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Bayesian analysis of a repairable system subject to overhauls with bounded failure intensity

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Abstract. This paper deals with the Bayesian analysis of the failure data of a repairable mechanical system subject to minimal repairs and periodic overhauls. The effect of overhauls on the reliability of the system is modeled by a proportional age reduction model and the failure process between two successive overhauls is assumed to be 2-parameter Engelhardt-Bain process (2-EBP). Power Law Process (PLP) model has a disadvantage which 2-EBP can overcome. On the basis of the observed data and of a number of suitable prior densities, point and interval estimation of model parameters, as well as quantities of relevant interest are found. Also hypothesis tests on the effectiveness of performed overhauls have been developed using Bayes factor. Sensitivity analysis of improvement parameter is carried out. Finally, a numerical application is used to illustrate the proposed method.

Key Words*: Non-homogeneous Poisson process, preventive maintenance, minimal repair, Bayesian inference, Bayes factor*

1. INTRODUCTION

Most repairable mechanical systems often undergo a maintenance policy because they degrade with operating time. Maintenance extends system's lifetime or at least the mean time to failure and if the maintenance policy is effective it reduces the frequency of failures and the undesirable consequences of such failures. Maintenance can be categorized into two classes: corrective and preventive actions.
1.Corrective maintenance: All actions performed to restore the system to functioning condition when it fails.
2. Preventive maintenance: All actions performed to prevent failures when the system is operating.

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Further, corrective and preventive maintenance actions are generally classified in terms of their effect on the operating conditions of the system. Pham, *et al* (1996) classified them as Perfect maintenance (good-as-new), Minimal maintenance (bad-as-old), Imperfect maintenance (between good-as-new and bad-as-old), Worse maintenance (worse condition than just prior to its failure). In real-world situations, maintenance generally enhances the condition of the equipment at a level between the extremes, i.e. Imperfect maintenance.

We consider a system that deteriorates with age and receives two kinds of maintenance actions: minimal repair and overhaul. When a failure occurs, minimal repair is carried out. The minimal repair is a corrective maintenance action that brings the repaired equipment to the condition~~s~~ it was just before the failure occurrence (bad-as-old). Hence, the reliability of the system decreases with operating time until it reaches unacceptable values. When it reaches unacceptable values or at prefixed epochs, preventive maintenance action (overhaul) is performed so as to improve the system condition and hence reduce the probability of failure occurrence in the following interval. However, overhaul cannot return the system to "good as new", and thus it can be treated as imperfect repair. When the overhaul is effective, the reliability of the system improves significantly. An overhaul usually consists of a set of preventive maintenance actions such as oil change, cleaning, greasing and replacing some worn components of the system.

A general approach to model the improvement effect of maintenance, where each maintenance reduces the age of the unit in the view of the rate of occurrences of failures has been proposed by Malik (1979). It is assumed that each maintenance reduces proportionally the operating time elapsed from the previous maintenance. The proportional age reduction (PAR) model for imperfect maintenance given by Malik (1979) is a generalization of good-as-new and bad-as-old. Shin, *et al* (1996) have used this general model to propose a PAR model which assumes that each major overhaul reduces proportionally the age of the equipment by a fraction of the epoch of the overhaul. Two classes of models have been proposed by Jack (1997) to model different effects of rejuvenation of the preventive maintenance.

Bayes approach has been used by several authors in contrast to the classical approach, as it helps in incorporating prior information and/or technical knowledge on the failure mechanism and on the overhaul effectiveness into the inferential procedure. Pulcini (2000) has dealt with the statistical analysis of the failure data of repairable mechanical units subjected to minimal repairs and periodic overhauls from a Bayes viewpoint. The Power Law process (PLP) is used to model the failure process between two successive overhauls and the effect of overhauls on the reliability of the system is modeled by PAR model. Further, Pulcini (2001a) has also delved upon the prediction of future failures of repairable mechanical units subjected to minimal repairs and periodic overhauls using PLP again to model the failure process between two successive overhaul epochs with the effect of overhauls on the reliability of the system modeled by proportional age reduction model from a Bayes viewpoint.

In PLP models, the increasing failure intensity tends to infinity as the system age increases. However, it is noted that when beginning from a given system age, repeated minimal repair actions are combined with some overhauls performed in order to oppose the growth of failure intensity with the operating time, the failure intensity of deteriorating repairable systems attains a finite bound. The average behavior of the intensity function due to the

consecutive steps with increasing intensity between two subsequent overhauls result in globally constant asymptotic intensity. If such a data is analyzed through models with unbounded increasing failure intensity, such as the PLP, then pessimistic estimates of the system reliability may arise and incorrect preventive maintenance policy may be defined.
NHPP with an increasing bounded intensity was first suggested by Engelhardt & Bain (1986) called Engelhardt-Bain process (EBP). But the mathematical simplicity of their 1-parameter model makes it inadequate to analyze failure data in many cases. Most recently, Pulcini (2001b) has proposed a 2-parameter NHPP called the Bounded Intensity Process (BIP) to fit the failure data of deteriorating repairable systems showing bounded intensity function. Attardi, *et al* (2005) have proposed a new bounded intensity process called 2-parameter Engelhardt-Bain process(2-EBP) whose failure intensity function is given by:

$$\lambda(t) = \eta\left(\frac{t}{t+\theta}\right); \eta, \theta > 0; t \geq 0 \quad . \tag{1.1}$$

The failure intensity in (1.1) tends to its asymptote more slowly than the BIP intensity, and is in a way, a compromise between the PLP and BIP models. When $\eta \equiv 1/\theta$, the intensity in (1.1) degenerates into the EBP intensity.
This paper deals with the Bayesian analysis of the failure data of a repairable mechanical system subjected to minimal repairs and periodic overhauls. To model the effect of overhauls on the reliability of the system a proportional age reduction model is assumed and the 2-parameter Engelhardt-Bain process (2-EBP) is used to model the failure process between two successive overhauls. Point and interval estimates of model parameters as well as other quantities of interest have been provided. Hypothesis tests on the effectiveness of performed overhauls have been provided using Bayes factor. Sensitivity analysis of improvement parameter is carried out. Finally, a numerical application is used to illustrate the proposed method.

Nomenclature

n	total no. of failures
k	total no. of overhauls
x_i	i^{th} overhaul epoch, i = 1….k
η, θ	parameters of 2-EBP
ρ	improvement factor
t_r	time at which the failure intensity is r times the asymptotic value
$\lambda_1(t)$	initial failure intensity
$\lambda_{j+1}(t \mid x_j)$	conditional intensity function at a generic time t in the interval (x_j, x_{j+1})
$E\{N(x_j, x_{j+1})\}$	expected number of failures between two successive overhaul epochs
$g(\eta)$	prior pdf on η
$g(t_r)$	prior pdf on t_r
$g(\rho)$	prior pdf on ρ

a, b	prior gamma parameters of η
c, d	prior gamma parameters of t_r
p, q	prior beta parameters of ρ
μ_η, σ_η^2	prior mean and variance of η
$\mu_{t_r}, \sigma_{t_r}^2$	prior mean and variance of t_r
μ_ρ, σ_ρ^2	prior mean and variance of ρ
$g(\eta, t_r, \rho)$	joint prior pdf of η, t_r, ρ
$\pi(\eta, \theta, \rho \mid \text{data})$	joint posterior pdf of η, θ, ρ
N_τ	expected number of failures from 0 up to a generic time τ in the interval (x_i, x_{i+1})
NHPP	Non Homogeneous Poisson Process
2-EBP	2-parameter Engelhardt-Bain process
PLP	Power Law Process

2. BASIC ASSUMPTIONS

- Failure rate of the system is an increasing function of time that attains a finite bound as t tends to infinity.
- System is subjected to two kinds of maintenance actions: minimal repair and overhaul.
- The times to perform maintenance actions are ignored.
- Minimal repair will restore the failure rate only to bad-as-old condition. But overhauls will improve the system to a condition between bad-as-old and good-as-new.
- The failure density function is not changed by overhauls.
- The quality of an overhaul is dependent on improvement factor $\rho(0 \le \rho \le 1)$.
- The improvement parameter ρ has a uniform value over all the overhaul actions.
- The j^{th} overhaul reduces the age t of the system by a fraction ρ of the epoch x_j of the overhaul.
- The effect of overhauls on the reliability of the system is modeled by proportional age reduction model and the 2-parameter Engelhardt-Bain process(2-EBP) is used to model the failure process between two successive overhauls, say (x_j, x_{j+1}).
- $x_1, x_2, \ldots, x_k$ are the k overhaul epochs, which may coincide with failure times.

3. MODEL FORMULATION

Let $t_1 < t_2 < ... < t_n$ denote the n failure times of the repairable system observed till T. If $T \equiv t_n$, i.e. the process is observed till n[th] failure, then it is failure truncated sampling and T is a random variable. If $T > t_n$, then it is time truncated sampling, where T is a prefixed quantity and n is a random variable.

The 2-EBP model is an NHPP whose failure intensity is of the form

$$\lambda(t) = \eta\left(\frac{t}{t+\theta}\right); \eta, \theta > 0; t \geq 0 \tag{3.1}$$

which is an increasing bounded intensity function with the operating time t, equal to 0 at t = 0 and approaching an asymptote of η as t increases. So, the parameter η is the asymptotic value of the intensity function: $\eta \equiv \lambda_\infty$, and the parameter θ is a measure of the initial increasing rate of $\lambda(t)$: the smaller θ is, the faster the failure intensity increases until it approaches η .

Thus, the initial failure intensity, i.e. the intensity function till the first overhaul epoch x_1, is:

$$\lambda_1(t) = \eta\left(\frac{t}{t+\theta}\right); \; t \leq x_1. \tag{3.2}$$

The conditional intensity function at a generic time t in the interval (x_j, x_{j+1}), given x_j is :

$$\begin{aligned}\lambda_{j+1}(t \mid x_j) &= \lambda_1(t - \rho x_j) \\ &= \eta\left(\frac{t-\rho x_j}{t-\rho x_j+\theta}\right); \quad x_j < t \leq x_{j+1}.\end{aligned} \tag{3.3}$$

The expected number of failures between two successive overhaul epochs is:

$$\begin{aligned}E\{N(x_j, x_{j+1})\} &= \int_{x_j}^{x_{j+1}} \lambda_{j+1}(t \mid x_j)dt \\ &= \eta\left[(x_{j+1} - x_j) - \theta\log\left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta}\right)\right].\end{aligned} \tag{3.4}$$

The cumulative value of expected number of failures from 0 up to a generic time τ in the interval (x_i, x_{i+1}) is:

$$N_{(\tau)} = \left[\sum_{j=0}^{i-1} E\{N(x_j, x_{j+1})\}\right] + E\{N(x_i, x_\tau)\}, \; x_i \leq \tau \leq x_{i+1}, \; x_0 = 0,$$

$$= \eta\left[\sum_{j=0}^{i-1}\left((x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right)\right]$$
$$+ \tag{3.5}$$
$$\eta\left[(\tau-x_i)-\theta\log\left(\frac{\tau-\rho x_i+\theta}{x_i-\rho x_i+\theta}\right)\right].$$

4. LIKELIHOOD FUNCTION

If the data is time truncated, the joint pdf of the failure times ($t_1, t_2,\ldots,t_n$) :

$$f(t_1, t_2, ..., t_n) = f(t_1)\, f(t_2 \mid t_1) \ldots f(t_n \mid t_1, t_2, ..., t_{n-1})\, \overline{F}(T \mid t_n)$$

$$= \lambda(t_1).e^{-\int_0^{t_1}\lambda(u)du} . \lambda(t_2).e^{-\left[\int_0^{t_2}\lambda(u)du-\int_0^{t_1}\lambda(u)du\right]} \ldots \lambda(t_n).e^{-\left[\int_0^{t_n}\lambda(u)du-\int_0^{t_{n-1}}\lambda(u)du\right]} .e^{-\int_{t_n}^{T}\lambda(u)du}$$

$$= \left[\prod_{i=1}^{n}\lambda(t_i)\right]. e^{-\int_0^{T}\lambda(u)du} , \tag{4.1}$$

where T is the truncation time.

As $x_1, x_2, ...x_k$ are the k overhaul epochs, they provide disjoint fixed intervals (x_0, x_1), (x_1, x_2),… (x_{k-1}, x_k). Since the overhaul epochs are prescheduled, they are not random. Let the time interval (x_j, x_{j+1}) be the (j+1)th period.

From (3.3) and (4.1) for $t \in \left(x_j, x_{j+1}\right)$, the joint pdf of failure times in (x_j, x_{j+1})

$$\left[\prod_{i=t_i \in (x_j, x_{j+1}]} \lambda_{j+1}(t_i \mid x_j)\right] e^{-\int_{x_j}^{x_{j+1}} \lambda_{j+1}(t|x_j)dt} . \tag{4.2}$$

Since the overhaul epochs are prescheduled, the failure times in disjoint epoch periods are independent and therefore, the joint pdf of the failure times ($t_1, t_2,\ldots,t_n$) is simply the product form of (4.2)., i.e.

$$f(t_1, t_2, ..., t_n) = L(\text{data} \mid \eta, \theta, \rho) = \left[\prod_{j=0}^{k}\prod_{t_i \in (x_j, x_{j+1})} \lambda_{j+1}(t_i \mid x_j)\right]. e^{-\sum_{j=0}^{k}\int_{x_j}^{x_{j+1}} \lambda_{j+1}(t|x_j)dt}$$

$$= \eta^n \left[\prod_{j=0}^{k} \prod_{t_i \in (x_j, x_{j+1})} \left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta} \right) \right] . e^{-\eta \sum_{j=0}^{k} \left[(x_{j+1} - x_j) - \theta \log \left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta} \right) \right]}, \tag{4.3}$$

where $x_{k+1} = T$.

5. BAYESIAN PROCEDURE

5.1 Informative prior

Suppose the analyst is able to anticipate:

a) the asymptotic value $\lambda_\infty \equiv \eta$ of the intensity function, and

b) the time t_r at which the failure intensity is r times the asymptotic value (r<1) i.e.

$$\frac{\lambda(t_r)}{\lambda_\infty} = r \; ; x_i < t_r < x_{i+1}$$

$$\Rightarrow t_r = \left(\frac{(1-r)\rho x_i + r\theta}{1-r} \right). \tag{5.1}$$

Now, we assume a Gamma density for η and t_r, both for the mathematical tractability and flexibility. Therefore,

$$g(\eta) = \frac{b^a}{\Gamma(a)} \eta^{a-1} \exp(-b\eta); \, a, b > 0, \tag{5.2}$$

and

$$g(t_r) = \frac{d^c}{\Gamma(c)} (t_r)^{c-1} \exp(-dt_r); \, c, d > 0, \tag{5.3}$$

where the gamma parameters are related to the prior mean and standard deviation by:

$$a = \left(\frac{\mu_\eta}{\sigma_\eta} \right)^2, \, b = \frac{\mu_\eta}{\sigma_\eta^2}, \tag{5.4}$$

and

$$c = \left(\frac{\mu_t}{\sigma_t} \right)^2, \, d = \frac{\mu_t}{\sigma_t^2}. \tag{5.5}$$

The analyst can use the fact that a ρ value closer to 0 (to 1) indicates less (more) effectiveness of the overhaul actions to formulate an informative prior on ρ. We assume that the analyst is able to anticipate both a prior mean and standard deviation on ρ, say μ_ρ and σ_ρ, then we formalize this information through a beta density:

$$g(\rho) = \frac{\rho^{p-1}(1-\rho)^{q-1}}{\beta(p,q)}; \, 0 \le \rho \le 1, \tag{5.6}$$

where the values of the parameters related to the prior mean and standard deviation are:

$$p = \frac{(1-\mu_\rho)\,\mu_\rho^2}{\sigma_\rho^2} - \mu_\rho,\; q = p\frac{(1-\mu_\rho)}{\mu_\rho}. \tag{5.7}$$

5.2 Joint prior density

Assuming the prior independence of the parameters (η, θ, ρ), and using (5.1) to find density of θ, the joint prior density $g(\eta, \theta, \rho)$ is given by:

$$g(\eta, \theta, \rho) \propto \eta^{a-1} e^{-b\eta} \left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)^{c-1} e^{-d\left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)} \rho^{p-1}(1-\rho)^{q-1}. \tag{5.8}$$

5.3 Joint posterior density on model parameters

By combining the likelihood (4.3) with the joint prior density (5.8), the joint posterior density on the model parameters is given by:

$$\pi(\eta, \theta, \rho \mid data) = \frac{1}{D}\eta^n.\left[\prod_{j=0}^{k}\prod_{t_i \in (x_j, x_{j+1})}\left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta}\right)\right]. e^{-\eta\sum_{j=0}^{k}\left[(x_{j+1}-x_j) - \theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j - \rho x_j + \theta}\right)\right]}.$$
$$\eta^{a-1}.e^{-b\eta}.\left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)^{c-1}.e^{-d\left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)}.\rho^{p-1}.(1-\rho)^{q-1}, \tag{5.9}$$

where

$$D = \Gamma(n+a)\int_0^\infty\int_0^1 \frac{f(\theta, \rho \mid data)}{\left(b + \sum_{j=0}^{k}\left[(x_{j+1} - x_j) - \theta\log\left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta}\right)\right]\right)^{n+a}} d\rho d\theta\,, \tag{5.10}$$

$$f(\theta, \rho \mid data) = \left[\prod_{j=0}^{k}\prod_{t_i \in (x_j, x_{j+1})}\left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta}\right)\right].\left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)^{c-1}.e^{-d\left(\frac{(1-r)\rho x_i + r\theta}{1-r}\right)}.\rho^{p-1}.(1-\rho)^{q-1}. \tag{5.11}$$

6. POSTERIOR INFERENCE ON THE PARAMETERS

$$h(\theta \mid data) = \frac{\Gamma(n+a)}{D}\int_0^1 \frac{f(\theta, \rho \mid data)}{\left(b + \sum_{j=0}^{k}\left[(x_{j+1} - x_j) - \theta\log\left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta}\right)\right]\right)^{n+a}} d\rho \tag{6.1}$$

$$h(\rho\,|\,\text{data}) = \frac{\Gamma(n+a)}{D}\int_0^{\infty}\frac{f(\theta,\rho\,|\,\text{data})}{\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)^{n+a}}d\theta \tag{6.2}$$

$$h(\eta\,|\,\text{data}) = \frac{1}{D}\int_0^{\infty}\int_0^{1} f(\theta,\rho\,|\,\text{data}).\eta^{n+a-1}.e^{-\eta\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)}d\rho d\theta \tag{6.3}$$

The point estimates (mean, median, and mode) of the parameters (θ, ρ, η) as well as the credibility intervals can be obtained from (6.1), (6.2), (6.3), respectively using numerical integration and iterative procedures.

7. POSTERIOR INFERENCE ON THE INTENSITY FUNCTION

The conditional intensity function at a generic time τ in the interval (x_i, x_{i+1}), given x_i is:

$$\begin{aligned}\lambda_\tau &\equiv \lambda_{i+1}(\tau\,|\,x_i)\\ &= \eta\left(\frac{\tau-\rho x_i}{\tau-\rho x_i+\theta}\right); \quad x_i < \tau \le x_{i+1}.\end{aligned} \tag{7.1}$$

$$\Rightarrow \eta = \lambda_\tau\left(\frac{\tau-\rho x_i+\theta}{\tau-\rho x_i}\right) . \tag{7.2}$$

Using (7.2), the posterior density on λ_τ is given by:

$$h(\lambda_\tau\,|\,\text{data}) = \frac{1}{D}.$$

$$\int_0^{\infty}\int_0^{1} f(\theta,\rho\,|\,\text{data}).\left(\frac{\tau-\rho x_i+\theta}{\tau-\rho x_i}\right)^{n+a}.(\lambda_\tau)^{n+a-1}.e^{-\lambda_\tau\left(\frac{\tau-\rho x_i+\theta}{\tau-\rho x_i}\right)\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)}d\rho d\theta\,. \tag{7.3}$$

The point estimates (mean, median, and mode) of λ_τ as well as the credibility intervals can be obtained from (7.3) using numerical integration and iterative procedures.

In particular, the posterior mean $E\{\lambda_\tau\,|\,\text{data}\}$ is:

$$E\{\lambda_\tau \mid data\} = \frac{\Gamma(n+a+1)}{D}.$$
$$\int_0^\infty\int_0^1 f(\theta,\rho \mid data).\left(\frac{\tau-\rho x_i}{\tau-\rho x_i+\theta}\right).\frac{1}{\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)^{n+a+1}}d\rho d\theta\ . \tag{7.4}$$

8. POSTERIOR INFERENCE ON THE EXPECTED NUMBER OF FAILURES

Using (5.3)

$$\eta = N_{(\tau)} / Z\ , \tag{8.1}$$

where

$$Z = \left[\sum_{j=0}^{i-1}\left((x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right)\right]+\left[(\tau-x_i)-\theta\log\left(\frac{\tau-\rho x_i+\theta}{x_i-\rho x_i+\theta}\right)\right]. \tag{8.2}$$

Using (8.1), the posterior density of expected number of failures in the time interval $(0,\tau)$ is:

$$h(N_{(\tau)} \mid data) = \frac{1}{D}.$$
$$\int_0^\infty\int_0^1 f(\theta,\rho \mid data).\left(\frac{1}{Z}\right)^{n+a}.\left(N_{(\tau)}\right)^{n+a-1}.e^{-\left(\frac{N_{(\tau)}}{Z}\right)\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)}d\rho d\theta\ . \tag{8.3}$$

The point estimates (mean, median, and mode) of $N_{(\tau)}$ as well as the credibility intervals can be obtained from (8.3) using numerical integration and iterative procedures.
In particular, the posterior mean $E\{N_{(\tau)} \mid data\}$ is:

$$E\{N_{(\tau)} \mid data\} = \frac{\Gamma(n+a+1)}{D}.\int_0^\infty\int_0^1 f(\theta,\rho \mid data).Z.\frac{1}{\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)^{n+a+1}}d\rho d\theta. \tag{8.4}$$

9. BAYES TESTING

To assess the effectiveness of the overhauls and evaluate evidence in favour of the assumed PAR-2EBP model, statistical procedures need to be carried out.

Further, $\rho\,(0 \le \rho \le 1)$ is the improvement parameter that measures the effect (on the average) of major overhauls on the reliability of the system. When $\rho = 0$, the overhaul effectiveness is null (minimal repair or bad-as-old maintenance), and the whole process reduces to 2-EBP. When $\rho = 1$, the overhaul is perfect (good-as-new maintenance), and each overhaul produces a renewal of the equipment. Thus, testing for the complete effectiveness or ineffectiveness of major overhauls means to test $\rho = 1$ or $\rho = 0$ in the presence of model parameters θ, η.

The Bayesian approach to hypothesis testing was developed by Jeffreys (1935, 1961). In this approach he introduced the statistical methods to represent the probability of the data according to each of the two competing theories, and the posterior probability that one of the theories is correct is computed using Bayes' theorem.

The posterior probability $\Pr\{H_k \mid \text{data}\}$ {k=0, 1} is:

$$\Pr\{H_k \mid \text{data}\} = \frac{\Pr\{\text{data} \mid H_k\}.\Pr\{H_k\}}{\Pr\{\text{data} \mid H_0\}.\Pr\{H_0\} + \Pr\{\text{data} \mid H_1\}.\Pr\{H_1\}}, \quad k=0, 1, \tag{9.1}$$

where

$\Pr\{\text{data} \mid H_k\}$ is the product of the likelihood function and the joint prior of the model parameters.

Bayes factor B_{01} is defined as:

$$B_{01} = \frac{\Pr\{H_0 \mid \text{data}\} / \Pr\{H_1 \mid \text{data}\}}{\Pr\{H_0\} / \Pr\{H_1\}} = \frac{\Pr\{\text{data} \mid H_0\}}{\Pr\{\text{data} \mid H_1\}}. \tag{9.2}$$

Large values of B_{01} provide evidence in favour of the null hypothesis H_0, whereas large values of $B_{10} = \frac{1}{B_{01}}$ provide evidence against H_0.

Kass, *et al* (1995) provided appropriate bounds for $2\log B_{10}$:

$2\ln B_{10}$	**Evidence against** H_0
0 to 2	Not worth more than a bare mention
2 to 6	Positive
6 to 10	Strong
> 10	Very Strong

9.1. Testing for perfect overhauls

$$H_0^{'} : \rho = 1$$
$$H_1 : 0 < \rho < 1$$

Evidence in favour of null hypothesis $H_0^{'}$ would affirm that the effectiveness of overhauls is complete whereas evidence in favour of alternate hypothesis would affirm that the effectiveness of overhauls is partial and PAR-2EBP model is more adequate to describe the underlying failure process.

Under $H_0^{'}$, the likelihood function is:

$$L(data\,|\,\eta,\theta)=\eta^{n}\left[\prod_{j=0}^{k}\prod_{t_i\in(x_j,x_{j+1})}\left(\frac{t_i-x_j}{t_i-x_j+\theta}\right)\right].\,e^{-\eta\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-x_j+\theta}{\theta}\right)\right]}. \tag{9.3}$$

Using (5.8) and (9.3),

$$\Pr\{data\,|\,H_0^{'}\}=\frac{b^a d^c\Gamma(n+a)}{\Gamma(a)\Gamma(c)}\int_0^{\infty}\frac{\left(\frac{(1-r)x_i+r\theta}{1-r}\right)^{c-1}.e^{-d\left(\frac{(1-r)x_i+r\theta}{1-r}\right)}.\left[\prod_{j=0}^{k}\prod_{t_i\in(x_j,x_{j+1})}\left(\frac{t_i-x_j}{t_i-x_j+\theta}\right)\right]}{\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-x_j+\theta}{\theta}\right)\right]\right)^{n+a}}d\theta\ , \tag{9.4}$$

$$\Pr\{data\,|\,H_1\}=\frac{b^a d^c\Gamma(n+a)}{\Gamma(a)\Gamma(c)B(p,q)}\int_0^{\infty}\int_0^{1}\frac{f(\theta,\rho\,|\,data)}{\left(b+\sum_{j=0}^{k}\left[(x_{j+1}-x_j)-\theta\log\left(\frac{x_{j+1}-\rho x_j+\theta}{x_j-\rho x_j+\theta}\right)\right]\right)^{n+a}}d\rho d\theta\ , \tag{9.5}$$

and the Bayes factor B_{01} is the ratio of (9.4) to (9.5).

9.2. Testing for null effectiveness of overhauls

$$H_0^{''}:\rho=0$$
$$H_1:0<\rho<1$$

Evidence in favour of null hypothesis $H_{0''}$ would affirm that the effectiveness of overhauls is null whereas evidence in favour of alternate hypothesis would affirm that the effectiveness of overhauls is partial and PAR-2EBP model is more adequate to describe the underlying failure process.

Under $H_0^{''}$, the likelihood function is:

$$L(data\,|\,\eta,\theta)=\eta^{n}\left[\prod_{i=0}^{n}\left(\frac{t_i}{t_i+\theta}\right)\right].\,e^{-\eta\left[T-\theta\log\left(1+\frac{T}{\theta}\right)\right]}. \tag{9.6}$$

Using (5.8) and (9.6)

$$\Pr\{data \mid H_0''\} = \frac{b^a d^c \Gamma(n+a)}{\Gamma(a)\Gamma(c)} \int_0^\infty \frac{\left(\frac{r\theta}{1-r}\right)^{c-1} . e^{-d\left(\frac{r\theta}{1-r}\right)} . \left[\prod_{i=0}^{n}\left(\frac{t_i}{t_i+\theta}\right)\right]}{\left(b+T-\theta\log\left(1+\frac{T}{\theta}\right)\right)^{n+a}} d\theta, \quad (9.7)$$

and the Bayes factor B_{01} is the ratio of (9.7) to (9.5).

10. SENSITIVITY ANALYSIS

Sensitivity analysis is carried out with respect to the prior information on ρ, namely the prior mean μ_ρ and the standard deviation σ_ρ, by evaluating the Bayes factor over a reasonable range of values for μ_ρ and σ_ρ. If the change in values of μ_ρ and σ_ρ does not have much effect on Bayes factor, then the proposed prior is robust.

11. NUMERICAL APPLICATION

Consider the following hypothetical data for illustrative purpose:
The failure times (n =18) and overhaul epochs are given in Table 1, major overhauls marked with *. We have assumed it to be a time truncated sample with failures observed for 1500 units. Four major overhauls are assumed to be performed at times different than failure times.

Table 1. Failure times and overhaul epochs

202	265	300*	363	508	571	600*	755	770	818	868
900*	999	1054	1068	1108	1200*	1230	1268	1330	1376	1447

Suppose that analyst is able to anticipate a prior mean μ_η = 0.133 and σ_η = .004 (a = 4, b = 30). In addition from previous experiences, the analyst possesses a vague belief that the failure intensity, at the time t_r = 753 units, is nearly half its asymptotic value: $\frac{\lambda(753)}{\lambda_\infty} = 0.5$. Then, he formalizes his prior knowledge on t_r through the exponential density having mean $\mu_t = 753$, so that: c = 1 and d = 0.001. As $x_i < t_r < x_{i+1}$, therefore 600 < 753 < 1200. Hence, x_i = 600.
Again, the analyst possesses a vague belief that the overhaul actions are quite effective, and then he chooses the beta density for the improvement parameter with prior mean $\mu_\rho = 0.6$ and standard deviation σ_ρ = 0.26 (p = 1.5 and q = 1). The posterior mean and the 0.95 equal-tails credibility intervals for θ, ρ are given in Table 2.

Table 2. Bayes estimates and credibility intervals

Bayes Estimates	θ	ρ
Point Estimate	1879.49	0.683
0.95 Lower Limit	515	0.236
0.95 Upper Limit	3940	0.956

Table 3 compares the observed failure data to the posterior mean and 0.80 equal-tails credibility intervals of the cumulative expected number of failures $N_{(\tau)}$.

In Table 4 and Table 5 the quantity $2\log B_{10}$ is given for the selected values of μ_ρ and σ_ρ respectively, when the null hypothesis is $H_0^{'}:\rho=1$ and $H_0^{''}:\rho=0$.
The values in Tables 4 and 5 are compared with the bounds given by Kass, *et al* (1995), thus affirming that there is positive evidence against the null hypothesis, so that neither the overhauls are completely effective nor have null effectiveness. So, PAR-2EBP is adequate to describe the failing process.

Moreover the change in values of μ_ρ and σ_ρ does not have much effect on Bayes factor, therefore, the proposed prior is robust.

Table 3. Posterior mean and Credibility Intervals of $N_{(\tau)}$

0- τ	Observed failures	Mean Number of Failures	0.80 Lower Limit	0.80 Upper Limit
0-300	2	2.137	1.44	2.76
0-400	3	2.763	2.03	3.44
0-500	3	3.815	2.88	4.53
0-600	5	5.241	4	6.41
0-700	5	6.182	4.84	7.42
0-800	7	7.514	5.95	8.97
0-900	9	9.191	7.31	10.95
0-1000	10	10.411	8.35	12.39
0-1100	12	11.993	9.66	14.2
0-1200	13	13.899	11.20	16.43
0-1300	15	15.368	12.41	18.16
0-1400	17	17.178	13.9	20.3
0-1500	18	19.290	15.62	22.77

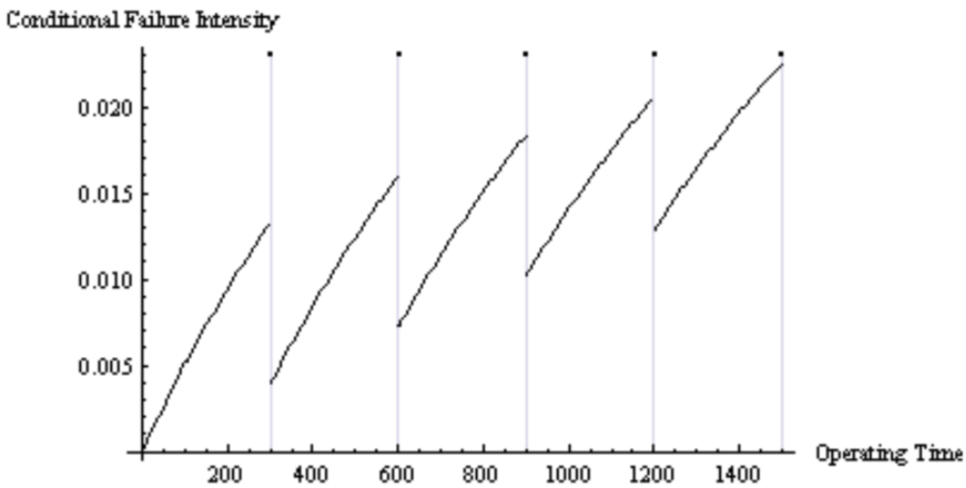

Figure 1. The posterior mean of the conditional failure intensity λ_τ

Table 4. Sensitivity Analysis on ρ over a reasonable range of values for μ_ρ

Percentage Deviation	μ_ρ	$2\log B_{10}$ $H_{0'}:\rho=1$	$H_{0''}:\rho=0$
-1%	0.594	4.94386	2.0820
+1%	0.606	4.9630	2.1012
-2%	0.588	4.9320	2.0702
+2%	0.612	4.9720	2.1102
-3%	0.582	4.9204	2.0586
+3%	0.618	4.9804	2.1186

Table 5. Sensitivity Analysis on ρ over a reasonable range of values for σ_ρ

Percentage Deviation	σ_ρ	$2\log B_{10}$ $H_{0'}:\rho=1$	$H_{0''}:\rho=0$
-1%	0.2574	4.9595	2.0977
+1%	0.2626	4.9468	2.0850
-2%	0.2548	4.9655	2.1037
+2%	0.2652	4.9402	2.0784
-3%	0.2522	4.9712	2.1094
+3%	0.2678	4.9334	2.0716

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