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An accelerated sequential sampling for estimating the reliability of N-parallel systems

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Abstract. The problem of designing an experiment to estimate the reliability of a system that has N subsystems connected in series where each subsystem n has T_n components connected in parallel is investigated both theoretically and by simulation. An accelerated sampling sheme is introduced. It is shown that the accelerated sampling scheme is asymptotically optimal as the total number of units goes to infinity. Numerical comparisons for a system that has two subsystems connected in series where each subsystem has two components connected in parallel are also given. They indicate that the accelerated sampling scheme performs better than the batch sequential sampling scheme and is nearly optimal.

Key Words: *Reliability, accelerated sampling scheme, optimal sampling scheme, batch sequential sampling scheme, Monte Carlo simulation.*

1. INTRODUCTION

Suppose that a system is composed of N independent subsystems connected in series, where each subsystem n has T_n components connected in parallel.
Denoting by $(1-q_{in})$ the reliability of component i where q_{in} is the probability that a unit of component i in subsystem n fails, then the reliability of each subsystem n is $(1-\prod_{i=1}^{T_n} q_{in})$. Therefore, the reliability of the system is the product of the reliability of each subsystem

$$R=\prod_{n=1}^{N}(1-\prod_{i=1}^{T_n} q_{in}).$$

Nearly optimal sampling schemes for estimating sytem reliabilites have been proposed by Berry (1977), Djerdjour and Rekab (2002), Enaya et al. (2010), Enaya et al. (2006),

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Kolowrocki (1994) and Rekab (1993). These papers dealt with series systems with many components. The system that we consider in this paper is more general, since it consists of subsytems in series where each susbystem has parallel components. To estimate the reliability of a system composed of two independent subsystems, where each subsytem has two components conncetd in paraller, Enaya et al. (2010) proposed a batch sequential sampling scheme. It was shown to be nearly optimal and peformed better than the balanced sampling scheme. In this article, an accelerated sampling scheme is introduced and compared with the batch and the optimal allocation schemes. Theoretical results show that the proposed accelerated sampliong scheme is asymptoticall optimal. Monte carlo studies indicate that it performs better than the batch sequential sampling scheme, and as well as the optimal sampling scheme.

For estimating the product of means (which has an application in reliability), using a Bayesian appraoch (Rekab (2000) and Rekab et al. (2007)) determined asymptotically optimal sampling schemes as the total nunmber of observations tends to infinity. Page (1990) determined a sampling scheme and a stopping rule. The sampling scheme was shown to be asymptotically optimal as the cost of sampling tends to zero. A similar formulation without a stopping rule was investigated earlier by Berry (1977). Exact optimal sampling schemes were determined by Hardwick and Stout (1992). However, exact optimality could only be achieved using dynamic programming.

The rest of the paper is organized as follows. In the next section, we compute the variance of the overall reliability and the variance incurred by the optimal sampling scheme. The accelerated sampling scheme is described in Section 3. Section 4 presents the results of the Monte Carlo simulations.

2. OPTIMAL SAMPLING SCHEME

The problem is how to allocate a fixed total number of T tests units from component i in subsystem n so as to minimize the variance of the system reliability estimate $\hat{R}$. Since the reliability of the system is

$$R = P[1^{st}\, subsystem\ works]\cdots P[N^{th} subsystem\ works]$$

$$= (1-\prod_{i=1}^{T_1} q_{i1})\cdots(1-\prod_{i=1}^{T_N} q_{iN}) = \prod_{n=1}^{N}(1-\prod_{i=1}^{T_n} q_{in}),$$

then

$$\hat{R} = (1-\prod_{i=1}^{T_1} \hat{q}_{i1})\cdots(1-\prod_{i=1}^{T_N} \hat{q}_{iN}) = \prod_{n=1}^{N}(1-\prod_{i=1}^{T_n} \hat{q}_{in}),$$

where $\hat{q}_{in}$ is the maximum likelihood estimator(MLE) of the probability of failure of component i in subsystem n, that is

$$\hat{q}_{in} = \sum_{j=1}^{M_{in}} (X_{inj}/M_{in})$$

where

$$X_{inj} = \begin{cases} 0 & \text{if the jth unit of component i in subsystem n functions,} \\ 1 & \text{if the jth unit of component i in subsystem n fails,} \end{cases} \quad (2.1)$$

and M_{in} represents the number of units to test from component i in subsystem n. The optimal sampling scheme is to select M_{in} so as to minimize the variance of $\hat{R}$.

$$Var(\hat{R}) = E[\prod_{n=1}^{N}(1-\prod_{i=1}^{T_n}\hat{q}_{in})^2] - E^2[\prod_{n=1}^{N}(1-\prod_{i=1}^{T_n}\hat{q}_{in})],$$

$$Var(\hat{R}) = \prod_{n=1}^{N}[1-2\prod_{i=1}^{T_n}E(\sum_{j=1}^{M_{in}}\frac{X_{inj}}{M_{in}}) + \prod_{i=1}^{T_n}E^2(\sum_{j=1}^{T_n}\frac{X_{inj}}{M_{in}})]$$

$$-\prod_{n=1}^{N}[1-\prod_{i=1}^{T_n}E(\sum_{j=1}^{M_{in}}\frac{X_{inj}}{M_{in}})]^2.$$

Theorem 2.1 *The variance of the optimal sampling scheme is given by*

$$Var(O) = \frac{1}{T}\{\sum_{n=1}^{N}(\sum_{j=1}^{T_n}(\sqrt{p_{jn}q_{jn}}\prod_{k=1 k\neq j}^{T_n}q_{kn})\prod_{l=1 l\neq n}^{N}(1-\prod_{i=1}^{T_l}q_{il}))\}^2.$$

Proof. After dropping higher order terms, the variance of the estimated reliability can be approximated by

$$Var(\hat{R}) \approx \sum_{n=1}^{N}[\prod_{l=1 l\neq n}^{N}(1-\prod_{i=1}^{T_l}q_{il})^2(\sum_{j=1}^{T_n}(\frac{p_{jn}q_{jn}}{M_{jn}}\prod_{k=1 k\neq j}^{T_n}q_{kn}^2))]$$

$$= \frac{1}{T}[\sum_{n=1}^{N}(\sum_{j=1}^{T_n}(\sqrt{p_{jn}q_{jn}}\prod_{k=1 k\neq j}^{T_n}q_{kn})\prod_{l=1 l\neq n}^{N}(1-\prod_{i=1}^{T_l}q_{il}))]^2$$

$$+\sum_{n\neq m}\sum_{j\neq i}\frac{(M_{jn}S_{im}-M_{im}S_{jn})^2}{2TM_{jn}M_{im}} + \sum_{n=m}\sum_{j\neq i}\frac{(M_{jn}S_{im}-M_{im}S_{jn})^2}{2TM_{jn}M_{im}},$$

$$+\sum_{n\neq m}\sum_{j=i}\frac{(M_{jn}S_{im}-M_{im}S_{jn})^2}{2TM_{jn}M_{im}},$$

where $S_{im} = \sqrt{p_{im}q_{im}}(\prod_{k=1 k\neq i}^{T_m}q_{km})\prod_{l=1 l\neq m}^{N}(1-\prod_{v=1}^{T_l}q_{vl}) \approx \sqrt{q_{im}/(1-q_{im})}$. Treating M_{in} as a continuous variables, then by differentiation with respect to M_{in} leads to a system of linear equations under the constraint $T = \sum_{n=1}^{N}\sum_{i=1}^{T_n}M_{in}$.

$$\frac{M_{im}}{M_{jn}} = \frac{S_{im}}{S_{jn}}.$$

Therefore, the optimal number of units to be tested from component i in subsystem n is

$$M_{in} = T \frac{S_{in}}{\sum_{n=1}^{N}\sum_{j=1}^{T_n} S_{jn}}.$$

3. ACCELERATED SEQUENTIAL SAMPLING SCHEME

Since the optimal sampling scheme depends on q_{in} which are unknown, then the optimal sampling scheme is not practical. To overcome this problem, we propose an accelerated sequential samplimg scheme that will mimick the optimal sampling scheme. Let $T = \sum_{n=1}^{N}\sum_{i=1}^{T_n} M_{in}$ be the total number of units to be tested and $S = \sum_{n=1}^{N} T_n$ be the total number of components.

The accelerated sequential sampling scheme is defined as follows:

(1). In the first stage, we start with an initial sample of L observations for each component in each subsystem, L must satisfy the following conditions:

a. $L \to \infty$ as $T \to \infty$.

b. $L \le (T/S)$.

c. L must be relatively small compared to T, that is $(L/T) \to 0$ as $T \to \infty$.

Then we split the remaining $T - SL$ units into two equal batches, and estimate $c_{in} = \sqrt{q_{in}/(1-q_{in})}$ by $\hat{c}_{inL} = \sqrt{\hat{q}_{inL}/(1-\hat{q}_{inL})}$, where $\hat{q}_{inL} = (\sum_{j=1}^{L} X_{inj})/L$.

(2). In the second stage, there are $(T - SL)/2$ remaining units to be tested. Since M_{in} is at least L and at most (T+SL)/2, then M_{in} is chosen in the following manner:

$$M_{in} = min\left\{\frac{T+SL}{2}, max\{L, \hat{M}_{inL}\}\right\},$$

where

$$\hat{M}_{inL} = T \frac{\hat{c}_{inL}}{\sum_{n=1}^{N}\sum_{j=1}^{T_n} \hat{c}_{jnL}}.$$

(3). Let M_{ink} denote the number units of component i of subsystem n that have been tested up to stage k, where $(T+SL)/2+1 \le k \le T$. We test one unit of component i of subsystem n at stage $k+1$ if

$$\frac{M_{ink}}{M_{jmk}} < \frac{\hat{c}_{jmk}}{\hat{c}_{ink}}.$$

We proceed sequentially until all $(T-SL)/2$ units are tested. If any ties occur, then they can be resolved in an arbitrary fashion by taking an independent random experiment or by by a fixed convention such as alternating the choice of the components.

Theorem 3.1 *The excess variance incurred by the accelerated sequential sampling scheme over the variance incurred by the optimal sampling scheme is in the order of* $1/T$ *, as* $T \to \infty$.

Proof. Since

$$Var(\hat{R}) \approx Var(O) + \sum_{n\neq m}\sum_{j\neq i} \frac{(M_{jn}S_{im} - M_{im}S_{jn})^2}{2TM_{jn}Mim} + \sum_{n=m}\sum_{j\neq i} \frac{(M_{jn}S_{im} - M_{im}S_{jn})^2}{2TM_{jn}Mim} + \sum_{n\neq m}\sum_{j=i} \frac{(M_{jn}S_{im} - M_{im}S_{jn})^2}{2TM_{jn}Mim},$$

then the proof will follow if we establish

$$\frac{M_{in}}{M_{jm}} \to \frac{c_{jm}}{c_{in}}.$$

The rest of the proof is similar to rekab [10]. For k large enough there exists

$$k^{(i)} = sup\left\{ t < k : \frac{M_{int}}{M_{jmt}} < \frac{\hat{c}_{jmM_{jmt}}}{\hat{c}_{inM_{int}}} \right\}$$

where $t \geq \dfrac{T+SL}{2} + 1$ and $k^{(i)} \to \infty$ as $k \to \infty$. Then

$$\frac{M_{ink}}{M_{jmk}} \leq \frac{M_{ink^{(i)}} + 1}{M_{jmk^{(i)}}} \leq \frac{\hat{c}_{jmM_{jmk^{(i)}}}}{\hat{c}_{inM_{ink^{(i)}}}} + \frac{1}{M_{jmk^{(i)}}}.$$

On the other hand,

$$\frac{M_{ink}}{M_{jmk}} \geq \frac{M_{ink^{(i)}}}{M_{jmk^{(i)}} + 1} = \left(1 - \frac{1}{M_{jmk^{(i)}} + 1}\right) \frac{M_{ink^{(i)}}}{M_{jmk^{(i)}}} \geq \frac{\hat{c}_{jmM_{jmk^{(i)}}}}{\hat{c}_{inM_{ink^{(i)}}}} \left(1 - \frac{1}{M_{jmk^{(i)}} + 1}\right).$$

The proof follows by the stong law of large numbers.

4. MONTE CARLO SIMULATION

In this section, we use Monte Carlo simulations with 5,000 replications to compare the variance incurred by the accelerated sequential sampling scheme (AS) with the variance incurred by the optimal sampling scheme and the variance incurred by the batch sequential sampling scheme (BS) proposed by Enaya et al. (2010). Let the speed of convergence of a sequential sampling scheme P be defined as: SPEED =

$T[V(P)-V(O)]$ where O denotes the optimal sampling scheme. We will consider a system that is composed of two independent subsystems connected in series, where each subsystem has two components connected in parallel.

Suppose that the two subsystems have equal probabilities $p_1 = p_2 = p_3 = p_4 = 0.95$, and suppose that a total of 100 units will be tested. The variance of the accelerated sequential sampling scheme is very close to the variance of the optimal sampling scheme, and it is smaller than the variance of the batch sequential sampling scheme. The speed of the accelerated sequential sampling scheme is much smaller than the speed of the batch sequential sequential sampling scheme ; see Table 1 below.

Table 1. Comparison of the accelerated sequential sampling scheme with the optimal sampling scheme and the batch sequential sampling scheme for equal probabilities (T =100 and with N=5,000 replications).

p_1, p_2, p_3, p_4	Var(O)/Var(AS)	Var(AS)/Var(BS)	Speed(AS)/Speed(BS)
0.10, 0.10, 0.10, 0.10	0.999484	0.968588	0.015651
0.30, 0.30, 0.30, 0.30	0.999544	0.985473	0.029962
0.50, 0.50, 0.50, 0.50	0.999693	0.998636	0.018320
0.70, 0.70, 0.70, 0.70	0.999951	0.985364	0.003254
0.90, 0.90, 0.90, 0.90	0.999934	0.961342	0.001637
0.95, 0.95, 0.95, 0.95	0.999937	0.965001	0.001712

Table 2. Comparison of the accelerated sequential sampling with the optimal sampling scheme and the batch sequential sampling sheme with different probabilities (T=100 and with N=5,000 replications).

p_1, p_2, p_3, p_4	Var(O)/Var(AS)	Var(AS)/Var(BS)	Speed(AS)/Speed(BS)
0.10, 0.20, 0.30, 0.30	0.994120	0.970301	0.161136
0.10, 0.20, 0.30, 0.40	0.989623	0.943249	0.147098
0.20, 0.30, 0.30, 0.40	0.997967	0.998618	0.594995
0.20, 0.30, 0.40, 0.50	0.993261	0.961056	0.142579
0.60, 0.60, 0.70, 0.80	0.986567	0.958598	0.237231
0.60, 0.70, 0.70, 0.80	0.993349	0.940493	0.095118
0.60, 0.70, 0.80, 0.90	0.936707	0.886949	0.331805
0.70, 0.80, 0.80, 0.90	0.976952	0.890114	0.157323
0.70, 0.80, 0.90, 0.95	0.877438	0.862390	0.434428
0.90, 0.90, 0.95, 0.95	0.943777	0.914446	0.375371
0.10, 0.20, 0.70, 0.80	0.825716	0.801810	0.413523
0.10, 0.20, 0.80, 0.90	0.766804	0.829911	0.532236
0.20, 0.30, 0.60, 0.70	0.933788	0.866344	0.300298
0.20, 0.30, 0.80, 0.90	0.794287	0.867072	0.572986
0.30, 0.40, 0.60, 0.70	0.968514	0.920151	0.266235
0.30, 0.40, 0.80, 0.90	0.821070	0.881773	0.571646

Consider now a system that has two subsystems connected in series with unequal reliabilities $p_1 = 0.70$, $p_2 = 0.80$, $p_3 = 0.80$, $p_4 = 0.90$, and suppose that a total of 100 units will be tested. The ratio of the variance incurred by the optimal sampling scheme over the variance incurred by the accelearted sequential sampling scheme is 0.976952. However, the ratio of the variance incuured by the accelerated sequential sampling scheme over the variance incurred by the batch sequential sampling scheme is 0.890114, and the ratio of the speed of the accelerated sequential sampling scheme over the speed of the batch sequential sequential sampling scheme is 0.15733; see Table 2.

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