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REPRESENTATION THEOREMS FOR MULTIVALUED PRAMARTS

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Abstract. Existence of pramarts selectors for multivalued pramart whose values are convex weakly compact subsets of a separable Banach space E (resp. subsets of a dual space E^*) are established. Representation theorems for multivalued pramarts are also presented.

1. Introduction

Representation theorems of multivalued martingales, submartingales, supermartingales and uniform-amarts have been extensively studied in recent years by A. Choukairi [7], C. Hess [11], D. Q. Luu [14], Z. P. Wang and X. H. Xue [19] and S. Li and Y. Ogura [13]. It is known that every multivalued martingale is a multivalued submartingale and is a supermartingale and also a uniform-amart. So, any uniform-amart is a pramart. A naturel questions raised by A. Choukairi in [8] is the existence of pramarts selectors for multivalued pramart. The main purpose of this work is not only to solve this problem but also prove that a multivalued pramart has a Castaing representation by pramarts selectors. The paper is organized as follows. In Section 2 we recall some notations and definitions and summarize needed results. In Section 3 we give some decomposition results for convex weakly compact valued pramarts. In Section 4 we discuss the existence of pramart selectors of convex weakly compact valued pramarts. In Section 5 we present a decomposition results for multivalued pramart whose values are convex weakly compact in the dual of a separable Banach space and we show the existence of pramarts selectors of the above class of pramarts.

2. Preliminaries and background

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_n)_{n\geq 1}$ an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\geq 1}\mathcal{F}_n$. E is a separable Banach space with the dual E^* and the strong dual E^*_b. $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) the closed unit ball of E (resp. E^*).

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2^E is the set of all subsets of E. Let $cc(E)$ (resp. $cwk(E)$) be the set of nonempty convex closed subsets of E (resp. weakly compact subsets of E). For $A \in 2^E \backslash \emptyset$, we denote by clA and $\overline{co}A$ the closure and the closed convex hull of A respectively, and define $|A| = \sup\{||x|| : x \in A\}$, the distance function and the support function associated with A are defined respectively by

$$d(x, A) = \inf\{\|x - y\|, \, y \in A\} \ (x \in E).$$

$$\delta^*(x^*, A) = \sup\{\langle x^*, y\rangle, \, y \in A\} \ (x^* \in E^*).$$

The Hausdorff distance between A and B is denoted by

$$\mathcal{H}(A, B) = \sup_{x^* \in \overline{B}_{E^*}} |\delta^*(x^*, A) - \delta^*(x^*, B)|.$$

The equivalent definition of Hausdorff distance is

$$\mathcal{H}(A, B) = \max\{\inf\{\lambda : B \subset A + \lambda\}, \inf\{\lambda : A \subset B + \lambda\}\},$$

where

$$A + \lambda = \{x : d(x, A) \leq \lambda\}.$$

A multifunction (mappings for short) X is a map from Ω into 2^E. The domain of X is defined by

$$\mathrm{dom}(X) = \{\omega \in \Omega : X(\omega) \neq \emptyset\}.$$

A selector of X is a function $f : \Omega \longrightarrow E$ such that $f(\omega)$ is a member of $X(\omega)$ for all $\omega \in \mathrm{dom}(X)$.

A multifunction $X : \Omega \longrightarrow 2^E$ is said to be measurable, if for every open set $U \subset E$, the set

$$X^- U = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{F}$ (see [6], [12]). A measurable multifunction is also called a random set. For each $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{L}^1_{cwk(E)}(\mathcal{F}_n)$ (resp. $\mathcal{L}^1_{cc(E)}(\mathcal{F}_n)$) the space of all $\mathcal{F}_n$-measurable $cwk(E)$-valued multifunctions $X : \Omega \to cwk(E)$ (resp. $cc(E)$-valued multifunctions $X : \Omega \to cc(E)$) such that $\omega \to |X(\omega)|$ is integrable. A sequence $(X_n)_{n\in\mathbb{N}}$ of $cc(E)$-valued multifunctions is adapted if each X_n is $\mathcal{F}_n$-measurable. A measurable selector of the random set X is an $(\mathcal{F}, \mathcal{B}(E))$-measurable selector of X. A Castaing representation [6] of X is a sequence $f_n : \Omega \longrightarrow E$ of measurable selectors of X such that

$$X(\omega) = cl\{f_k(\omega) : k \geq 1\} \quad \text{for all} \quad \omega \in \mathrm{dom}(X).$$

We denote by $L^1_E(\mathcal{F})$ the space of (equivalence classes of) $(\mathcal{F}, \mathcal{B}(E))$-measurable functions $f : \Omega \longrightarrow E$ such that $\omega \longrightarrow \|f(\omega)\|$ is integrable. Such an f is said to be Bochner integrable. For every multifunction $X : \Omega \longrightarrow 2^E$ and every sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$, we set

$$S^1_X(\mathcal{B}) = \{f \in L^1_E(\mathcal{B}) : f(\omega) \in X(\omega) \text{ a.s.}\}.$$

It is known that $S^1_X(\mathcal{B})$ characterizes X up to P-null sets (see [12]). A measurable multifunction X such that $S^1_X(\mathcal{B})$ is nonempty is declared integrable. Using Hiai and Umegaki [12, Theorem 2.2], it is readily seen that X is integrable if and only if $d(0, X(\cdot)) \in L^1_E$. Now, consider an integrable $\mathcal{F}$-measurable multifunction $X : \Omega \longrightarrow cc(E)$. Following Hiai and Umegaki we define the multivalued conditional expectation of X relative to $\mathcal{B}$ as the $\mathcal{B}$-measurable random set $G = E^{\mathcal{B}}X$ such that $S^1_G(\mathcal{B}) = cl\{E^{\mathcal{B}}f : f \in S^1_X(\mathcal{F})\}$, the closure being taken in L^1_E (where $E^{\mathcal{B}}f$ denotes the usual conditional expectation relative to $\mathcal{B}$ of a Bochner integrable function f). In the special case where $\mathcal{B} = \{E, \emptyset\}$, $E^{\mathcal{B}}X$ is simply denoted by $E(X)$ and is equal to $cl\{E(f) : f \in S^1_X(\mathcal{F})\}$. New existence results of conditional expectation for convex weakly compact valued multifunctions and its applications to martingales are available in [1, 3].

We denote by $\mathbb{T}$ the set of all bounded stopping times. A sequence $(X_n)_{n\geq 1}$ in $L^1_E(\mathcal{F})$ is of class (B) if

$$\sup_{\tau\in\mathbb{T}} \int_\Omega \|X_\tau\| dP < \infty.$$

3. Decomposition theorems for multivalued pramarts

Before going further, let us introduce the definitions of pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $L^1_E(\mathcal{F})$.

Definition 3.1. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ in $L^1_E(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in \mathbb{T}$ such that

$$\forall \sigma, \tau \in \mathbb{T}, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\|X_\sigma - E^{\mathcal{F}_\sigma} X_\tau\| > \varepsilon]) < \varepsilon.$$

Definition 3.2. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in \mathbb{T}$ such that

$$\forall \sigma, \tau \in \mathbb{T}, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\mathcal{H}(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon]) < \varepsilon.$$

It is clear that if $(X_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ is a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, then, for each $x^* \in \overline{B}_{E^*}$, the adapted sequence $(\delta^*(x^*, X_n), \mathcal{F}_n)_{n\in\mathbb{N}}$ is a real-valued pramart in $L^1_{\mathbb{R}}(\mathcal{F})$ because

$$|\delta^*(x^*, X_\sigma) - E^{\mathcal{F}_\sigma} \delta^*(x^*, X_\tau)| \leq \mathcal{H}(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau).$$

Definition 3.3. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ in $L^1_{\mathbb{R}}(\mathcal{F})$ is a subpramart, if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in \mathbb{T}$ such that

$$\forall \sigma, \tau \in \mathbb{T}, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P(\{(X_\sigma - E^{\mathcal{F}_\sigma} X_\tau)^+ \geq \varepsilon\}) \leq \varepsilon.$$

Definition 3.4. Let $(X^m_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of real subpramarts. It is called a uniform sequence of positive subpramarts if for every $\varepsilon > 0$, there is $\sigma_0 \in \mathbb{T}$ such that if $\sigma, \tau \in \mathbb{T}$ with $\tau \geq \sigma \geq \sigma_0$, then

$$P(\{\sup_{m\in\mathbb{N}} (X^m_\sigma - E^{\mathcal{F}_\sigma} X^m_\tau)^+ \geq \varepsilon\}) \leq \varepsilon.$$

Now we proceed to the decomposition of $cwk(E)$-valued pramarts.

Theorem 3.5. *Assume that E_b^* is separable. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that there exists a $cwk(E)$-valued multifunction $K : \Omega \Longrightarrow cwk(E)$ satisfying $X_n(\omega) \subset K(w)$ $\forall n \in \mathbb{N}, \forall \omega \in \Omega$. Then there exists a multifunction $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that:*

$$\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$$

Proof. *Step* 1 **Claim:** $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$ a.s. $\forall x^* \in \overline{B}_{E^*}$.

Let $M_1^* = (f_j^*)_{j\in\mathbb{N}}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology $\tau(E^*, E)$. Since $(X_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, that is,

$$\sup_{n\in\mathbb{N}} \int_\Omega |X_n| dP = \sup_{n\in\mathbb{N}} \int_\Omega \sup_{x^*\in\overline{B}_{E^*}} |\delta^*(x^*, X_n)| dP < \infty$$

for each $j \in \mathbb{N}$, the L^1-bounded pramart $(\delta^*(f_j^*, X_n))_{n\in\mathbb{N}}$ converge a.s. to an integrable function $\varphi_{x^*} \in L^1_{\mathbb{R}}(\mathcal{F})$. Let $\omega \in \Omega$, define the function $s(\cdot)$ by

$$s(f_j^*) = \lim_{n\to\infty} \delta^*(f_j^*, X_n(\omega)) \quad (j \in \mathbb{N}).$$

s is sublinear and continuous for the Mackey topology $\tau(E^*, E)$. Consequently, there is $X_\infty(\omega) \in cwk(E)$ with $X_\infty(\omega) \subset K(\omega)$ such that

$$s(f_j^*) = \delta^*(f_j^*, X_\infty(\omega)) \quad (j \in \mathbb{N}).$$

Then there exists a negligible set $N \in \mathcal{F}$ such that for all $\omega \in \Omega \backslash N$

$$\lim_{n\to\infty} \delta^*(f_j^*, X_n(\omega)) = \delta^*(f_j^*, X_\infty(\omega)) \quad \forall j \in \mathbb{N}.$$

Since the functions $\delta^*(\cdot, X_n(\omega))$ and $\delta^*(\cdot, X(\omega))$ are continuous for the Mackey topology $\tau(E^*, E)$. So, we deduce that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad \text{a.s.} \quad \forall x^* \in \overline{B}_{E^*}.$$

We check that $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Indeed, X_∞ is measurable and for fixed $x^* \in \overline{B}_{E^*}$ and $\omega \in \Omega$, the function $n \longrightarrow \delta^*(x^*, X_n(\omega))$ is continuous from $\mathbb{N} \cup \{+\infty\}$ into $\mathbb{R}$, therefore the function $n \to \sup\{\delta^*(x^*, X_n(\omega)) : x^* \in \overline{B}_{E^*}\}$ is lower semi continuous on $\mathbb{N} \cup \{+\infty\}$ and so

$$|X_\infty|(\omega) \leq \liminf_n |X_n|(\omega),$$

by Fatou Lemma we have

$$\int_\Omega |X_\infty| dP \leq \int_\Omega \liminf_n |X_n| dP$$
$$\leq \liminf_n \int_\Omega |X_n| dP \leq \sup_n \int_\Omega |X_n| dP < +\infty.$$

Step 2 **Claim:** $\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0$ a.s.

Let $D_1^* = (e_j^*)_{j\in\mathbb{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$. We have

$$\mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = \sup_{j\in\mathbb{N}} |\delta^*(e_j^*, X_n) - \delta^*(e_j^*, E^{\mathcal{F}_n} X_\infty)|.$$

As $(\delta^*(e_j^*, X_n) - \delta^*(e_j^*, E^{\mathcal{F}_n}X_\infty))_{n\in\mathbb{N}}$ are real-valued pramarts in $L^1_{\mathbb{R}}(\mathcal{F})$ which converges a.s. to 0, and $((|\delta^*(e_j^*, X_n) - \delta^*(e_j^*, E^{\mathcal{F}_n}X_\infty)|)_{n\in\mathbb{N}})_{j\in\mathbb{N}}$ is a uniform sequence of positive subpramarts, applying lemma VIII.1.15 in [9] we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n}X_\infty) &= \lim_{n\to\infty} \sup_{j\in\mathbb{N}} |\delta^*(e_j^*, X_n) - \delta^*(e_j^*, E^{\mathcal{F}_n}X_\infty)| \\ &= \sup_{j\in\mathbb{N}} \lim_{n\to\infty} |\delta^*(e_j^*, X_n) - \delta^*(e_j^*, E^{\mathcal{F}_n}X_\infty)| = 0\end{aligned}$$

almost surely. □

Now we give a quasi-decomposition theorem for convex weakly compact valued pramart.

Theorem 3.6. *Assume that E_b^* is separable. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that, there exists a $cwk(E)$-valued multifunction $K : \Omega \Longrightarrow cwk(E)$ satisfying $X_n(\omega) \subset K(w)$ $\forall n \in \mathbb{N}, \forall \omega \in \Omega$. Then there exist a multivalued martingale $(M_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that*

$$X_n(w) \subset M_n(w) + Z_n(w) \quad a.s.$$

$$|Z_n| \longrightarrow 0 \quad a.s. \quad as \quad n \to +\infty.$$

Proof. By Theorem 3.5 there exists $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n}X_\infty) = 0 \quad \text{a.s.}$$

Let $M_n = E^{\mathcal{F}_n}X_\infty$, then $(M_n)_{n\in\mathbb{N}}$ is a multivalued martingale, and

$$\lim_{n\to\infty} \mathcal{H}(X_n, M_n) = 0 \quad \text{a.s.}$$

If we set $\rho_n = \mathcal{H}(X_n, M_n)$, define Z_n by

$$Z_n = \{x \in E \setminus \|x\| \leq \mathcal{H}(X_n, M_n) = \rho_n\} = \overline{B}_E(0, \rho_n).$$

Then by definition of the Hausdorff distance we have

$$X_n(w) \subset M_n(w) + \mathcal{H}(X_n(w), M_n(w)) = M_n(w) + \rho_n(w) \ \text{ a.s.}$$

So $X_n(w) \subset M_n(w) + Z_n(w)$ a.s. Indeed, we must prove that

$$\{x : \ d(x, M_n(w)) \leq \rho_n(w)\} = M_n(w) + \overline{B}_E(0, \rho_n(w)).$$

First, if $d(x, M_n(w)) \leq \rho_n(w)$, then for each $k > 0$, there exists $a_k \in M_n(w)$ such that

$$\|x - a_k\| \leq \rho_n(w) + \frac{1}{k}.$$

That is

$$x - a_k \in \overline{B}_E(0, \rho_n(w) + \frac{1}{k}).$$

So, there exists $y_k \in \overline{B}_E(0, \rho_n(w) + \frac{1}{k})$ such that $x - a_k = y_k$ and

$$x = a_k + y_k \in M_n(w) + \overline{B}_E(0, \rho_n(w) + \frac{1}{k}), \quad \forall k > 0.$$

Since $a_k \in M_n(w)$ and $y_k \in \overline{B}_E(0, \rho_n(w) + \frac{1}{k})$ this implies that there exists $a \in M_n(w)$ such that

$$\lim_{k\to\infty} \langle x^*, a_k\rangle = \langle x^*, a\rangle \quad \forall x^* \in E^*,$$

and

$$\begin{aligned}\lim_{k\to\infty} \langle x^*, x - a_k\rangle &= \langle x^*, x - a\rangle \\ &= \lim_{k\to\infty} \langle x^*, y_k\rangle \\ &\le \limsup_{k\to\infty} \delta^*(x^*, \overline{B}_E(0, \rho_n(w) + \frac{1}{k})).\end{aligned}$$

That is

$$\langle x^*, x - a\rangle \le \delta^*(x^*, \overline{B}_E(0, \rho_n(w))) \quad \forall x^* \in E^*.$$

According to Proposition III.35 in [6], we deduce that $y = x - a \in \overline{B}_E(0, \rho_n(w))$. Finally $x \in M_n(w) + \overline{B}_E(0, \rho_n(w))$.

Conversely, if $x \in M_n(w) + \overline{B}_E(0, \rho_n(w))$, this implies that $\exists a \in M_n(w)$, $\exists z \in \overline{B}_E(0, \rho_n(w))$ such that

$$x = a + z.$$

Indeed, since $x \in M_n(w) + \overline{B}_E(0, \rho_n(w))$, then there exists $(x_k)_{k\ge1}$ such that $x = \lim_k x_k$ and $x_k = a_k + z_k$ with $a_k \in M_n$ and $z_k \in \overline{B}_E(0, \rho_n(w))$, hence there exist k_j subsequence of k and $a \in M_n$ such that

$$\lim_{j\to\infty} \langle x^*, a_{k_j}\rangle = \langle x^*, a\rangle.$$

On the other hand $x_{k_j} = a_{k_j} + z_{k_j}$ thus

$$\lim_{j\to\infty} \langle x^*, x_{k_j} - a_{k_j}\rangle = \langle x^*, x - a\rangle = \lim_{j\to\infty} \langle x^*, z_{k_j}\rangle = \langle x^*, z\rangle.$$

So, $z \in \overline{B}_E(0, \rho_n(w))$ and $x = a + z$. Consequently

$$\|x - a\| = \|z\| \le \rho_n(w)$$

and

$$d(x, M_n(w)) \le \rho_n(w).$$

Finally, $X_n(w) \subset M_n(w) + Z_n(w)$ a.s. and

$$|Z_n| \le \mathcal{H}(X_n, M_n) \longrightarrow_{n\to+\infty} 0 \quad \text{a.s.}$$

□

The following result is a consequence of Theorem 3.5.

Corollary 3.7. *Assume that E_b^* is separable. Let $(X_n, \mathcal{F}_n)_{n\ge1}$ be a bounded pramart in $L^1_E(\mathcal{F})$ such that there exists a multifunction $K \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying $X_n(\omega) \in K(w)$ $\forall n \in \mathbb{N}, \forall \omega \in \Omega$. Then there are a unique regular martingale (Y_n) in $L^1_E(\mathcal{F})$ and a pramart (Z_n) in $L^1_E(\mathcal{F})$ such that*

$$X_n = Y_n + Z_n, \quad \forall n \in \mathbb{N},$$

$$|Z_n| \longrightarrow 0 \quad a.s. \quad as \quad n \to +\infty.$$

Proof. By Theorem 3.5 there exists X_∞ in $L^1_E(\mathcal{F})$ such that

$$\lim_n \|X_n - E^{\mathcal{F}_n} X_\infty\| = 0 \quad \text{a.s.}$$

Then, by setting $Y_n = E^{\mathcal{F}_n} X_\infty$ for all $n \in \mathbb{N}$, we have $X_n = Y_n + X_n - Y_n = Y_n + Z_n$, where $Z_n = X_n - Y_n$ is obviously a pramart and $\lim_n \|Z_n\| = \lim_n \|X_n - Y_n\| = 0$ a.s. The uniqueness is more or less classical. Suppose

$$X_n = Y'_n + Z'_n$$

with the required properties in the corollary. Then

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} Z'_n = 0 \quad \text{a.s.}$$

For each m fixed in $\mathbb{N}$ we have

$$Y_m - Y'_m = \lim_{n\to\infty} (E^{\mathcal{F}_n} Y_m - E^{\mathcal{F}_n} Y'_m)$$

$$= \lim_{n\to\infty} (Y_n - Y'_n) = \lim_{n\to\infty} (Z_n - Z'_n) = 0 \quad \text{a.s.}$$

for every $m \in \mathbb{N}$. □

4. Representation theorems for multivalued pramarts

In this present section we give our first results of existence of pramart selectors for $cwk(E)$-valued pramart.

Definition 4.1. A sequence $(f_n, \mathcal{F}_n)_{n\geq 1}$ is called a pramart selector of $(X_n, \mathcal{F}_n)_{n\geq 1}$ if

(i) $f_n \in S^1_{X_n}(\mathcal{F}_n)$ for all $n \in \mathbb{N}$.

(ii) $(f_n, \mathcal{F}_n)_{n\geq 1}$ is a pramart in $L^1_E(\mathcal{F})$. In this case we write $(f_n, \mathcal{F}_n)_{n\geq 1} \in PS(X_n)$ and let $PS(X_n)$ denote the set of all pramart selectors of $(X_n, \mathcal{F}_n)_{n\geq 1}$.

To get further representation theorem, we need the following lemmas.

Lemma 4.2. *Let $(X_n)_{n\geq 1}$ be a sequence in $L^1_E(\mathcal{F})$. If $(X_n)_{n\geq 1}$ is of class (B) and (X_n) converge in probability. Then $(X_n)_{n\geq 1}$ is a pramart in $L^1_E(\mathcal{F})$.*

Proof. See [20, Lemma 6]. □

Lemma 4.3. *If $\mathcal{B}_1 \subset \mathcal{B}_0$ are two sub-σ-fields of $\mathcal{F}$, $X \in \mathcal{L}^1_{cc(E)}(\mathcal{B}_1)$, $Y \in \mathcal{L}^1_{cc(E)}(\mathcal{B}_0)$ and $\theta : \Omega \longrightarrow \mathbb{R}^+\backslash\{0\}$ is a $\mathcal{B}_1$-measurable function, then for each $f \in S^1_X(\mathcal{B}_1)$ we can find $g \in S^1_Y(\mathcal{B}_0)$ such that*

$$\|f(w) - E^{\mathcal{B}_1} g(w)\| \leq \mathcal{H}(X(w), E^{\mathcal{B}_1} Y(w)) + \theta(w) \quad a.s.$$

Consequently, if Y is $\mathcal{B}_1$-measurable, then there exist some $g \in S^1_Y(\mathcal{B}_1)$ such that

$$\|f(w) - g(w)\| \leq \mathcal{H}(X(w), Y(w)) + \theta(w) \quad a.s.$$

Proof. See [16, Lemma 3.3]. □

Example 4.4. Let $(f_n, \mathcal{F}_n)_{n\geq 1}$ be a vector valued pramart and $(r_n, \mathcal{F}_n)_{n\geq 1}$ be a real valued pramart. Take $\overline{B}_E$ the closed unit ball of E, let $A \in \mathcal{F}$. Define

$$X_n = f_n 1_A + r_n 1_{A^c} \overline{B}_E.$$

Then $(X_n)_{n\geq 1}$ is a multivalued pramart. Indeed, for $\tau, \sigma \in \mathbb{T}\,(\tau \geq \sigma)$

$$\begin{aligned}
&P(\mathcal{H}(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon)\\
&= P(\mathcal{H}(f_\sigma 1_A + r_\sigma 1_{A^c}\overline{B}_E, E^{\mathcal{F}_\sigma}(f_\tau 1_A + r_\tau 1_{A^c}\overline{B}_E)) > \varepsilon)\\
&\leq P(\|f_\sigma 1_A - E^{\mathcal{F}_\sigma} f_\tau 1_A\| + |\overline{B}_E|.|r_\sigma 1_{A^c} - E^{\mathcal{F}_\sigma} r_\tau 1_{A^c}| > \varepsilon)\\
&\leq P(\|f_\sigma 1_A - E^{\mathcal{F}_\sigma} f_\tau 1_A\| + |r_\sigma 1_{A^c} - E^{\mathcal{F}_\sigma} r_\tau 1_{A^c}| > \varepsilon)\\
&\leq P(\|f_\sigma 1_A - E^{\mathcal{F}_\sigma} f_\tau 1_A\| > \frac{\varepsilon}{2}) + P(|r_\sigma 1_{A^c} - E^{\mathcal{F}_\sigma} r_\tau 1_{A^c}| > \frac{\varepsilon}{2})\\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

It is easy to see that every sequence $(g_n)_{n\geq 1}$ define by

$$g_n = f_n 1_A + r_n 1_{A^c} x \quad \text{for each } x \in \overline{B}_E$$

is a pramart selector of $(X_n)_{n\geq 1}$.

Definition 4.5. Given $(X_n)_{n\geq 1}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. We say that assumption (A) holds, if every sequence of selectors of $(X_n)_{n\geq 1}$ is of class (B).

Theorem 4.6. *Assume that E^*_b is separable. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that there exists a $cwk(E)$-valued multifunction $K : \Omega \Longrightarrow cwk(E)$ satisfying $X_n(\omega) \subset K(w)$ $\forall n \in \mathbb{N}, \forall \omega \in \Omega$ and if assumption (A) holds. Then*

$$S^1_{X_k}(\mathcal{F}_k) = \pi_k(PS(X_n)).$$

Where for every $(f_n) \in PS(X_n)$, $\pi_k((f_n)) = f_k$ (π_k is the usual projection to the kth element of the sequence $(f_n)_{n\geq 1}$).

Proof. By Theorem 3.5 there is $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \mathcal{H}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad \text{a.s.}$$

Let $M_n = E^{\mathcal{F}_n} X_\infty$, for each $n \in \mathbb{N}$ we set $\rho_n(w) = \mathcal{H}(X_n(w), M_n(w))$, and let $r_n(w) = \rho_n(w) + \frac{1}{2^n}$, $\forall n \geq 1$.

Now, let $k \geq 1$ and let $\widehat{f_k} \in S^1_{X_k}(\mathcal{F}_k)$. From [14] see also [11], we know that there exists a sequence $(h^i_n, \mathcal{F}_n)_{i\geq 1}$ in $MS(M_n)$ (here $MS(M_n)$ is the set of all martingales selectors of M_n) such that for every $n \geq 1$,

$$M_n(w) = cl\{h^i_n(\omega);\quad i \geq 1\}, \quad \forall \omega \in \Omega.$$

Define $\tau : \Omega \longrightarrow \mathbb{N}$ and $h^\tau_k : \Omega \longrightarrow E$ by

$$\tau(w) = \inf\{i \geq 1,\ \|\widehat{f_k}(\omega) - h^i_k(\omega)\| \leq d(\widehat{f_k}(w), M_k(w)) + \frac{1}{2^k}\}$$

and

$$h_k(w) = \sum_{i=1}^{+\infty} 1_{\{\tau=i\}}(\omega) h_k^i(w) = h_k^{\tau(\omega)}(w).$$

Obviously $\tau \in \mathcal{F}_k$ and $h_k(w) \in M_k(w)$. Also we have

$$\begin{aligned} \|\widehat{f}_k(w) - h_k(w)\| &= \sum_{i\geq 1} 1_{\{\tau=i\}}(\omega)\|\widehat{f}_k(w) - h_k^i(w)\| \\ &\leq d(\widehat{f}_k(w), M_k(w)) + \frac{1}{2^k} \leq r_k(w) \quad \text{a.s.} \end{aligned} \tag{4.6.1}$$

Next define

$$h_n(w) = \begin{cases} \sum_{i\geq 1} 1_{\{\tau=i\}}(\omega) h_n^i(w), & \text{if} \quad n \geq k; \\ \\ E(h_k(w)/\mathcal{F}_n), & \text{if} \quad n < k. \end{cases}$$

Then $(h_n, \mathcal{F}_n)_{n\geq 1}$ is in $MS(M_n)$. For each $h_n(w) \in M_n(w)$ by using Lemma 4.3 we can find a sequence $f_n \in S^1_{X_n}(\mathcal{F}_n)$ such that

$$\|f_n(w) - h_n(w)\| \leq r_n(w) \quad \text{a.s.} \tag{4.6.2}$$

Next we shall prove that $(f_n, \mathcal{F}_n)_{n\geq 1} \in PS(X_n)$. Indeed, firstly we can write $f_n = h_n + (f_n - h_n) = h_n + z_n$ where $z_n = f_n - h_n$, on the other hand from (4.6.2) z_n converge to zero a.s. as $n \to +\infty$, and

$$\int_\Omega \|z_\tau\| dP \leq \int_\Omega \|f_\tau\| dP + \int_\Omega |X_\infty| dP$$

then

$$\sup_{\tau\in\mathbb{T}} \int_\Omega \|z_\tau\| dP \leq \sup_{\tau\in\mathbb{T}} \int_\Omega \|f_\tau\| dP + \int_\Omega |X_\infty| dP < \infty.$$

Hence by Lemma 4.2, z_n is a pramart. This with $(h_n)_{n\geq 1}$ being a martingale, implies that $(f_n)_{n\geq 1}$ is a pramart and it's martingale component in the decomposition of Corollary 3.7 is given by $(h_n)_{n\geq 1}$. By (4.6.1) and (4.6.2) we can take $f_k = \widehat{f}_k$ and so $\widehat{f}_k \in \pi_k((f_n)) \in \pi_k(PS(X_n))$. Hence $S^1_{X_k}(\mathcal{F}_k) \subset \pi_k(PS(X_n))$. It is obvious that $S^1_{X_k}(\mathcal{F}_k) \supset \pi_k(PS(X_n))$. So we have the result. □

Now we are ready to state the following representation theorem of $cwk(E)$-valued pramarts.

Theorem 4.7. *Assume that E_b^* is separable. Let $(X_n)_{n\geq 1}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that, there exists a $cwk(E)$-valued multifunction $K : \Omega \Longrightarrow cwk(E)$ satisfying $X_n(\omega) \subset K(w) \quad \forall n \in \mathbb{N}, \forall \omega \in \Omega$ and if assumption (A) holds. Then there exists a sequence $(f_n^k)_{k\geq 1}$ in $PS(X_n)$ such that for every $n \geq 1$,*

$$X_n(w) = cl\{f_n^k(w),\, k \geq 1\}, \quad \forall \omega \in \Omega.$$

Proof. By Castaing representation theorem, we have that, for any $k \in \mathbb{N}$, there exists a sequence $\{g^{k,i} : i \in \mathbb{N}\} \subset S^1_{X_k}(\mathcal{F}_k)$ such that $X_k(w) = cl\{g^{k,i}(w) : i \in \mathbb{N}\}$ for all $\omega \in \Omega$. By virtue of Theorem 4.6, there exists a sequence of pramart selectors $\{h_n^{k,i,j} : j \in \mathbb{N}\}$ in $PS(X_n)$ such that

$$\lim_{j\to\infty} \|\pi_k(h_n^{k,i,j}) - g^{k,i}\|_1 = 0 \quad \text{for all} \quad k, i \in \mathbb{N}.$$

Then

$$\lim_{j\to\infty} \|h_k^{k,i,j} - g^{k,i}\|_1 = 0 \quad \text{for all} \quad k, i \in \mathbb{N}. \tag{4.7.1}$$

But as from every L^1-convergent sequence we can extract an almost surely convergent subsequence, so by (4.7.1) without any loss of generality we have

$$X_k(w) = cl\{h_k^{k,i,j}(w) : i, j \in \mathbb{N}\} \quad \text{for all} \quad k \in \mathbb{N}.$$

Finally, if $\{f_n^l : l \in \mathbb{N}\}$ denotes the sequence $\{h_n^{k,i,j} : k, i, j \in \mathbb{N}\}$, then the result is automatically satisfied. □

5. Multivalued pramarts in dual space

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\geq 1}\mathcal{F}_n$. Let E be a separable Banach space, E^* the topological dual of E, $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) the closed unit ball of E (resp. E^*), $D = (x_p)_{p\in\mathbb{N}}$ a dense sequence in $\overline{B}_E$. We denote by E^*_b the strong dual endowed with the topology associated with the dual norm $||\cdot||_{E^*_b}$, by E^*_s the topological dual E^* endowed with the topology $\sigma(E^*, E)$ of pointwise convergence, alias w^* topology. Noting that E^* is the countable union of closed balls, we deduce that the space E^*_s is Suslin. A $2^{E^*_s}$-valued multifunction (alias mapping for short) $X : \Omega \rightrightarrows E^*_s$ is $\mathcal{F}$-measurable if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Given a $\mathcal{F}$-measurable mapping $X : \Omega \rightrightarrows E^*_s$ and a Borel set $G \in \mathcal{B}(E^*_s)$, the set

$$X^- G = \{\omega \in \Omega : X(\omega) \cap G \neq \emptyset\}$$

is $\mathcal{F}$-measurable, that is $X^- G \in \mathcal{F}$. In view of the completeness hypothesis on the probability space, this is a consequence of the Projection Theorem (see e.g. Theorem III.23 of [6]) and of the equality

$$X^- G = \mathrm{proj}_\Omega \{Gr(X) \cap (\Omega \times G)\}.$$

In particular, if $X : \Omega \rightrightarrows E^*_s$ is $\mathcal{F}$-measurable, the domain of X, defined by

$$\mathrm{dom}\, X = \{\omega \in \Omega : X(\omega) \neq \emptyset\}$$

is $\mathcal{F}$-measurable, because $\mathrm{dom}\, X = X^- E^*_s$. Here $L^1_{E^*}[E](\mathcal{F})$ is the space of all $\mathcal{F}$-measurable mappings $u : \Omega \longrightarrow E^*_s$ such that the function $|u| : \omega \mapsto ||u(\omega)||_{E^*_b}$ is integrable. For any $2^{E^*_s}$-valued mapping $X : \Omega \rightrightarrows E^*_s$, we denote by $\mathcal{S}^1_X(\mathcal{F})$ the set of all $L^1_{E^*}[E](\mathcal{F})$-selectors of X. By $cwk(E^*_s)$ we denote the set of all nonempty convex $\sigma(E^*, E)$-compact subsets of E^*_s. A mapping $X : \Omega \to cwk(E^*_s)$ is scalarly $\mathcal{F}$-measurable if the function $\omega \to$

$\delta^*(x, X(\omega))$ is $\mathcal{F}$-measurable for every $x \in E$. Let us recall that any scalarly $\mathcal{F}$-measurable $cwk(E^*_s)$-valued mapping is $\mathcal{F}$-measurable. Indeed, let $(e_k)_{k\in\mathbb{N}}$ be a sequence in E which separates the points of E^*, then we have $x \in X(\omega)$ if and only if, $\langle e_k, x\rangle \le \delta^*(e_k, X(\omega))$ for all $k \in \mathbb{N}$. Further, we denote by $\mathcal{L}^1_{cwk(E^*_s)}(\Omega, \mathcal{F}, P)$ (shortly $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$) the space of all integrably bounded multifunctions X such that the function $|X| : \omega \to |X(\omega)|$ is integrable, here $|X(\omega)| := \sup_{y^*\in X(\omega)} ||y^*||_{E^*_b}$, by the above consideration, it is easy to see that $|X|$ is $\mathcal{F}$-measurable. Let $\mathcal{H}^*_{E^*_b}$ be the Hausdorff distance associated with the dual norm $||\cdot||_{E^*_b}$ on bounded closed convex subsets in E^*, and X, Y be two convex weak* compact valued measurable mapping, then $\mathcal{H}^*_{E^*_b}(X, Y)$ is measurable because $\mathcal{H}^*_{E^*_b}(X, Y) = \sup_{j\in\mathbb{N}}[\delta^*(e_j, X) - \delta^*(e_j, Y)]$, where $(e_j)_{j\in\mathbb{N}}$ is a dense sequence in $\overline{B}_E$. A sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is bounded if $(|X_n|)_{n\in\mathbb{N}}$ is bounded in $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ (shortly $L^1_{\mathbb{R}}(\mathcal{F})$). For the existence and uniqueness of the conditional expectation in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and $L^1_{E^*}[E](\mathcal{F})$, we refer the reader to [3, 18]. In the sequel we assume that E^*_b is separable.

Before going further, let us introduce the definition of pramarts in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and $L^1_{E^*}[E](\mathcal{F})$.

Definition 5.1. An adapted sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is a pramart if for every $\varepsilon > 0$, there exists $\sigma_\varepsilon \in \mathbb{T}$ such that

$$\forall \sigma, \tau \in \mathbb{T}, \quad \tau \ge \sigma \ge \sigma_\varepsilon \Rightarrow P(\mathcal{H}^*_{E^*_b}(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon) < \varepsilon,$$

where $\mathcal{H}^*_{E^*_b}$ stands for the Hausdorff distance associated with the dual norm $||\cdot||_{E^*_b}$ on $cwk(E^*_s)$.

Further, if $(X_n)_{n\in\mathbb{N}}$ is single-valued, definition 5.1 is reduced to:

Definition 5.2. An adapted sequence $(X_n)_{n\in\mathbb{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ is a pramart if for every $\varepsilon > 0$, there exists $\sigma_\varepsilon \in \mathbb{T}$ such that

$$\forall \sigma, \tau \in \mathbb{T}, \quad \tau \ge \sigma \ge \sigma_\varepsilon \Rightarrow P(||X_\sigma - E^{\mathcal{F}_\sigma} X_\tau||_{E^*_b} > \varepsilon) < \varepsilon.$$

Similarly if $(X_n)_{n\in\mathbb{N}}$ is a pramart in $L^1_{E^*}[E](\mathcal{F})$, then for every x in the unit ball $\overline{B}_E$ of E, the sequence $(\langle x, X_n\rangle)_{n\in\mathbb{N}}$ is a pramart in $L^1_{\mathbb{R}}(\mathcal{F})$, since we have

$$||X_\sigma - E^{\mathcal{F}_\sigma} X_\tau||_{E^*_b} = \sup_{x\in\overline{B}_E} [\langle x, X_\sigma - E^{\mathcal{F}_\sigma} X_\tau\rangle].$$

Here we give an elementary lemma that we will be needed later.

Lemma 5.3. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $cwk(E^*_s)$ and D denotes a dense sequence in $\overline{B}_E$ such that*

(i) *For every $x \in D$, $\lim_{n\to\infty} \delta^*(x, A_n)$ exist a.s.*

(ii) $\sup_{n\in\mathbb{N}} |A_n| < \infty$.

Then there exists $A_\infty \in cwk(E^*_s)$ satisfying

$$\lim_{n\to\infty} \delta^*(x, A_n) = \delta^*(x, A_\infty) \quad (x \in E).$$

Proof. For each $x \in D$, define the function $r(\cdot)$ by

$$r(x) = \lim_{n\to+\infty} \delta^*(x, A_n).$$

r is sublinear and continuous because by condition (ii) we have

$$\begin{aligned}\sup_{\|x\|\le 1} |r(x)| &= \sup_{\|x\|\le 1} |\lim_{n\to\infty} \delta^*(x, A_n)| \\ &\le \sup_{n\in\mathbb{N}} |A_n| < \infty.\end{aligned}$$

Hence by [17, Lemma 1] there exists $A_\infty \in cwk(E_s^*)$ such that

$$r(x) = \delta^*(x, A_\infty) \quad \forall x \in \overline{B}_E.$$

□

Now we are ready to state the decomposition of $cwk(E_s^*)$-valued pramarts.

Theorem 5.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$. Then there exists $X_\infty \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$ such that*

(i) $$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s. \quad \forall x \in \overline{B}_E,$$

(ii) $$\lim_{n\to\infty} \mathcal{H}^*_{E_b^*}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$$

Proof. Step 1 **Claim:** $\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty)$ a.s. $\forall x \in \overline{B}_E$.

Let $D_1 = (e_j)_{j\in\mathbb{N}}$ denotes a dense sequence in $\overline{B}_E$. As $(X_n)_{n\in\mathbb{N}}$ is a bounded pramart in $\mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$, for each $j \in \mathbb{N}$, $(\delta^*(e_j, X_n))_{n\in\mathbb{N}}$ is a bounded real-valued pramart in $L^1_{\mathbb{R}}(\mathcal{F})$. So for each $j \in \mathbb{N}$, $(\delta^*(e_j, X_n))_{n\in\mathbb{N}}$ converges a.s. to an integrable function m_j in $L^1_{\mathbb{R}}(\mathcal{F})$. By hypotheses of theorem, since $(X_n, \mathcal{F}_n)_{n\ge 1}$ is a multivalued pramart, then for every $n \in \mathbb{N}$, $(|X_n|, \mathcal{F}_n)_{n\ge 1}$ is a sequence of real subpramarts which are bounded in $L^1_{\mathbb{R}}(\mathcal{F})$. So by [9, Lemma VIII.2.4.1] we deduce that

$$\sup_{n\in\mathbb{N}} |X_n| < \infty \quad \text{a.s.}$$

Hence Lemma 5.3 gives $X_\infty \in cwk(E_s^*)$ such that

$$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad \text{a.s.} \quad \forall x \in \overline{B}_E.$$

Finally by Fatou Lemma, We check that $X_\infty \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$.

Step 2 **Claim:** $\lim_{n\to\infty} \mathcal{H}^*_{E_b^*}(X_n, E^{\mathcal{F}_n} X_\infty) = 0$ a.s.

Let $D_1 = (e_j)_{j\in\mathbb{N}}$ denote a dense sequence in $\overline{B}_E$. We have

$$\mathcal{H}^*_{E_b^*}(X_n, E^{\mathcal{F}_n} X_\infty) = \sup_{j\in\mathbb{N}} |\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty)|.$$

As $(\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty))_{n\in\mathbb{N}}$ are real-valued pramarts in $L^1_{\mathbb{R}}(\mathcal{F})$ which converges a.s. to 0, and $((|\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty)|)_{n\in\mathbb{N}})_{j\in\mathbb{N}}$ is a uniform sequence of positive subramarts, applying Lemma VIII.1.15 in [9] we have

$$\lim_{n\to\infty} \mathcal{H}^*_{E_b^*}(X_n, E^{\mathcal{F}_n} X_\infty) = \lim_{n\to\infty} \sup_{j\in\mathbb{N}} |\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty)|$$

$$= \sup_{j\in\mathbb{N}} \lim_{n\to\infty} |\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n} X_\infty)| = 0$$

almost surely. □

Comments: In [4] when the Banach space is weakly compactly generated (WCG), the authors give the weak star Kuratowski (w^*K for short) converges of pramarts in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Recall that the Banach space E is weakly compactly generated (WCG) if there exists a weakly compact subset of E whose linear span is dense in E.

Corollary 5.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a bounded pramart in $L^1_{E^*}[E](\mathcal{F})$. Then there exist a martingale $(Y_n)_{n\in\mathbb{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ and a pramart $(Z_n)_{n\in\mathbb{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ such that $X_n = Y_n + Z_n$, $\forall n \in \mathbb{N}$ and such that $(Z_n)_{n\in\mathbb{N}}$ norm converges to 0 a.s.*

Proof. As $(\langle x, X_n\rangle)_{n\in\mathbb{N}}$ is a real-valued bounded pramart in $L^1_{\mathbb{R}}$ for each $x \in \overline{B}_E$, $(\langle x, X_n\rangle)_{n\in\mathbb{N}}$ converges a.s to an integrable function m_x. Then Theorem 5.4(ii) provides a $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ such that

$$\lim_{n\to\infty} \|X_n - E^{\mathcal{F}_n} X_\infty\|_{E^*_b} = 0 \quad \text{a.s.}$$

the result follows by putting $Y_n = E^{\mathcal{F}_n} X_\infty$ and $Z_n = X_n - E^{\mathcal{F}_n} X_\infty$. □

Now we state the existence of martingales selectors for $cwk(E^*_s)$-valued supermartingales $(X_n)_{n\geq 1}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ via a projective limit technique. See ([11, Proposition 3.7]) for details. For this purpose we shall recall the definition of the projective limit of a sequence of sets. Let $(\Gamma_n)_{n\geq 1}$ be a sequence of sets and for any $m, n \geq 1$ such that $m \leq n$, a map $u_{mn} : \Gamma_n \longrightarrow \Gamma_m$. Also assume the two following hypotheses:

(i) $\forall m \geq 1$, $u_{mm} = id_{\Gamma_m}$ = the identity map of Γ_m.

(ii) $\forall m, n, p \geq 1$ such that $m \leq n \leq p$, $u_{mp} = u_{mn} \circ u_{np}$.

The sequence $(\Gamma_n)_{n\geq 1}$, together with the maps u_{mn} is called a projective system. If the Γ_n are topological spaces and if the u_{mn} are continuous we speak of a projective system of topological spaces. Let Γ be the cartesian product of the Γ_n for $n \geq 1$ and pr_n the projection from Γ onto Γ_n. The subset S of Γ defined by

$$S := \{x = (x_n)_{n\geq 1} / \; pr_m(x) = u_{mn} \circ pr_n(x) \quad \forall m, n \geq 1, \, m \leq n\}$$

is called the projective limit of the projective system defined above.

Lemma 5.6. *Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a supermartingales in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then $(X_n, \mathcal{F}_n)_{n\geq 1}$ admits a martingale selector $(f_n, \mathcal{F}_n)_{n\geq 1}$ in $L^1_{E^*}[E](\mathcal{F})$.*

Proof. By [6, Theorem VIII.34], for $m < n$ and for $f \in L^1_{E^*}[E](\mathcal{F}_n)$, the conditional expectation $E^{\mathcal{F}_m} f$ exists and belongs to the space $L^1_{E^*}[E](\mathcal{F}_m)$. Now thinks to Proposition VIII.33 in [6]

$$E^{\mathcal{F}_m} : L^1_{E^*}[E](\mathcal{F}_n) \longrightarrow L^1_{E^*}[E](\mathcal{F}_m)$$

is continuous for the topologies

$$\sigma(L^1_{E^*}[E](\mathcal{F}_n), L^\infty_E(\mathcal{F}_n))$$

and

$$\sigma(L^1_{E^*}[E](\mathcal{F}_m), L^\infty_E(\mathcal{F}_m))$$

respectively. Further by [6, Theorem VIII.34] the $L^1_{E^*}[E](\mathcal{F}_n)$-selectors set $S^1_{X_n}(\mathcal{F}_n)$ is convex $\sigma(L^1_{E^*}[E](\mathcal{F}_n), L^\infty_E(\mathcal{F}_n))$ compact. Now let us set

$$u_{mn}(f) := E^{\mathcal{F}_m} f, \quad \forall f \in S^1_{X_n}(\mathcal{F}_n).$$

Then by the supermartingale property we have

$$u_{mn}(S^1_{X_n}(\mathcal{F}_n)) \subset (S^1_{X_m}(\mathcal{F}_m)).$$

Therefore the sequence $(S^1_{X_n}(\mathcal{F}_n))_{n\geq 1}$ together with the sequence of continuous mappings $(u_{mn})_{m<n}$ is a projective system of compact spaces in $L^1_{E^*}[E](\mathcal{F})$. Therefore, by [2, Proposition 8, p.I.64] this system admits a nonempty projective limit which is the set $MS(X_n)$ of $L^1_{E^*}[E](\mathcal{F})$-martingales selectors of (X_n). (i.e., any member $(f_k)_{k\geq 1}$ of the projective limit satisfies, for any $m, n \in \mathbb{N}$ such that $m < n$,

$$pr_m((f_k)) = u_{mn} \circ pr_n((f_k))$$

or, equivalently, $f_m = E^{\mathcal{F}_m} f_n$. So we have $(f_k) \in MS(X_n)$). □

Now we proceed to the existence of pramart selector of pramart in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$.

Theorem 5.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, if assumption (A) holds. Then $(X_n)_{n\in\mathbb{N}}$ admits a pramart selector $(f_n)_{n\geq 1}$ in $L^1_{E^*}[E](\mathcal{F})$, that is $(f_n)_{n\in\mathbb{N}}$ is an integrable pramart and $f_n(\omega) \in X_n(\omega)$ for all $n \geq 1$ and for all $\omega \in \Omega$.*

Proof. Applying Theorem 5.4 to $(X_n)_{n\geq 1}$ which provides a multifunction $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad \text{a.s.}$$

let us set $M_n = E^{\mathcal{F}_n} X_\infty$ and $r_n = \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n} X_\infty) + \frac{1}{2^n}$. Since $(M_n)_{n\geq 1}$ is a $cwk(E^*_s)$-valued martingale, by Lemma 5.6 there exists $(g_n)_{n\geq 1}$ a martingale selector of M_n. By same argument in the proof of Theorem 4.6, for each g_n we pick a $\mathcal{F}_n$-measurable selector of X_n such that

$$\|f_n - g_n\|_{E^*_b} \leq r_n. \tag{5.7.1}$$

Writing $f_n = g_n + (f_n - g_n) = g_n + z_n$ where $z_n = f_n - g_n$, by (5.7.1) $(z_n)_{n\geq 1}$ converge a.s to zero when n goes to ∞, and for all $\tau \in \mathbb{T}$ we have

$$\int_\Omega \|z_\tau\| dP \leq \int_\Omega \|f_\tau\| dP + \int_\Omega |X_\infty| dP < \infty,$$

$$\sup_{\tau\in\mathbb{T}}\int_\Omega \|z_\tau\|dP \le \sup_{\tau\in\mathbb{T}}\int_\Omega \|f_\tau\|dP + \int_\Omega |X_\infty|dP < \infty,$$

then

$$\sup_{\tau\in\mathbb{T}}\int_\Omega \|z_\tau\|dP < \infty.$$

Hence by Lemma 4.2, z_n is a pramart. By Corollary 5.5 $(f_n)_{n\geq 1}$ is a pramart selector, because $(g_n)_{n\geq 1}$ is a $L^1_{E^*}[E](\mathcal{F})$ martingale and $(z_n)_{n\geq 1}$ is a pramart norm converges to 0 a.s. □

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