t e^{-2\omega s}\eta(s)\Psi(\mu(s))ds \\ + 3M^4e^{2|\omega|T}\int_0^t e^{-2\omega s}(c_1\mu(s) + c_2)ds. \end{aligned}$$

By (H6), we have

$$\mu(t) \leq c_0 + c_3 \int_0^t e^{-2\omega s}\eta(s)\Psi(\mu(s))ds + c_4 \int_0^t e^{-2\omega s}(c_1\mu(s) + c_2)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have $\mu(t) \leq v(t)$ for all $t \in [0,T]$ with $v(0) = c_0$ and

$$v'(t) = c_3 e^{-2\omega t}\eta(t)\Psi(\mu(t)) + c_4 e^{-2\omega t}(c_1\mu(t) + c_2).$$

Using that Ψ is increasing, we get

$$\begin{aligned} v'(t) &\leq c_3 e^{-2\omega t}\eta(t)\Psi(v(t)) + c_4 e^{-2\omega t}(c_1 v(t) + c_2) \\ &\leq \overline{m}(t)(1 + v(t) + \Psi(v(t))). \end{aligned}$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{1 + v(s) + \Psi(v(s))}ds \leq \int_0^t \overline{m}(s)ds.$$

By a change of variable and (H6), we obtain

$$\int_{v(0)}^{v(t)} \frac{d\tau}{1 + \tau + \Psi(\tau)} \leq \int_0^T \overline{m}(s)ds < \int_{c_0}^{\infty} \frac{d\tau}{1 + \tau + \Psi(\tau)}.$$

This inequality implies that there exists a constant $\overline{L}$ such that $v(t) \leq \overline{L},\ t \in [0,T]$ and hence $\mu(t) \leq \overline{L},\ t \in [0,T]$. Since for every $t \in [0,T]$, $E|x(t)|^2 \leq \mu(t)$, we have $\|x\|^2 = \sup\{E|x(t)|^2 : 0 \leq t \leq T\} \leq \overline{L}$, where $\overline{L}$ depends only on T and on the functions η and Ψ. This shows that U is bounded.

As a consequence of Theorem 2.7, we conclude that Φ has a fixed point which is the integral solution of problem (1.1). This completes the proof. □

4. Controllability results

In this section, we prove controllability results for stochastic semilinear evolution differential equations with nonlocal conditions of the form

$$(4.1) \quad \begin{cases} dx(t) \in [Ax(t) + F(t, x(t)) + Bu(t)]dt + g(t, x(t))dW(t),\ t \in [0,T], \\ x(0) + \theta(x) = x_0, \end{cases}$$

where A, F, g and θ are as in system (1.1), the control operator $u(\cdot)$ take values in $L^2([0,T], U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H.

Definition 4.1. We say that $x(t) \in C$ is an integral solution of problem (4.1) if

(i) $\int_0^t x(s)ds \in D(A),\ \ t \in [0,T]$,

(ii) there exists a function $f \in L^2([0,T], H)$ such that $f(t) \in F(t, x(t))$ a.e. $t \in [0,T]$ and

$$x(t) = x_0 - \theta(x) + A\int_0^t x(s)ds + \int_0^t [f(s) + (Bu)(s)]ds$$

$$+\int_0^t g(s,x(s))dW(s), \quad t\in[0,T],$$

where C is as in Section 3.

Lemma 4.2. *If x is an integral solution of problem* (4.1)*, then for $t\in[0,T]$, $x(t)$ is given by*

$$\begin{aligned} x(t) &= S'(t)(x_0-\theta(x)) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda (Bu)(s)ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda g(s,x(s))dW(s). \end{aligned}$$

The proof is similar to the proof of Lemma 3.2, we omit it here.

Definition 4.3. The system (4.1) is said to be controllable on $[0,T]$ if for every $x_1\in L^2(\Omega,\overline{D(A)})$, there exists a control $u\in L^2([0,T],U)$ such that the integral solution $x(t)$ of system (4.1) satisfies $x(T)+\theta(x)=x_1$.

Theorem 4.4. *Assume that hypotheses* (H1)-(H5) *hold. In addition, we suppose that*

(H7) *the linear operator $W: L^2([0,T],U)\to D(A)$ defined by*

$$Wu=\int_0^t S'(t-s)B_\lambda(Bu)(s)ds$$

has a pseudo-invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2([0,T],U)\backslash \mathrm{Ker}W$ and there exist positive constants M_1 and M_2 such that $\|B\|\le M_1$ and $\|\widetilde{W}^{-1}\| \le M_2$;

(H8) $|F(t,x)|^2 = \sup\{|v|^2 : v\in F(t,x)\} \le \eta(t)\Psi((|x|^2))$ *for almost all $t\in[0,T]$ and $x\in H$, where $\eta\in L^1([0,T],\mathbb{R}^+)$ and $\Psi:\mathbb{R}^+\to(0,\infty)$ is continuous concave and increasing, moreover, there exists a positive $\overline{L}$ such that*

$$\frac{(1-c_4)\overline{L}}{c_3+c_5\Psi(\overline{L})\int_0^T\eta(s)ds}>1,$$

where

$$\begin{aligned} c_3 &= 8M^2e^{2|\omega|T}(E|x_0|^2+L)+4M^4e^{2|\omega|T}Tc_2 \\ &\quad + 4M^4M_1^2e^{2|\omega|T}T^2M_2^2\Big[8(E|x_1|^2+L)+8e^{2|\omega|T}(E|x_0|^2+L) \\ &\quad + 4M^4e^{2|\omega|T}Tc_2\Big], \\ c_4 &= 4M^4e^{2|\omega|T}Tc_1+4M^4M_1^2e^{2|\omega|T}T^2M_2^2 4M^4e^{2|\omega|T}Tc_1, \\ c_5 &= 4M^4Te^{2|\omega|T}+4M^4M_1^2e^{2|\omega|T}T^2M_2^2 4M^4Te^{2|\omega|T}. \end{aligned}$$

Then problem (4.1) *is controllable on* $[0,T]$.

Proof. Using hypothesis (H7) for an arbitrary $x(\cdot)$ and $x_1 \in L^2(\Omega, \overline{D(A)})$, define the control

$$u_x(t) = \widetilde{W}^{-1}\bigg[B_\lambda(x_1 - \theta(x)) - S'(T)B_\lambda(x_0 - \theta(x)) - \int_0^T S'(T-s)B_\lambda f(s)ds - \int_0^T S'(T-s)B_\lambda g(s, x(s))dW(s)\bigg](t),$$

where $f \in N_{F,x}$.

Consider the multivalued map $\Phi : C_0 \to \mathcal{P}(C_0)$ defined by

$$(\Phi x)(t) = \bigg\{h \in C_0 : B_\lambda h(t) = S'(t)B_\lambda(x_0 - \theta(x)) + \int_0^t S'(t-s)B_\lambda f(s)ds + \int_0^t S'(t-s)B_\lambda Bu_x(s)ds + \int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s), \quad t \in [0,T]\bigg\},$$

where $f \in N_{F,x}$.

Since $\lim_{\lambda\to\infty} B_\lambda h = h$ for $h \in C_0$, it is clear that the fixed points of Φ are integral solutions to problem (4.1). Thus if we prove Φ has one fixed point, then the problem (4.1) is controllable on $[0,T]$. We still use Theorem 2.7 to prove Φ has a fixed point. The proofs that Φ is a completely continuous multivalued map, u.s.c., with convex closed values are similar to the proof in Theorem 3.3 and are omitted. We only prove that the set $U = \{x \in C_0 : \delta x \in \Phi x \text{ for some } \delta > 1\}$ is bounded.

Let $x \in U$, then $\delta x \in \Phi x$ for some $\delta > 1$. Thus there exists $f \in N_{F,x}$ such that for $t \in [0,T]$,

$$B_\lambda x(t) = \delta^{-1}S'(t)B_\lambda(x_0 - \theta(x)) + \delta^{-1}\int_0^t S'(t-s)B_\lambda f(s)ds + \delta^{-1}\int_0^t S'(t-s)B_\lambda Bu_x(s)ds + \delta^{-1}\int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s).$$

From (H4), (H5) and (H7), we have that for each $t \in [0,T]$,

$$\begin{aligned}
&E|x(t)|^2 \\
&= \lim_{\lambda\to\infty} E\bigg|\delta^{-1}S'(t)B_\lambda(x_0 - \theta(x)) + \delta^{-1}\int_0^t S'(t-s)B_\lambda f(s)ds \\
&\quad + \delta^{-1}\int_0^t S'(t-s)B_\lambda Bu_x(s)ds + \delta^{-1}\int_0^t S'(t-s)B_\lambda g(s, x(s))dW(s)\bigg|^2 \\
&\le \lim_{\lambda\to\infty} 4|S'(t)|^2 E|B_\lambda(x_0 - \theta(x))|^2 + 4M^4 e^{2|\omega|T}T\int_0^t E|f(s)|^2 ds \\
&\quad + 4\lim_{\lambda\to\infty} E\bigg|\int_0^t S'(t-s)B_\lambda Bu_x(s)ds\bigg|^2 + 4M^4e^{2|\omega|T}\int_0^t E|g(s, x(s))|^2 ds
\end{aligned}$$

$$\le 8M^2e^{2|\omega|T}(E|x_0|^2+L)+4M^4Te^{2|\omega|T}\int_0^t \eta(s)\Psi(E|x(s)|^2)ds$$
$$+4M^4e^{2|\omega|T}\int_0^t (c_1E(|x(s)|^2)+c_2)ds+4\lim_{\lambda\to\infty}E\left|\int_0^t S'(t-s)B_\lambda Bu_x(s)ds\right|^2.$$

On the other hand, we have

$$\lim_{\lambda\to\infty}E\left|\int_0^t S'(t-s)B_\lambda Bu_x(s)ds\right|^2$$
$$\le \lim_{\lambda\to\infty}M^4Te^{2|\omega|T}M_1^2\int_0^t E|u_x(s)|^2ds$$
$$\le M^4M_1^2e^{2|\omega|T}T^2M_2^2\Big[8(E|x_1|^2+L)+8e^{2|\omega|T}(E|x_0|^2+L)$$
$$+4M^4e^{2|\omega|T}T\int_0^T E|f(s)|^2ds+4M^4e^{2|\omega|T}\int_0^T E|g(s,x(s))|^2ds\Big]$$
$$\le M^4M_1^2e^{4|\omega|T}T^2M_2^2\Big[8(E|x_1|^2+L)+8e^{2|\omega|T}(E|x_0|^2+L)$$
$$+4M^4Te^{2|\omega|T}\int_0^T \eta(s)\Psi(E|x(s)|^2)ds+4M^4e^{2|\omega|T}\int_0^T (c_1E|x(s)|^2+c_2)ds\Big].$$

Then we have that

$$E|x(t)|^2\le 8M^2e^{2|\omega|T}(E|x_0|^2+L)+4M^4Te^{2|\omega|T}\int_0^t \eta(s)\Psi(E|x(s)|^2)ds$$
$$+4M^4e^{2|\omega|T}\int_0^t (c_1E(|x(s)|^2)+c_2)ds$$
$$+4M^4M_1^2e^{2|\omega|T}T^2M_2^2\Big[8(E|x_1|^2+L)+8e^{2|\omega|T}(E|x_0|^2+L)$$
$$+4M^4Te^{2|\omega|T}\int_0^T \eta(s)\Psi(E|x(s)|^2)ds$$
$$+4M^4e^{2|\omega|T}\int_0^T (c_1E(|x(s)|^2)+c_2)ds\Big].$$

In view of $\|x\|^2=\sup_{0\le t\le T}E|x(t)|^2$, we have

$$\|x\|^2$$
$$\le 8M^2e^{2|\omega|T}(E|x_0|^2+L)+4M^4Te^{2|\omega|T}\int_0^T \eta(s)\Psi(\|x\|^2)ds$$
$$+4M^4e^{2|\omega|T}\int_0^T (c_1(\|x\|^2)+c_2)ds+4M^4M_1^2e^{2|\omega|T}T^2M_2^2\Big[8(E|x_1|^2+L)$$
$$+8e^{2|\omega|T}(E|x_0|^2+L)+4M^4Te^{2|\omega|T}\int_0^T \eta(s)\Psi(\|x\|^2)ds$$

$$
\begin{aligned}
&\quad + 4M^4 e^{2|\omega|T} \int_0^T (c_1(\|x\|^2) + c_2) ds\Bigg] \\
&= 8M^2 e^{2|\omega|T}(E|x_0|^2 + L) + 4M^4 e^{2|\omega|T} T c_2 \\
&\quad + 4M^4 M_1^2 e^{2|\omega|T} T^2 M_2^2 \Bigg[8(E|x_1|^2 + L) + 8e^{2|\omega|T}(E|x_0|^2 + L) \\
&\quad + 4M^4 e^{2|\omega|T} T c_2\Bigg] \\
&\quad + \Bigg[4M^4 T e^{2|\omega|T} + 4M^4 M_1^2 e^{2|\omega|T} T^2 M_2^2 4M^4 T e^{2|\omega|T}\Bigg] \Psi(\|x\|^2) \int_0^T \eta(s) ds \\
&\quad + \Bigg[4M^4 e^{2|\omega|T} T c_1 + 4M^4 M_1^2 e^{2|\omega|T} T^2 M_2^2 4M^4 e^{2|\omega|T} T c_1\Bigg] \|x\|^2 \\
&= c_3 + c_4\|x\|^2 + c_5 \Psi(\|x\|^2) \int_0^T \eta(s) ds.
\end{aligned}
$$

Then we have

$$
\frac{(1 - c_4)\|x\|^2}{c_3 + c_5 \Psi(\|x\|^2) \int_0^T \eta(s) ds} \leq 1.
$$

Using (H8) and concavity of Ψ, we obtain there exists some constant $\overline{L}$ such that $\|x\|^2 \leq \overline{L}$. This shows that U is bounded. □

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