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On simple estimation technique for the reliability of exponential lifetime model

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Abstract. Exponential distribution plays a key role in engineering reliability and its applications. The exponential failure model has been studied for years. This article introduces two new preliminary test estimators for the reliability function ($R(t)$) in complete and censored samples from the exponential model with the use of a prior estimation (θ_0) of the mean (θ). The proposed preliminary test estimators are studied and compared numerically with the existing estimators. Computer-intensive calculations for bias and relative efficiency show that for, different values of levels of significance and for varying constants involved in the proposed estimators, the proposed estimators are far better than classical and existing estimators.

KeyWords: *survival function, exponential lifetime distribution, preliminary test of significance, shrinkage, Bessel's function, mean squared error, bias ratio*

1. THE MODEL THAT INCORPORATES PRIOR INFORMATION

Indeed, the one- parameter exponential life-model is one of the most important models in many areas. In the context of reliability evaluation and life testing, a number of engineering data have been examined (see, Davis, 1952; Gnedenko, et al., 1969; Johnson, et al., 2005) and it was shown that the exponential distribution gives quite a good fit for the most cases. Examples of random variables that have been modelled successfully by the exponential lifetime distribution are "the lifetimes of transistors, batteries, tubes, bearings and the time to decay of radioactive atom, various queuing networks and Markov chains" (see, Forbes et al., 2011). Moreover, the cost of experimental units is one of the most important factors in life testing and reliability evaluation problems. Indeed, the objective

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of utilizing the prior estimate and using the single stage preliminary procedure is to minimize the cost of experimental units and the mean squared error of the new estimator.
In order to define the reliability of a given system (or unit); suppose that the unit begins to function at the instant $t = 0$ and that a failure occurs at the instant $t = x$. Assume that x is a random variable which is characterized by the exponential lifetime model with distribution given by,

$$F(t \mid \theta) = P(x \le t). \tag{1.1}$$

The reliability function of the unit during the time t is given by,

$$R(t \mid \theta) = 1 - F(t \mid \theta)$$
$$= \exp(-t|\theta), \quad t \ge 0, \quad \theta > 0 . \tag{1.2}$$

It is worth mentioning that different papers have discussed the problem of the estimation of the parameters and reliability function of exponential lifetime distribution (see Bain and Engelhardt (1991), Sinha (1986), Gnedenko et al. (1969) and Davis (1952)).
In many problems, the experimenter has a prior point estimate (or initial value) θ_o about the unknown parameter of exponential lifetime distribution θ. Suppose, for example that n lamps are tested and that a lamp will fail after 2000 hours of testing process. Here, we may take $\theta_0 = 2000$ hours. Similarly, based on an extensive testing, it is determined that the time t (in years), before a major repair is required for a manufacturer of a washing machine, is characterized by the exponential lifetime model with $\theta = 4$ years. In this case, θ_0 can be taken to be equal to 4 years. Based on Thompson (1968), Kambo et. al., (1990, 1992), Al-Hemyari (2010) and Al-Hemyari and Al-Dabag (2012) θ_o is a 'natural origin' and such natural origins may arise for any one of number of reasons, e.g., we are estimating θ and:

(i) **"**the prior value (θ_o) of θ in many practical problems exists".

(ii) "we believe θ_0 is close to the true value of θ_o, " or

(iii) "we fear that θ_0 may be near the true value of θ, i.e., something bad happens if $\theta_0 \cong \theta$, and we do not know about it".

In such cases, it is natural to utilize θ_o in the estimation of θ and it may improve the estimation procedure, i.e., reducing the mean squared error (MSE) of the new estimators or it may give a saving in the sample size. The new estimator of θ, which utilizes the initial estimate θ_o, is called the preliminary test (PT) estimator.

NOTATIONS

$B(\tilde{R}_i(t); \hat{R}(t) \mid R(t); R)$	Bias of $\tilde{R}_i(t)$
$F(t \mid \theta)$	Cumulative distribution function
d	Degree of freedom
$U_d(0 \le \hat{\theta} \le \chi^2 \mid \theta)$	Distribution function of a Chi-square random variable with d

$\Gamma(.)$	Gamma function
$\overline{K}_n(z)$	Incomplete modified Bessel's function of order *n*
α	Level of significance
$K_n(z)$	Modified Bessel's function of order *n*
MLE	Maximum likelihood estimator
$MSE(\tilde{R}_i(t); \hat{R}(t) \mid R(t); R)$	Mean squared error of $\tilde{R}_i(t)$
$MSE(\hat{\theta} \mid \theta)$	Mean squared error of $\hat{\theta}$
$MSE(\tilde{\theta} \mid \theta, R)$	Mean squared error of $\tilde{\theta}$
$\hat{\theta}$	MLE of θ
$\hat{R}(t)$	MLE of $R(t)$
R	Preliminary test region
$RE(\tilde{R}_I(t) \mid \hat{R}(t))$	Relative efficiency of $\tilde{R}_i(t)$
$R(t)$	Reliability of a given system (or component) for a given time
θ	Parameter of exponential distribution
$\tilde{R}(t)$	Preliminary test estimator of $R(t)$
θ_0	Prior value
k	Shrinkage factor
λ	Ratio (θ_0 / θ)
ϕ	Ratio (t / θ)

2. REVIEW OF LITERATURE AND THE AIM OF THE PAPER

Many papers have studied the problem of constructing preliminary test estimators of the parameter θ of the exponential distribution for different estimation problems and different sampling schemes (see Kambo et al., (1990), Chiou (1990), Jani (1991), Kambo et al., (1992), Kourouklis (1994) , Srivastiva and Kapasi (1999) and Lemer (2006)). Numerous papers have been produced in respect to the preliminary test estimator of the survival (reliability) function of the exponential lifetime model. Some of the work done in this direction is as follows: Chiou (1987) considered a preliminary test estimator for $R(t)$ of the exponential lifetime model given by,

$$\bar{R}(t) = \begin{cases} xp(-t/\theta_0), & if \;\; C_1 \le 2T/\theta_0 \le C_2, \\ \breve{R}(t), & if \;\; C_1 > 2T/\theta_0 \;\; or \;\; 2T/\theta_0 > C_2 \end{cases} \tag{2.1}$$

where $T=\sum_{i=1}^{r} X_{(i)}+(n-r)X_{(r)}$, $\hat{\theta}=T/r$ is the MLE of θ, C_1 and C_2 respectively are the lower and upper 100(1-$\alpha/2$) percentile points of the Chi-square distribution of $2r$ degrees of freedom and the minimum variance unbiased estimator $\breve{R}(t)$ (see Basu (1964)) is,

$$\breve{R}(t)=\begin{cases}(1-t/T)^{r-1}, if \quad t<T,\\ 0, \qquad if \quad t\geq T.\end{cases} \tag{2.2}$$

Chiou (1992) suggests the following estimator as a modification of the above estimator,

$$\bar{R}(t)=\begin{cases}k\exp(-t/\theta_0)+(1-k)\breve{R}(t), \ if \ C_1\leq 2T/\theta_0\leq C_2,\\ \breve{R}(t), \qquad if \ C_1>2T/\theta_0 \ or \ 2T/\theta_0>C_2,\end{cases} \tag{2.3}$$

where k $(0<k<1)$ is a constant factor.

Chiou (1993) proposes an empirical Bayes shrinkage estimation for $R(t)$.

Recently, Al-Hemyari and Al-Dolami (2012) have studied the problem of estimating the reliability function of exponential model and proposed a shrinkage estimator of estimating $R(t)$ given by,

$$\tilde{R}(t)=\{[\exp(-t/\theta_0)]I_R+[\exp(-t/k(\sum_{i=1}^{n} X_i/n))]I_{\bar{R}}\}, \tag{2.4}$$

where $k(0\leq k\leq 1)$, represents a shrinkage weighting factor, I_R is the indicator function of the acceptance region R and $I_{\bar{R}}$ is the indicator function of the rejection region $\bar{R}$.

Utilizing the prior value in the estimation of exponential model and employing the preliminary estimator for the unknown parameters have been also discussed for other estimation problems by many authors. Al-Hemyari (2009) and Al-Hemyari and Jehel (2011) have studied the following polling shrinkage estimators for exponential distribution by using two-stage complete data and two-stage time censored data respectively,

$$\tilde{R}(t)=\left\{\left[\exp\left(-t\left(k(n_1/\sum_{i=1}^{n_1}X_{1i})-\theta_o\right)+\theta_o\right)\right]I_R+\left[\exp\left(-t\left(n_1(n_1/\sum_{i=1}^{n_1}X_{1i})+n_2(n_2/\sum_{i=1}^{n_2}X_{2i})\right)/(n_1+n_2)\right)I_{\bar{R}}\right]\right\}, \tag{2.5}$$

and

$$\tilde{R}(t)=\left\{\left[\exp\left(-t\left(k({}_1r_0/n_{1}\,{}_1T_0)-\theta_o\right)+\theta_o\right)\right]I_R+\left[\exp\left(-t\left(({}_1r_0/{}_1T_0)+({}_2r_0/{}_2T_0)\right)/(n_1+n_2)\right)I_{\bar{R}}\right]\right\}, \tag{2.6}$$

where $({}_jr_0/{}_jT_0)$ and $(n_j/\sum_{i=1}^{n_j}X_{ji})$ are the ML estimators of $\hat{\theta}_j$ based on complete and time censored data respectively.

Furthermore, Baklizi and Ahmed (2008) have proposed a similar estimator to Chiou's (1992) for the two- parameter Weibull model. Al-Hemyari (2010) discusses two classes of preliminary test shrinkage estimators of estimating the scale parameter and reliability function of Weibull lifetime model in right censored samples.

This study concerns itself with investigating the problem of estimating the reliability function $R(t)$ in the exponential distribution when the information regarding θ is available in the form of an initial guess value θ_O, by using complete and censored samples where the aim is to find new estimators of $R(t)$, which offer some improvement over the classical and similar estimators. To be more specific, two appropriate choices of simple weighting functions are used in two proposed preliminary test estimators and the expressions of bias ratio, mean squared error and the relative efficiency of the proposed estimators are obtained and computed. The discussion regarding the comparisons with the earlier known results are then made. Some numerical results and conclusions drawn there from are presented.

3. THE FIRST SIMPLE PT ESTIMATOR of $R(t)$

As noted earlier, the main objective of this study is to draw and develop some preliminary test estimators for the reliability (survival) function. The aim of this technique is to improve the estimation procedure and to achieve some desirable properties, i.e., reducing the mean squared error of the new estimator, saving the sampling size and earning asymptotical properties (see Thompson (1968), Kambo et. al., (1990, 1992), Al-Hemyari and Husain (2012)).
The reliability function of the exponential model is defined as “the probability that the system has when the life length is greater than t of a given system” (alhemyari, 2010) and given in (1.2). Traditionally, the unknown parameters are estimated by using the classical methods (MLE, LSE, …), i.e. the sample data alone. The MLE of $R(t)$ (see, Sinha (1986), Bain and M. Engelhardt (1991)) is given by,

$$\hat{R}(t) = \exp(-(t/\hat{\theta})), \tag{3.1}$$

where $\hat{\theta}$ is calculated by $\sum_{i=1}^{n} X_i / n.$
“However, it is well known that the inclusion of prior information in the estimation of parameters is likely to improve the quality of the estimator in terms of desirable statistical properties” (seeAl-Hemyari and Husain (2012)).
In this section, we will discuss and develop two simple preliminary test (PT) estimators for the reliability function $R(t)$ of the exponential model, when a prior guess value θ_o of the scale parameter θ is previously available, and when the complete sample observations are available. It is worth mentioning here that the final proposed preliminary test (PT) estimator of $R(t)$ will take one of two alternatives depending on the preliminary test of hypothesis. Using θ_o and a test criterion, construct a region R in the space of parameter $\theta.$ If $\hat{\theta} \in R,$ use $\exp\{-(t/\hat{\theta})\}$ as an estimator of $R(t).$ Otherwise, take $\exp\{-(t/\theta_0)\}$ as an estimator of $R(t).$ The explicit expressions are obtained for the bias ratio and MSE of the proposed estimator.

The first proposed PT estimator of $R(t)$ denoted by $\tilde{R}_1(t)$, is then given by,

$$\tilde{R}_1(t)=\begin{cases}\exp\{-(t/\hat{\theta})\}, & if\ \hat{\theta}\in R,\\ \exp\{-(t/\theta_0)\}, & if\ \hat{\theta}\notin R,\end{cases} \tag{3.2}$$

where R is the commonly used acceptance region of the null hypothesis $H_0:\theta=\theta_0$ against the alternative $H_1:\theta\neq\theta_0$. If α is the level of significance of the test, then the preliminary test region R is given by,

$$R=\{\hat{\theta}:T(\hat{\theta})\in[L_{1-\alpha/2},\ U_{\alpha/2}]\}, \tag{3.3}$$

where $L_{1-\alpha/2}$ and $U_{\alpha/2}$ are the lower and upper 100(1-α/2) percentile points of the statistic $T(\hat{\theta})$, which are used for testing the above hypothesis. If the chi-square statistic $T(\hat{\theta})=2n\hat{\theta}/\theta$ is used, the region R is given by,

$$R=[(\theta_0/2n)\chi^2_{1-\alpha/2,2n},(\theta_0/2n)\chi^2_{\alpha/2,2n}], \tag{3.4}$$

where $0\le a<b$, $\chi^2_{1-\alpha/2,2n}$ and $\chi^2_{\alpha/2,2n}$ are the lower and upper 100(α/2) percentile points of the Chi-square distribution with $2n$ degrees of freedom. Let $R=[a,\ b]$, the bias of $\tilde{R}_1(t)$ then is defined by,

$$B(\tilde{R}_1(t),\hat{R}(t)\mid R(t);R)=E(\tilde{R}_1(t))-R(t)$$
$$=\int_R \exp\{-(t/\hat{\theta})\}f(\hat{\theta}\mid\theta)\ d\hat{\theta}+\int_{\bar{R}} \exp\{-(t/\theta_0)\}f(\hat{\theta}\mid\theta)d\hat{\theta}-\exp\{-(t/\theta)\}. \tag{3.5}$$

In fact, the classical estimator $\hat{R}(t)$ is easy to compute and like any preliminary test estimator, the proposed PT $\tilde{R}_1(t)$ is biased. It is not always easy to calculate the exact expressions of complicated PT estimators. In addition, it is not easy to calculate the equations of $B(\tilde{R}_1(t);\hat{R}(t)\mid R(t);R)$, $MSE(\tilde{R}_1(t);\hat{R}(t)\mid R(t);R)$ and $RE(\tilde{R}_1(t)\mid\hat{R}(t))$ of a complicated PT estimator like $\tilde{R}_1(t)$. As a result of some algebraic calculations, it follows that the bias ratio of $\tilde{R}(t)$ is reduced to,

$$B(\tilde{R}_1(t),\hat{R}(t)\mid R(t);R)l\theta=\exp\left\{-\left(\frac{t}{\theta}\left(\frac{1}{\lambda}+1\right)\right)\right\}(U_{2n}(a^*\le\hat{\theta}\le b^*\mid\theta)-1)+\bar{K}_n(2\sqrt{cd})(1/\exp(-t/\theta))-1, \tag{3.6}$$

since under $H_0, T(\hat{\theta})$ has a chi-square distribution with $2n$ degrees of freedom, $R=[a,\ b],\ a<b,\ R^*=[a^*,b^*],\ a^*<b^*,\ a^*=2na/\theta,\ b^*=2nb/\theta,$

$$U_i(0\le\hat{\theta}\le\chi^2\mid\theta)=\int_0^{\chi^2}\frac{\hat{\theta}^{(n+i-1)-1}}{\Gamma(n+i-1)}\left(\frac{n}{\theta}\right)^n\exp\left(-n\left(\frac{\hat{\theta}}{\theta}\right)\right)\ d\hat{\theta},\ i=1,2, \tag{3.7}$$

$$U_d(a^*\le\hat{\theta}\le b^*\mid\theta)=U_d(0\le\hat{\theta}\le b^*\mid\theta)-U_d(0\le\hat{\theta}\le a^*\mid\theta), \tag{3.8}$$

and $U_d(\hat{\theta})$ represents the cumulative distribution function of a Chi-square random variable with d degrees of freedom, $\overline{K}_n(z)$ is an incomplete modified Bessel's function. Denoting the modified Bessel's function of the second kind of order n by $K_n(z)$ where $K_n(z) = K_{-n}(z)$ for $n = 0,1,2, \; ... \;,$. (Watson, 1952; Sinha, 1986) and

$$\int_0^\infty y^{-l} e^{-(cy+d/y)} dy = 2(c/d)^{(l-1)/2} K_{l-1}(2\sqrt{cd}), \;\; l = 0,1,2, \tag{3.9}$$

where c and d are constants. Now we evaluate the integral

$$\int_0^\infty \exp\{-(t/\hat{\theta})\} f(\hat{\theta} \mid \theta) \; d\hat{\theta}, \tag{3.10}$$

by making use of the result of the modified Bessel's function in (3.12), and by making the change of constant $-l = n-1, \; c = 1, \; d = nt/\theta,$ and the variable $y = n\hat{\theta}/\theta,$ we get,

$$\int_0^\infty (\hat{\theta})^{n-1} \left(\frac{n}{\theta}\right)^n \exp\left\{-\left(\frac{n\hat{\theta}}{\theta} + \frac{t}{\hat{\theta}}\right)\right\} \; d\hat{\theta} = \int_0^\infty y^{n-1} e^{-(y+(nt/\theta)/y)} dy$$

$$= 2(nt/\theta)^{(n-1)/2} K_n(2\sqrt{nt/\theta}), \tag{3.11}$$

$y = n\hat{\theta}/\theta$. $\overline{K}_n(2\sqrt{nt/\theta})$ is an incomplete modified Bessel's function of the second kind of order n, where the integration is over the interval $R^* = [a^*, \; b^*], \; a^* = \lambda\chi^2_{1-\alpha/2,2r}$, and $b^* = \lambda\chi^2_{\alpha/2,2r}$. The expression of the mean squared error of $\tilde{R}_1(t)$ is given by,

$$MSE(\tilde{R}_1(t); \hat{R}(t) \mid R(t); R) = E(\tilde{R}_1(t) - R(t))^2$$

$$= \int_{\hat{\theta} \in R} \left[\exp\{-(t/\hat{\theta})\} - \exp\{-(t/\theta)\}\right]^2 f(\hat{\theta} \mid \theta) \;\; d\hat{\theta}$$

$$+ \int_{\theta \in \overline{R}} \left[\exp\{-(t/\theta_0)\} - \exp\{-(t/\theta)\}\right]^2 f(\hat{\theta} \mid \theta) \;\; d\hat{\theta}. \tag{3.12}$$

As a result of the expressions (3.10) and (3.11), we obtain,

$MSE(\tilde{R}_1(t); \hat{R}(t) \mid R(t); R)$

$$= (2/\Gamma(r))\left(\frac{2nt}{\theta}\right)^{n/2} \overline{K}_n\left(2\sqrt{\frac{2nt}{\theta}}\right) - (4/\Gamma(r)).\exp\{-(t/\theta)\}\left(\frac{nt}{\theta}\right)^{n/2} \overline{K}_n\left(2\sqrt{\frac{nt}{\theta}}\right) + \left[\exp\{-(t/\theta)\}\right]^2. \tag{3.13}$$

The efficiency of $\tilde{R}_1(t)$ relative to MLE $\hat{R}(t)$ is defined (ibid) by,

$$RE(\tilde{R}_1(t) \mid \hat{R}(t)) = MSE(\hat{R}(t) \mid R(t)) / MSE(\tilde{R}_1(t); \hat{R}(t) \mid R(t); R)), \tag{3.14}$$

where

$$MSE(\hat{R}(t) \mid R(t)) = E(\hat{R}(t) - R(t))^2$$
$$= (2/\Gamma(r))\left(\frac{2nt}{\theta}\right)^{n/2} K_r\left(2\sqrt{\frac{2nt}{\theta}}\right) - (4/\Gamma(r))\exp\{-(t/\theta)\}\left(\frac{nt}{\theta}\right)^{n/2} \times$$
$$\times K_r\left(2\sqrt{\frac{nt}{\theta}}\right) + \left[\exp\{-(t/\theta)\}\right]^2. \tag{3.15}$$

4. THE SECOND SIMPLE PT ESTIMATOR OF $R(t)$

In section 3, we undertook the study of PT estimator for the reliability function of the exponential model. The PT estimator to be considered in this section is a slight modification of $\tilde{R}_1(t)$, i.e. in the present section instead of $\exp\{-(t/\hat{\theta})\}$, the estimator $\exp\{-(t/k\hat{\theta})\}$, where k represents the shrinkage factor (see remark 1), is used in the estimation of $R(t)$. The justification of including k in PT estimator is to reduce the mean squared error of the new PT estimator.
The proposed PT estimator, which is now denoted by $\tilde{R}_2(t)$, takes the form,

$$\tilde{R}_2(t) = \begin{cases} \exp(-(t|k\hat{\theta})), & if\ \hat{\theta} \in R, \\ \exp(-(t|\theta_0)), & if\ \hat{\theta} \notin R, \end{cases} \tag{4.1}$$

with the interval R defined by (3.4). Setting $R = [a, \ b]$. The expression of the bias ratio and mean squared error of $\tilde{R}_2(t)$ are derived by using the same method as in the last section, where we obtain the following results:

$$B(\tilde{R}_2(t); \hat{R}(t) \mid R(t); R) l\theta = \overline{K}_n(2\sqrt{cd/k})(1/\exp(-(t/\theta))$$
$$+\exp\left\{-\left(\frac{t}{\theta}\left(\frac{1}{\lambda}+1\right)\right)\right\}(1 - U_{2n}(a^* \le \hat{\theta} \le b^* \mid \theta)) - 1, \tag{4.2}$$

and

$$MSE(\tilde{R}_2(t); \hat{R}(t) \mid R(t); R) = \exp\{-(t/\theta_0) G_{2r}(\theta \mid (a^*, b^*))$$
$$+\left[\exp\{-(t/\theta_0)\} - \exp\{-(t/\theta)\}\right]^2 (1 - U_{2n}(a^* \le \hat{\theta} \le b^* \mid \theta))$$
$$+\overline{K}_n(2\sqrt{2cd/k})) - 2\exp\{-(t/\theta)\}\overline{K}_n(2\sqrt{cd/k}), \tag{4.3}$$

where $c = 1$, $d = nt/\theta$ and $\overline{K}_n(nt/k\theta)$ is given by,

$$\overline{K}_n(nt/k\theta) = \int_0^\infty (\hat{\theta})^{n-1}\left(\frac{n}{\theta}\right)^n \exp\left\{-\left(\frac{n\hat{\theta}}{\theta} + \frac{jt}{k\hat{\theta}}\right)\right\} \ d\hat{\theta}, \quad j = 3, 4. \tag{4.4}$$

$\overline{K}_n(2\sqrt{nt/k\theta})$ is also an incomplete Bessel function, where the integration is over the same interval $R^* = [a^*, \; b^*],$ and

$$RE(\tilde{R}_2(t) \mid \hat{R}(t)) = MSE(\hat{R}(t) \mid R(t)) / MSE(\tilde{R}_2(t); \hat{R}(t) \mid R(t); R)). \tag{4.5}$$

Remark 1: The performance of $\tilde{R}_2(t)$ depends on a proper choice of the shrinkage factor k and the region $R.$ Following Al-Hemyari and Al-Dolami (2012), the suggested weighting function k for $\tilde{R}_2(t)$ satisfies the criterion,

$$\{k : MSE(\hat{\theta} \mid \theta_0) - MSE(\tilde{\theta} \mid \theta_0, R) \le k \quad MSE(\hat{\theta} \mid \theta_0)\}. \tag{4.6}$$

Simple calculations lead to,

$$k = 1 - \frac{\int_R (\hat{\theta} - \theta_0)^2 f((\hat{\theta} \mid \theta_0) d\hat{\theta}}{\int_0^\infty (\hat{\theta} - \theta_0)^2 f((\hat{\theta} \mid \theta_0) d\hat{\theta}} \le 1, \tag{4.7}$$

where

$$\int_a^b (\hat{\theta} - \theta_0)^i \frac{(\hat{\theta})^{(n-1)}}{\Gamma(n+i-1)} \left(\frac{n}{\theta}\right)^n \exp\left[-n\left(\frac{\hat{\theta}}{\theta}\right)\right] \; d\hat{\theta}$$

$$= (\theta_0)^i \sum_{i=1}^{l} (-1)^i C_i^l (r\lambda)^{l-i} \frac{\Gamma(l+r-i)}{\Gamma(r)} U_{2n}(a^* \le \hat{\theta} \le b^* \mid \theta), \quad i = 1, 2, \tag{4.8}$$

a^*, b^* and $U_d(a^* \le \hat{\theta} \le b^* \mid \theta)$ are defined in (3.6), (3.7) and (3.8).

Remark 2: We note that k given by (4.7) is $k \le 1.$ Thus, if $k = 0,$ the estimator $\tilde{R}_2(t)$ is a constant, and therefore we shall not consider it.

Remark 3: The results derived in sections 3 and 4 for the PT $\tilde{R}_i(t), \;\; i = 1,2$ can be used in the following estimation problems by using the other censored schemes (the details are not given for space consideration).

i) Right censored samples.
ii) Left censored samples.
iii) Doubly censored samples.
iv) Censored sample with replacement.

5. APPLICATION

The calculation and explanation of the proposed preliminary test estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are discussed in this section. In this study, the survival times were illustrated and investigated in days of a device. Moreover, the proposed preliminary test estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are computed for some given values involved in those estimators. In addition, two sets of life times of advice were given (see http://theriac.org/DeskReference/viewDocument.php?id=195), and the MLE of θ and $R(t)$ (see equations (2.1) and (3.1)) are calculated for different samples. In this example, we assumed that 20 items (the first set of observations) are placed on a life test, and the test is terminated after all items failed, and the mean life $\hat{\theta} = 460.14,$ is calculated. Here we may take $\theta_o = 460.14$. The computations of PT estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are based on the second data set and utilizes the prior value $\theta_0 = 460.14$. Select a sample of size $n = 4, 6, 8, 10, 12,$ from the second data set. Based on a complete sample of size $n = 4, 6, 8, 10, 12,$ $\hat{R}(t), \hat{\theta}, R, \tilde{R}_1(t)$ and $\tilde{R}_2(t),$ are to be computed and $\hat{R}(t), \tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are given below (Table 1) for a number of values assigned for n, $t = 25, 50, 57, 100$ and $\alpha = 0.01, 0.05, 0.1.$ The corresponding values of $RE(\tilde{R}(t) \mid \hat{R}(t))$ and $B(\tilde{R}(t); \hat{R}(t) \mid R(t); R),$ can be obtained from Figures 1 to 6 using the corresponding constants $\alpha, r, t, \phi = (t/\theta),$ and λ. The observations and comparisons between the estimators $\hat{R}(t), \tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are given.

Based on Table 1, we observe the following:

i) The values of classical estimator $\hat{R}(t)$ (see equation (3.1)) are independent of α, and are increasing functions of $\hat{\theta}$ and decreasing functions of t.

ii) The numerical values of $\tilde{R}_i(t), \ i = 1, 2$ are decreasing function of α, i.e., $\alpha = 0.01$ which gives the high value of $\tilde{R}_i(t), \ i = 1, 2$ than the other values of α.

iii) For fixed α, the numerical values of $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are also decreasing function of n, i.e., $n = 4$, which yields the higher values of $\tilde{R}_i(t), \ i = 1, 2$ than the other values of n.

iv) The computed values of the improved estimators $\tilde{R}_1(t)$ are greater than the values of the classical estimator $\hat{R}(t)$ for all values of n, α and t; whereas $\tilde{R}_2(t)$ are generally greater than the values of the classical estimator $\hat{R}(t)$ when $t < 100$.

v) It is observed that the numerical values of $\tilde{R}_2(t)$ are equal or greater than that of $\tilde{R}_1(t)$.

Table 1. $\hat{\theta}, \hat{R}(t), R$ and $\tilde{R}_i(t), \ i = 1,2$ for different values of α, n and t.

α	n	t	$\hat{\theta}$	$\hat{R}(t)$	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	n	t	$\hat{\theta}$	$\hat{R}(t)$	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$
0.01	4	25	328.86	0.926	0.928	0.928	6	75	298.87	0.778	0.785	0.779
0.05	4	25	328.86	0.926	0.927	0.927	6	75	298.87	0.778	0.782	0.777
0.1	4	25	328.86	0.926	0.927	0.927	6	75	298.87	0.778	0.778	0.774
0.01	4	50	328.86	0.858	0.862	0.861	6	100	298.87	0.715	0.724	0.717
0.05	4	50	328.86	0.858	0.860	0.860	6	100	298.87	0.715	0.720	0.714
0.1	4	50	328.86	0.858	0.859	0.859	6	100	298.87	0.716	0.716	0.711
0.01	4	75	328.86	0.796	0.801	0.799	8	25	294.30	0.918	0.921	0.916
0.05	4	75	328.86	0.796	0.798	0.797	8	25	294.30	0.918	0.920	0.917
0.1	4	75	328.86	0.796	0.796	0.796	8	25	294.30	0.918	0.919	0.917
0.01	4	100	328.86	0.737	0.744	0.742	8	50	294.30	0.844	0.849	0.844
0.05	4	100	328.86	0.737	0.741	0.739	8	50	294.30	0.844	0.846	0.842
0.1	4	100	328.86	0.737	0.738	0.737	8	50	294.30	0.844	0.844	0.840
0.01	6	25	300.06	0.920	0.922	0.921	8	75	294.30	0.775	0.782	0.775
0.05	6	25	300.06	0.920	0.921	0.920	8	75	294.30	0.775	0.779	0.772
0.1	6	25	300.06	0.920	0.920	0.920	8	75	294.30	0.775	0.776	0.770
0.01	6	50	300.06	0.846	0.851	0.848	8	100	294.30	0.712	0.721	0.712
0.05	6	50	300.06	0.846	0.849	0.846	8	100	294.30	0.712	0.717	0.708
0.1	6	50	300.06	0.846	0.847	0.846	8	100	294.30	0.712	0.713	0.706
0.01	6	75	300.06	0.778	0.785	0.781	10	25	268.79	0.911	0.915	0.911
0.05	6	75	300.06	0.778	0.782	0.779	10	25	268.79	0.911	0.913	0.910
0.1	6	75	300.06	0.778	0.779	0.778	10	25	268.79	0.911	0.912	0.909
0.01	6	100	300.06	0.716	0.725	0.720	10	50	268.79	0.830	0.837	0.831
0.05	6	100	300.06	0.716	0.720	0.7120	10	50	268.79	0.830	0.834	0.828
0.1	6	100	300.06	0.716	0.717	0.715	10	50	268.79	0.830	0.831	0.826
0.01	8	25	298.87	0.919	0.922	0.920	10	75	268.79	0.756	0.766	0.757
0.05	8	25	298.87	0.919	0.921	0.919	10	75	268.79	0.756	0.761	0.753
0.1	8	25	298.87	0.919	0.920	0.918	10	75	268.79	0.756	0.757	0.750
0.01	8	50	298.87	0.845	0.851	0.846	10	100	268.79	0.689	0.701	0.690
0.05	8	50	298.87	0.845	0.848	0.845	10	100	268.79	0.689	0.695	0.685
0.1	8	50	298.87	0.845	0.846	0.844	10	100	268.79	0.689	0.690	0.681

6. SIMULATION AND NUMERICAL RESULTS

To study the behaviour of the proposed PT estimators empirically and to give a useful comparison between the proposed, classical and existing estimators, we perform some computer calculations for the bias ratio and the relative efficiency. The following numerical results and conclusions are based on these computations:

6.1 The PT estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$

For the PT estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ the computations are done for $n = 4(2)12$ $(i.e.,4,6,8,10,12)$, $\phi = (t/\theta) = (1/6)(1/6)(36/6)$ $(i.e.,1/6,2/6,3/6,\ ...,\ 35/6,36/6)$, $\alpha = 0.01,0.02,0.05,0.1,0.15$ and $\lambda = 0.1(0.1)3(i.e.,\ 0.1,0.2,0.3,0.4,\ ...,\ ,2.9,3)$. Some of these computations are given in Figures 1 to 6. We make the following observations from the Graphs of computations presented in this paper.

1) As expected, the PT estimators ($\tilde{R}_1(t)$ and $\tilde{R}_2(t)$) give high relative efficiency than the classical estimator $\hat{R}(t)$ in the neighbourhood of $\lambda \cong 1$. Both PT estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ give the highest relative efficiency when $\lambda \cong 1$ and decrease as $|\lambda - 1|$ increases (see Figure1).

2) The relative efficiency of $\tilde{R}_i(t),\ i = 1,\ 2$ is a concave function of λ, i.e., the proposed PT estimators have a maximum efficiency in the neighbourhood of $\lambda \cong 1$.

3) The relative efficiency of $\tilde{R}_i(t),\ i = 1,\ 2$ is an increasing function of ϕ with the maximum of $\phi = (36/6)$.

4) From Figure 3 we note that the relative efficiency of $\tilde{R}_i(t),\ i = 1,\ 2$ is a decreasing function of α; where the relative efficiency is also a decreasing function of n i.e., $\alpha = 0.01, \phi = (36/6),$ and $n = 4$ yields the highest efficiency.

5) The PT estimators $\tilde{R}_i(t),\ i = 1,\ 2$ are biased. From Figure 4 we observe that the bias ratio is reasonably small when θ is sufficiently close to θ_0 .

6.2 Comparisons

From Figures 1-6, we observe the following comparisons between the proposed PT estimators, classical and similar estimators.

1) The proposed PT estimators $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ have smaller mean squared error in the neighbourhood of $\lambda \cong 1$ than the classical estimator $\hat{R}(t)$ for $0.1 \le \lambda < 3$.

2) In comparing $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$, we observe from the computations that the PT estimator $\tilde{R}_2(t)$ has a higher relative efficiency than $\tilde{R}_1(t)$ for $0.1 \le \lambda < 3$ (Figures 1-4).

6.3 Conclusion

Two modified preliminary test shrinkage estimators of the reliability function of the exponential model have been suggested for complete and censored data and by choosing region R as a preliminary test region of the hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \ne \theta_0$.

The behavioural pattern of the proposed estimators when a prior guess value of θ is available have been studied on the bases of bias ratio and relative efficiency.

The simulation approach shows that the proposed estimators, $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are better than the classical estimators and that of Al-Hemyari and Al-Dolami (2012), Chiou (1992), Jani (1991) and Chiou (1990) and (1987) and both in terms of higher relative efficiency and boarder range of λ for which efficiency is greater than the unity for a number of choices of n, α and ϕ when it is felt that the prior value θ_o does not deviate much from the true value θ.

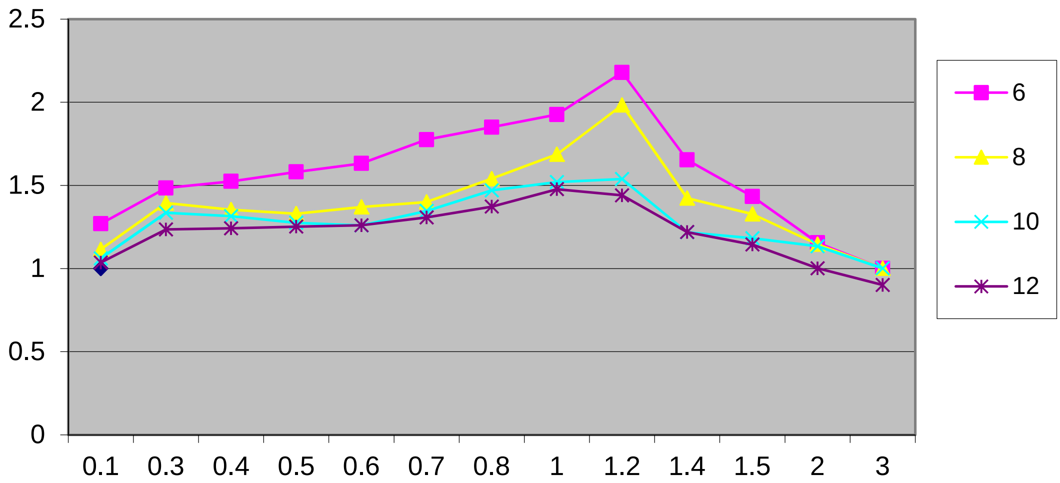

Figure 1. $RE(\tilde{R}_1(t)\,|\,\tilde{R}(t))$ *for* $\alpha = 0.01$, $\phi = 36/6$, $n = 4(2)12$, and $\lambda = 0.1(0.1)3$

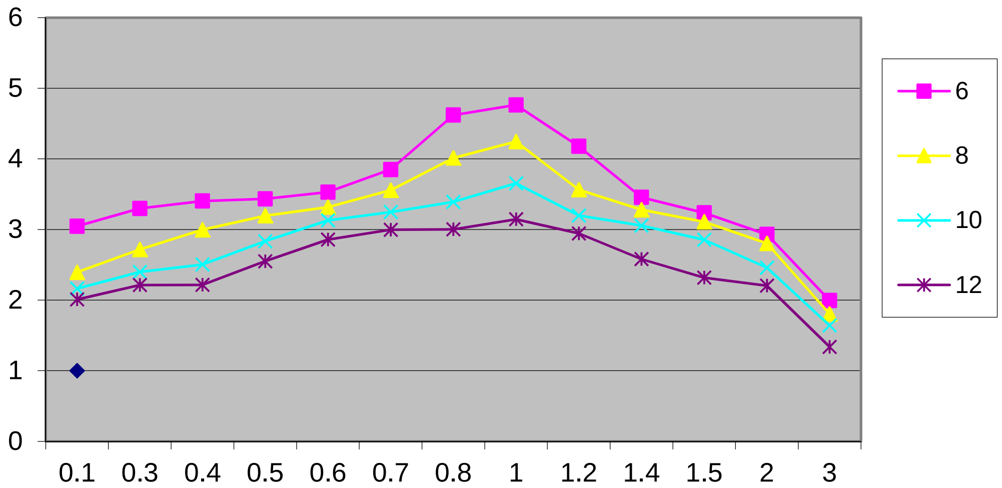

Figure 2. $RE(\tilde{R}_2(t)\,|\,\tilde{R}(t))$ *for* $\alpha = 0.01$, $\phi = 36/6$, $n = 4(2)12$, and $\lambda = 0.1(0.1)3$

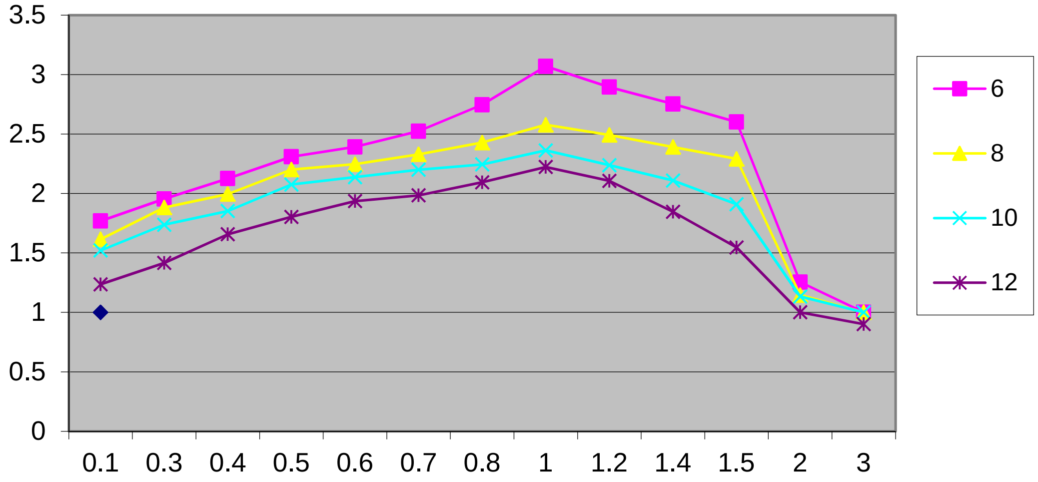

Figure 3. $RE(\tilde{R}_2(t) \mid R(t))$ *for* $\alpha = 0.05$, $\phi = 36/6$, $n = 4(2)12$, and $\lambda = 0.1(0.1)3$

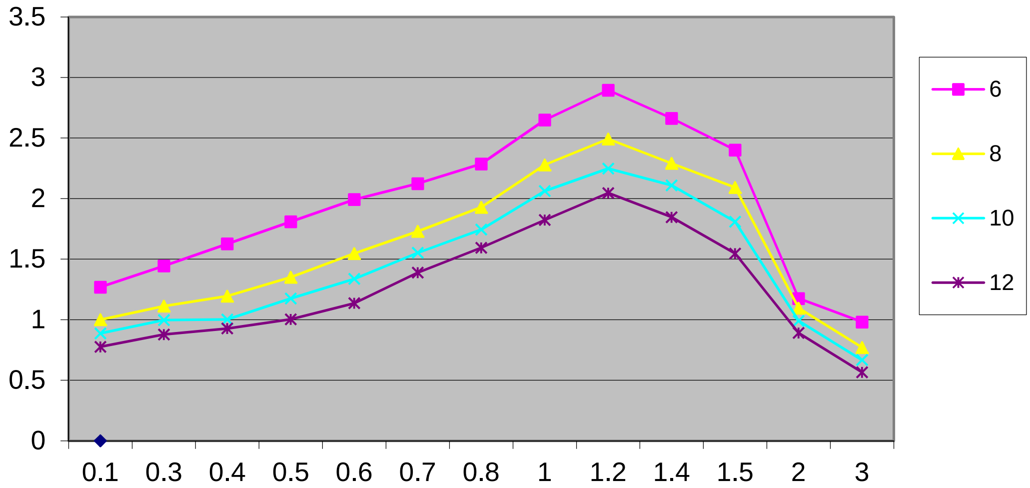

Figure 4. $RE(\tilde{R}_2(t) \mid R(t))$ *for* $\alpha = 0.1$, $\phi = 36/6$, $n = 4(2)12$, and $\lambda = 0.1(0.1)3$

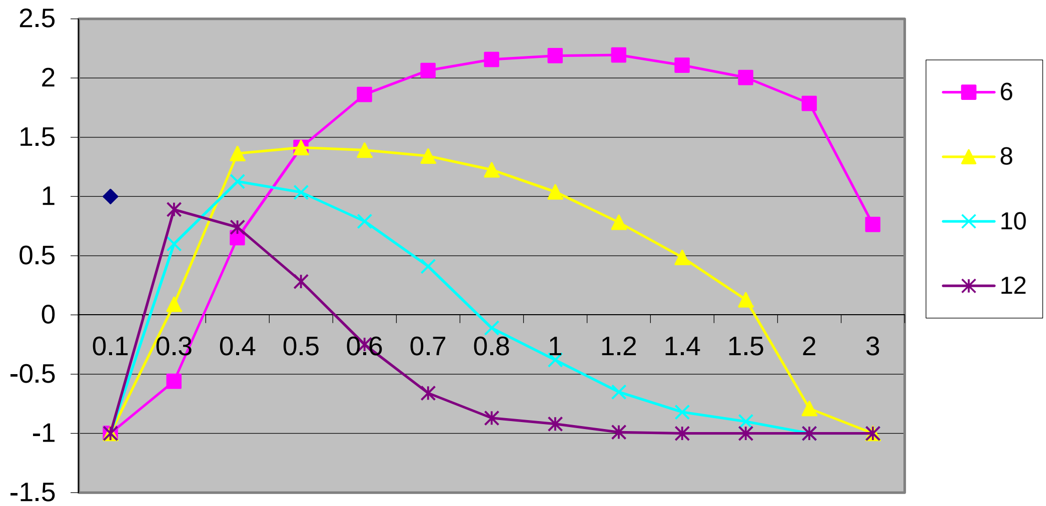

Figure 5. $B(\tilde{R}_1(t) \mid R(t))$ *for* $\alpha = 0.01, \phi = 36/6, n = 4(2)12$, and $\lambda = 0.1(0.1)3$

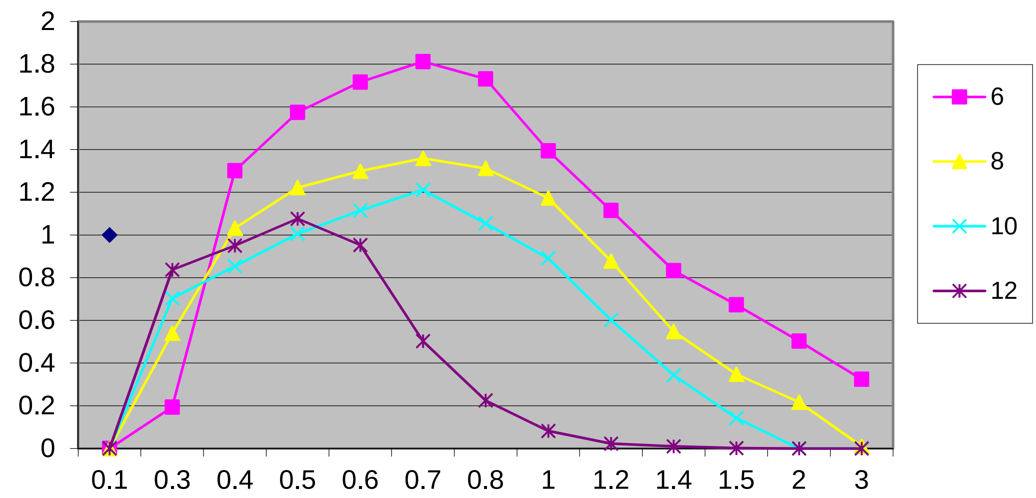

Figure 6. $B(\tilde{R}_2(t) \mid R(t))$ *for* $\alpha = 0.01, \phi = 36/6, n = 4(2)12$, and $\lambda = 0.1(0.1)3$

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